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SOLUTION OF GENERALIZED PARTIAL DIFFERENTIAL EQUATIONS BY USING DOUBLE q-INTEGRAL TRANSFORM

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Abstract. In this article, we present the meaning of two Laplace-Sumudu transforms in q-calculus using the two variable functions. This binary transformation is a new combination of the Laplace transform and the Sumudu transform. The main aim of this work is to demonstrate a new efficient binary q-transformation for solving differential equations. To present this new change, several problems are discussed to understand the effectiveness and efficiency of the plan. Also, this article surveys recent applications to solve generalized diffusion, wave and space-time telegraphic equations.

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1. Introduction

Many physical problems of interest are defined by ODEs or PDEs with appropriate initial or boundary conditions. These problems are often formulated in practice and engineering sciences as threshold problems, boundary value problems, or boundary value threshold problems, which tend to be more mathematically rigorous and physically more realistic [22]. The Laplace transform method is particularly useful for solving these problems. This method is useful for resolving the response of systems managed by the variable as equal to the initial data [21].

A partial differential equation is very important in mathematical physics [10], wave equation is known as the fundamental equation in mathematical physics and appears in many branches of physics such as mathematics and engineering [18]. Although it is important to obtain exact solutions to partial equations in applied mathematics, finding new ways of discovering new realities or approximate solutions for [30] is still a difficult problem.

In recent years, many authors have devoted to studying the solution of differential equations using different methods. Among them, experiments are Laplace variational iteration method, differential transform method [3], Laplace [17], Fourier, double Laplace transform [13], Sumudu transform [12, 17] and Adomian decomposition method. This article discusses the solutions of differential equations and partial differential equations that arise in mathematics, physics, and engineering sciences. We present new methods based on Laplace and Sumudu transforms to be used in modifications or analogues of Laplace-Sumudu transforms.

The Laplace transform of the $\zeta(x)$ function in [8] is defined as:

$$L\{\zeta(x)\} = \int_0^\infty e^{-\rho x}\zeta(x)dx, \qquad Re(\rho) > 0 \tag{1.1}$$

and its inverse denoted by L^{-1} is defined by

$$\zeta(x) = L^{-1}\{\bar{\zeta}(\rho)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\rho x}\bar{\zeta}(\rho)d\rho, \quad c \geq 0.$$

In [7, 19, 20] the authors describe the q-analogue of the famous Laplace transform of q-Jackson [15] integrals

$${}_qL_s\{\zeta(\rho)\} = \frac{1}{(1-q)}\int_0^\infty e_q(-s\rho)\zeta(\rho)d_q\rho. \tag{1.2}$$

The Sumudu transform was introduced by Watugala [26], its recorded frequency is more than Laplace, than an average

$$S\{\zeta(\rho); u\} = \int_0^\infty e^{-\rho}\zeta(u\rho)d\rho, \qquad u \in (-\tau_1, \tau_2).$$

The Sumudu transform for a time function $\zeta(\rho)$ is calculated by factoring Sumudu's transformation variable u as part of the $\zeta(\rho)$ function and then integrating against $e^{-\rho}$. This u is significant in the first function and $\zeta(\rho)$ becomes $\zeta(u\rho)$ to preserve units and dimensions.

In [2], the authors describe the q-analogue of the Sumudu transform as follows:

$$S_q\{\zeta(\rho); s\} = \frac{1}{(1-q)s}\int_0^\infty e_q(-\frac{\rho}{s})\zeta(\rho)d_q\rho, \quad s \in (\tau_1, \tau_2).$$

For further details in q-calculus go through [1, 4].

The paper is organized section wise. In the next section, we introduce some of the key points and results that are important to provide important results. In section 3, we introduce the q-Laplace-Sumudu transform, which provides some advantages such as convergence, absolute convergence. In section 4, we examine the convolution product and in subsection 4.1, we provide some properties of q-Laplace-Sumudu transform. In section 5, we give some examples to illustrate the main results. Finally, in section 6, the method has been used to solve some well-known partial differential equations.

2. Preliminaries

In this section, we list important terms and symbols used in this paper.

The $q-$shifted factorials for $q \in (0,1)$ and $\alpha \in \mathbb{C}$ are defined as

$$(\alpha; q)_0 = 1, \quad (\alpha; q)_n = \prod_{k=0}^{n-1}(1-\alpha q^k), \quad n = 1, 2, ...,$$

$$(\alpha; q)_\infty = \lim_{n\to\infty}(\alpha; q)_n = \prod_{k=0}^{\infty}(1-\alpha q^k).$$

Also we write

$$[\alpha]_q = \frac{1-q^\alpha}{1-q}, \quad [\alpha]_q! = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.$$

The $q-$derivatives of a function f is given in [16] as:

$$(D_q\zeta)(x) = \frac{\zeta(x)-\zeta(qx)}{(1-q)x}, \quad \text{if } x \neq 0$$

and $(D_q\zeta)(0) = \zeta'(0)$ provided $\zeta'(0)$ exists.
If ζ is differentiable, then $(D_q\zeta)(x)$ tend to $\zeta'(x)$ as q tends to 1. For $n \in \mathbb{N}$, we have

$$D_q^1 = D_q, \qquad (D_q^+)^1 = D_q^+.$$

The $q-$derivative of product of two functions is defined as

$$D_q(\zeta \cdot \eta)(x) = \eta(x)D_q\zeta(x) + \zeta(qx)D_q\eta(x).$$

The $q-$integrals from 0 to a and from 0 to ∞ is called the $q-$Jackson integral, defined in [15] by

$$\int_0^a \zeta(x)d_qx = (1-q)a\sum_{n=0}^{\infty}\zeta(aq^n)q^n$$

and

$$\int_0^\infty \zeta(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty}\zeta(q^n)q^n$$

provided these sums converge absolutely. The integration by parts in terms of $q-$calculus is given by

$$\int_a^b \eta(x)D_q\zeta(x)d_qx = \zeta(b)\eta(b) - \zeta(a)\eta(a) - \int_a^b \zeta(qx)D_q\eta(x)d_qx. \tag{2.1}$$

The $q-$analogues of the exponential function is described in [14, 16] as

$$E_q^z = \sum_{n=0}^{\infty} q^{n(n-1)/2}\frac{z^n}{[n]_q!} = (-(1-q)z;q)_\infty$$

and

$$e_q^z = \sum_{n=0}^{\infty}\frac{z^n}{[n]_q!} = \frac{1}{((1-q)z;q)_\infty}, \quad |z| < \frac{1}{1-q},$$

these q-exponentials are analogues of classical exponential functions and satisfies the relationship.

$$D_qe_q^z = e_q^z, \quad D_qE_q^z = E_q^{qz}$$

and

$$e_q^zE_q^{-z} = E_q^{-z}e_q^z = 1.$$

Jackson also describes the q-analogue of the classical gamma function in [24, 25, 27, 28, 29, 31, 32, 33].

$\Gamma(t) = \int\limits_0^\infty x^{t-1}e^{-x}dx$ by

$$\Gamma_q(t) = \frac{(q;q)_\infty}{(q^t;q)_\infty}(1-q)^{1-t}, \quad t \neq 0, -1, -2, \cdots .$$

This also satisfies

$$\Gamma_q(t+1) = [t]_q\Gamma(t),\ \Gamma_q(1) = 1 \ \text{ and } \lim_{q\to 1^-} \Gamma_q(t) = \Gamma(t),\ Re(t) > 0.$$

The Γ_q function has the q-integral representation as

$$\begin{aligned}\Gamma_q(s) &= \int\limits_0^{1/(1-q)} t^{s-1}E_q^{-qt}d_qt \\ &= \int\limits_0^{\infty/(1-q)} t^{s-1}E_q^{-qt}d_qt.\end{aligned}$$

The q-integral representation of Γ_q based on q-exponential function e_q^x and q-integral representation of q-beta function is defined in [23] as follows:

For all $s, t > 0$, we have

$$\Gamma_q(s) = K_q(s) \int\limits_0^{\infty/(1-q)} x^{s-1}e_q^{-x}d_qx$$

and

$$B_q(t,s) = K_q(t)\int\limits_0^\infty x^{t-1}\frac{(-xq^{s+t};q)_\infty}{(-x;q)_\infty}d_qx,$$

where in [6], $\quad K_q(t) = \dfrac{(-q,-1;q)_\infty}{(-q^t,-q^{1-t};q)_\infty}.$

If $\quad \dfrac{\log(1-q)}{\log(q)} \in \mathbb{Z}$, we obtain

$$\begin{aligned}\Gamma_q(s) &= K_q(s)\int_0^\infty x^{s-1}e_q^{-x}d_qx \\ &= \int\limits_0^\infty t^{s-1}E_q^{-qt}d_qt.\end{aligned}$$

3. The $\tilde{q}$-Laplace-Sumudu transform

Definition 3.1. Let $\tilde{q} = (q_1, q_2) \in (0,1),\ (s,t) \in \mathbb{C}$ and f be a function of two variables x and y defined on $\mathbb{R}_{q_1,+} \times \mathbb{R}_{q_2,+}$. Then the $\tilde{q}$-Laplace-Sumudu transform of f is defined by the double integral in the form:

$$ {}_{\tilde{q}}L_x S_y(f)(s,t) = H_{\tilde{q}}(s,t) = \frac{1}{(1-q)^2 t} \int_0^\infty \int_0^\infty e_q^{-sx-y/t} f(x,y) d_q x d_q y \tag{3.1} $$

provided the integral exists, where $\mathbb{R}_{q,+} = \{q^n, n \in \mathbb{Z}\}$ and

$$ e_q^{-x} = \begin{cases} [1-(1-q)x]^{1/1-q}, & \text{for } 0 < x < \dfrac{1}{1-q},\ q < 1, \\ [1-(q-1)x]^{-1/q-1}, & \text{for } x \geq 0,\ q > 1. \end{cases} $$

Remark 3.2. For suitable function f, $H_{\tilde{q}}(f)(s,t) = H(f)(s,t)$, when $\tilde{q}$ tends to $(1,1)$.

3.1. Convergence of $\tilde{q}$-Laplace-Sumudu transform.

Lemma 3.3. *If the integral* $\frac{1}{(1-q)} \int_0^\infty e_q^{-\frac{y}{t}} f(x,y) d_q y$ *converges at* $\frac{1}{t} = t_0$ *say, then this integral converges for* $\frac{1}{t} > t_0$.

Proof. Assume $\alpha(x,y) = \frac{1}{(1-q)} \int_0^y e_q^{-t_0 v} f(x,v) d_q v, \quad 0 < t < \infty$. Then, clearly $\alpha(x,0) = 0$ and $\lim_{t\to\infty} \alpha(x,y)$ exists, because integral

$$ \frac{1}{(1-q)} \int_0^\infty e_q^{-\frac{y}{t}} f(x,y) d_q y $$

converges at $\frac{1}{t} = t_0$.

By fundamental theorem of calculus

$$ \alpha_y(x,y) = \frac{1}{(1-q)} e_q^{-t_0 y} f(x,y). $$

Choose ϵ and R such that $0 < \epsilon < R$, then

$$\begin{aligned}\frac{1}{(1-q)}\int_{\epsilon}^{R} e_q^{-y/t} f(x,y)d_qy &= \frac{1}{(1-q)}\int_{\epsilon}^{R} e_q^{-\frac{y}{t}} \alpha_y(x,y)(1-q)e_q^{t_0y}d_qy \\ &= \int_{\epsilon}^{R} e_q^{-(1/t-t_0)y}\alpha_y(x,y)d_qy \\ &= e_q^{-(1/t-t_0)y}\alpha(x,qy)|_{\epsilon}^{R} \\ &\quad - \int_{\epsilon}^{R} \alpha(x,qy)(-1/t+t_0)e_q^{-(t-t_0)y}d_qy \\ &= e_q^{-(1/t-t_0)R}\alpha(x,qR) - e_q^{-(1/t-t_0)\epsilon}\alpha(x,q\epsilon) \\ &\quad - (1/t-t_0)\int_{\epsilon}^{R} \alpha(x,qy)e_q^{-(1/t-t_0)y}d_qy.\end{aligned}$$

Now let $\epsilon \to 0$ on both sides

$$\frac{1}{(1-q)}\int_0^R e_q^{-\frac{y}{t}} f(x,y)d_qy = e_q^{-(1/t-t_0)R}\alpha(x,qR) - (1/t-t_0)\int_0^R \alpha(x,qy)e_q^{-(t-t_0)y}d_qy.$$

Again let $R \to \infty$, if $\frac{1}{t} > t_0$ the first term on right side approaches to 0, then we have

$$\frac{1}{(1-q)}\int_0^{\infty} e_q^{-\frac{y}{t}} f(x,y)d_qy = -(1/t-t_0)\int_0^{\infty} \alpha(x,qy)e_q^{-(1/t-t_0)y}d_qy, \quad \text{for} \quad \frac{1}{t} > t_0,$$

this proves lemma if right side integral converges.

But by limit test obviously as y approaches to ∞, that is, $\lim_{y\to\infty} \alpha(x,y) = 0$. Therefore, integral on right converges for $\frac{1}{t} > t_0$. Hence

$$\int_0^{\infty} e_q^{-\frac{y}{t}} f(x,y)d_qy \quad \text{converges} \quad \text{for} \quad \frac{1}{t} > t_0.$$

□

Lemma 3.4. *If integral* $\zeta(x,s) = \int\limits_0^\infty e_q^{-\frac{y}{s}} f(x,y)d_qy$ *converges for* $\frac{1}{s} \geq s_0$ *and if* $\int\limits_0^\infty e_q^{-tx}\zeta(x,s)d_qx$ *converges at* $t = t_0$*, then* $\int\limits_0^\infty e_q^{-tx}\zeta(x,s)d_qx$ *converges for* $t > t_0$*.*

Proof. The proof is same as above Lemma 3.3. □

Theorem 3.5. *Let* $f(x,y)$ *be function of two variables continuous in* $\mathbb{R}_{q_1,+} \times \mathbb{R}_{q_2,+}$ *or continuous in the positive quadrant of xy-plane and is of exponential order* e^{cx+dy}*. Then the integral*

$$\frac{1}{(1-q)^2 t}\int\limits_0^\infty\int\limits_0^\infty e_q^{-sx-\frac{y}{t}} f(x,y)d_qxd_qy \tag{3.2}$$

exists for all s *and* $\frac{1}{t}$ *provided* $Re(s) > c$ *and* $Re(\frac{1}{t}) > d$*.*

Proof. We proceed to prove this theorem by using above Lemmas.

$$\begin{aligned}
&\frac{1}{(1-q)^2 t}\int\limits_0^\infty\int\limits_0^\infty e_q^{-sx-y/t} f(x,y)d_qxd_qy \\
&\quad = \frac{1}{(1-q)t}\int\limits_0^\infty e_q^{-sx}\left\{\frac{1}{(1-q)t}\int\limits_0^\infty e_q^{-y/t} f(x,y)d_qy\right\}d_qx \\
&\quad = \frac{1}{(1-q)}\int\limits_0^\infty e_q^{-sx}\zeta(x,t)d_qx, \qquad (3.3)
\end{aligned}$$

where $\zeta(x,t) = \frac{1}{(1-q)}\int\limits_0^\infty e_q^{-y/t} f(x,y)d_qy$, which converges by Lemma 3.4. And by Lemma 3.3, $\int\limits_0^\infty e_q^{-y/t} f(x,y)d_qy$ converges for $\frac{1}{t} > t_0$ and $q \in (0,1)$. Therefore, the integral on right side of (3.3) converges for $s > s_0$, $\frac{1}{t} > t_0$. Hence, the integral

$$\frac{1}{(1-q)^2 t}\int\limits_0^\infty\int\limits_0^\infty e_q^{-sx-y/t} f(x,y)d_qxd_qy$$

converges for $s > s_0$ and $\frac{1}{t} > t_0$. This completes the proof. □

Corollary 3.6. *If the integral* (3.3) *diverges at* $s = s_0$ *and* $t = t_0$*, then the integral* (3.3) *diverges at* $s > s_0$ *and* $t > t_0$*.*

Corollary 3.7. *The region of the convergence of the integral* (3.3) *is the positive quadrant of the* $xy-$*plane.*

Theorem 3.8. *If the integral* (3.3) *converges absolutely at* $s = s_0$ *and* $\frac{1}{t} = t_0$*, then integral* (3.3) *converges absolutely for* $s \geq s_0$ *and* $\frac{1}{t} \geq t_0$*.*

Proof. We can write

$$e_q^{-sx-y/t}|f(x,y)| \leq e_q^{-s_0x} \quad \text{for} \ \ s_0 \leq s < \infty, \ \ t_0 \leq \frac{1}{t} < \infty.$$

Therefore,

$$\frac{1}{(1-q)^2t}\int_0^\infty\int_0^\infty e_q^{-sx-y/t}|f(x,y)|d_qxd_qy \leq \frac{1}{(1-q)^2t}\int_0^\infty\int_0^\infty e_q^{-s_0x-t_0y}|f(x,y)|d_qxd_qy$$

from hypothesis $\frac{1}{(1-q)^2t}\int_0^\infty\int_0^\infty e_q^{-s_0x-t_0y}|f(x,y)|d_qxd_qy$ converges for $s > s_0$ and $\frac{1}{t} > t_0$. Hence, (3.3) converges absolutely for $s > s_0$ and $\frac{1}{t} > t_0$. □

4. $\tilde{q}$-Laplace-Sumudu convolution product

Definition 4.1. The convolution of two functions $g(x,y)$ and $h(x,y)$ is denoted by $(g**h)(x,y)$ and defined as

$$(g**h)(x,y) = \frac{1}{(1-q)^2}\int_0^x\int_0^y g(x-\zeta, y-\eta)h(\zeta,\eta)d_q\zeta d_q\eta.$$

Proposition 4.2. (Convolution Theorem) *Let* ${}_qL_xS_y[g(x,y)] = G(s,t)$ *and* ${}_qL_xS_y[h(x,y)] = H(s,t)$ *be two positive scalar functions of* x *and* y*. Then*

$${}_qL_xS_y[(g**h)(x,y)] = tG(s,t)H(s,t),$$

where $g(x,y)**h(x,y) = \frac{1}{(1-q)^2}\int_0^x\int_0^y g(x-\zeta,y-\eta)h(\zeta,\eta)d_q\zeta d_q\eta$*.*

Proof. We know that

$$
\begin{aligned}
&{}_qL_xS_y\{g(x,y)**h(x,y)\}(s,t)\\
&\quad=\frac{1}{(1-q)^2t}\int_0^\infty\int_0^\infty e_q^{-sx-y/t}(g**h)(x,y)d_qxd_qy,
\end{aligned}
\tag{4.1}
$$

$$
\begin{aligned}
&{}_qL_xS_y\{g(x,y)**h(x,y)\}(s,t)\\
&=\frac{1}{(1-q)^2t}\int_0^\infty\int_0^\infty e_q^{-sx-y/t}\Big\{\frac{1}{(1-q)^2}\int_0^x\int_0^y g(\zeta,\eta)\\
&\qquad\times h(x-\zeta,y-\eta)d_q\zeta d_q\eta\Big\}d_qxd_qy\\
&=\frac{1}{(1-q)^4t}\int_0^\infty\int_0^\infty\int_0^x\int_0^y[1+(q-1)(sx+y/t)]^{-1/q-1}g(\zeta,\eta)\\
&\qquad\times h(x-\zeta,y-\eta)d_q\zeta d_q\eta d_qxd_qy\\
&=\frac{1}{(1-q)^4t}\int_0^\infty\int_0^\infty g(\zeta,\eta)\Big\{\int_{x=\zeta}^\infty\int_{y=\eta}^\infty[1+(q-1)(sx+y/t)]^{-1/q-1}\\
&\qquad\times h(x-\zeta,y-\eta)d_q\zeta d_q\eta\Big\}d_qxd_qy.
\end{aligned}
\tag{4.2}
$$

Let

$$I=\int_{x=\zeta}^\infty\int_{y=\eta}^\infty[1+(q-1)(sx+y/t)]^{-1/q-1}h(x-\zeta,y-\eta)d_q\zeta d_q\eta.$$

Putting $x-\zeta=u$ and $y-\eta=v$, then

$$
\begin{aligned}
I=[1+(q-1)(s\zeta+\eta/t)]^{-1/q-1}&\int_0^\infty\int_0^\infty[1+(q-1)(su+v/t)]^{-1/q-1}\\
&\times[1+(q-1)(s\zeta+\eta/t)]^{1/q-1}[1+(q-1)(su+v/t)]^{1/q-1}\\
&\times[1+(q-1)\{(u+\zeta)s+(v+\eta)/t\}]^{-1/q-1}h(u,v)d_qud_qv.
\end{aligned}
$$

Let

$$
\begin{aligned}
&[1+(q-1)(s\zeta+\eta/t)]^{1/q-1}[1+(q-1)(su+v/t)]^{1/q-1}\\
&\quad\times[1+(q-1)\{(u+\zeta)s+(v+\eta)/t\}]^{-1/q-1}f_2(u,v)=h^*(u,v).
\end{aligned}
$$

Then

$$h(x-\zeta,y-\eta)=[1+(q-1)(s\zeta+\eta/t)]^{-1/q-1}[1+(q-1)(su+v/t)]^{-1/q-1}\times[1+(q-1)\{(u+\zeta)s+(v+\eta)/t\}]^{1/q-1}h^*(x-\zeta,y-\eta),$$

$$\begin{aligned}&{}_qL_xS_y\{g(x,y)**h(x,y)\}(s,t)\\&=\frac{1}{(1-q)^4t}\int_0^\infty\int_0^\infty g(\zeta,\eta)\Big\{[1+(q-1)(s\zeta+\eta/t)]^{-1/q-1}\\&\quad\times\int_0^\infty\int_0^\infty[1+(q-1)(su+v/t)]^{-1/q-1}h^*(x-\zeta,y-\eta)d_qxd_qy\Big\}d_q\zeta d_q\eta\\&={}_q L_xS_y\{g(\zeta,\eta)\}\Big\{\frac{1}{(1-q)^2}\frac{t}{t}\\&\quad\times\int_0 \infty\int_0^\infty[1+(q-1)\{(u+\zeta)s+(v+\eta)/t\}]^{-1/q-1}h^*(x-\zeta,y-\eta)d_qxd_qy\Big\}\\&={}_q L_xS_y\{g(\zeta,\eta)\}\cdot t\cdot{}_q L_xS_y\{h^*(x,y)\}{}_qL_xS_y\{g(x,y)**h(x,y)\}(s,t)\\&=tG(s,t)\cdot H(s,t).\end{aligned}$$

□

4.1. **Properties.** In this section, some interesting properties of $\tilde{q}$-Laplace-Sumudu transform are given which intersects with classical ones when $\tilde{q}$ tends to $(1,1)$.

Remark 4.3. (Scaling) For a real number k,

$$\begin{aligned}{}_qL_xS_y[kf(x,y)](s,t)&=\frac{1}{(1-q)^2t}\int_0^\infty\int_0^\infty kf(x,y)e_q^{-sx-y/t}d_qxd_qy\\&=\frac{k}{(1-q)^2t}\int_0^\infty\int_0^\infty f(x,y)e_q^{-sx-y/t}d_qxd_qy\\&=k\cdot_q L_xS_y[f(x,y)](s,t).\end{aligned}$$

Remark 4.4. (Linearity) We have

$$\begin{aligned}&{}_qL_xS_y[mf(x,y)+ng(x,y)](s,t)\\&\quad=m\cdot_q L_xS_y[f(x,y)]+n\cdot_q L_xS_y[g(x,y)](s,t).\end{aligned}$$

$$
\begin{aligned}
&{}_qL_xS_y[mf(x,y)+ng(x,y)](s,t)\\
&\quad=\frac{1}{(1-q)^2t}\int_0^\infty\int_0^\infty[mf(x,y)+ng(x,y)]e_q^{-sx-y/t}d_qxd_qy\\
&\quad=\frac{1}{(1-q)^2t}\left\{\int_0^\infty\int_0^\infty mf(x,y)e_q^{-sx-y/t}d_qxd_qy+\int_0^\infty\int_0^\infty ng(x,y)e_q^{-sx-y/t}d_qxd_qy\right\}\\
&\quad=m\,{}_qL_xS_y[f(x,y)]+n\,{}_qL_xS_y[g(x,y)](s,t).
\end{aligned}
$$

Remark 4.5. For $a>0$ and $b>0$, we have

$$
\begin{aligned}
{}_qL_xS_y[e_q^{-ax-by}f(x,y)](s,t)&=\bar{\bar{f}}(s+a,1/t+b)\\
&=\frac{1}{(1-q)^2t}\int_0^\infty\int_0^\infty e_q^{-ax-by}e_q^{-sx-y/t}f(x,y)d_qxd_qy\\
&=\frac{1}{(1-q)^2t}\int_0^\infty\int_0^\infty e_q^{-x(s+a)}e_q^{-y(1/t+b)}f(x,y)d_qxd_qy\\
&=\bar{\bar{f}}(s+a,1/t+b).
\end{aligned}
$$

Remark 4.6. For $a>0$ and $b>0$, we have

$$
{}_qL_xS_y[f(ax)g(by)](s,t)=\frac{1}{(1-q)^2t}\int_0^\infty\int_0^\infty e_q^{-sx-y/t}f(ax)g(by)d_qxd_qy.
$$

Put $ax=n$ and $by=m$, then

$$
\begin{aligned}
{}_qL_xS_y[f(ax)g(by)](s,t)&=\frac{1}{(1-q)^2t}\frac{1}{ab}\int_0^\infty\int_0^\infty e_q^{-sn/a}e_q^{-m/bt}f(n)g(m)d_qnd_qm\\
&=\frac{1}{(1-q)}\frac{1}{ab}\int_0^\infty e_q^{-sn/a}f(n)d_qn\frac{1}{(1-q)t}\int_0^\infty e_q^{-m/bt}g(m)d_qm\\
&=\frac{1}{a}\bar{f}(s/a)\cdot\frac{1}{b}\bar{g}(1/bt).
\end{aligned}
$$

Remark 4.7.

$$\begin{aligned}
{}_qL_xS_y[f(x)] &= \frac{1}{(1-q)^2t}\int_0^\infty\int_0^\infty e_q^{-sx-y/t}f(x)d_qxd_qy\\
&= \frac{1}{(1-q)}\int_0^\infty e_q^{-sx}f(x)d_qx\cdot\frac{1}{(1-q)t}\int_0^\infty e_q^{-y/t}d_qy\\
&= \bar{f}(s)\cdot\frac{1}{(1-q)t}\left[\frac{e_q^{-y/t}}{-1/t}\right]_0^\infty\\
&= \bar{f}(s)\cdot\frac{1}{(1-q)}t^2\\
&= \frac{\bar{f}(s)t^2}{(1-q)}.
\end{aligned}$$

Similarly, we have ${}_qL_xS_y[f(y)] = \dfrac{\bar{f}(t)}{(1-q)s}$.

5. Examples

Example 5.1. If $f(x,y)=1$ for $x>0$, $y>0$, then for $1<q<2$,

$$ {}_qL_xS_y\{1\} = \frac{1}{(1-q)^2(2-q)^2s}. $$

In fact,

$$\begin{aligned}
{}_qL_xS_y\{1\} &= \frac{1}{(1-q)^2t}\int_0^\infty\int_0^\infty e_q^{-sx-y/t}d_qxd_qy\\
&= \frac{1}{(1-q)^2t}\int_0^\infty\int_0^\infty e_q^{-sx}[1+(q-1)y/t]^{-1/q-1}d_qxd_qy\\
&= \frac{1}{(1-q)^2t}\int_0^\infty e_q^{-sx}\frac{[1+(q-1)y/t]^{q-2/q-1}}{\frac{q-2}{q-1}(q-1)/t}\,|_0^\infty d_qx\\
&= \frac{1}{(1-q)^2}\int_0^\infty e_q^{-sx}\frac{1}{(2-q)}d_qx
\end{aligned}$$

$$= \frac{1}{(1-q)^2}\frac{1}{(2-q)}\int_0^\infty e_q^{-sx} d_q x$$
$$= \frac{1}{(1-q)^2}\frac{1}{(2-q)}\frac{1}{(2-q)s}$$
$$= \frac{1}{(1-q)^2(2-q)^2 s}.$$

Example 5.2. If $f(x,y) = e_q^{ax+by}$ for all x, y, then

$$_qL_xS_y\{e_q^{ax+by}\} = \frac{1}{(1-q)^2}\frac{1}{(2-q)^2}\frac{1}{(s-a)(1/t-b)}.$$

In fact,

$$_qL_xS_y\{e_q^{ax+by}\} = \frac{1}{(1-q)^2 t}\int_0^\infty\int_0^\infty e_q^{ax+by} e_q^{-sx-y/t} d_q x d_q y$$
$$= \frac{1}{(1-q)^2 t}\int_0^\infty\int_0^\infty e_q^{-x(s-a)} e_q^{-y(1/t-b)} d_q x d_q y$$
$$= \frac{1}{(1-q)^2}\frac{1}{(2-q)^2 t}\frac{1}{(s-a)(1/t-b)}.$$

Example 5.3. If $f(x,y) = e_q^{i(ax+by)}$, then

$$_qL_xS_y\{e_q^{i(ax+by)}\} = \frac{1}{(1-q)^2 t}\int_0^\infty\int_0^\infty e_q^{i(ax+by)} e_q^{-sx-y/t} d_q x d_q y$$
$$= \frac{1}{(1-q)^2}\cdot\frac{1}{(2-q)^2 t}\cdot\frac{1}{(s-ia)(1/t-ib)}$$
$$= \frac{1}{(1-q)^2}\cdot\frac{1}{(2-q)^2 t}\cdot\frac{(s+ia)(1/t+ib)}{(s^2+a^2)(1/t^2+b^2)}$$
$$= \frac{1}{(1-q)^2(2-q)^2}\cdot\frac{(s/t-ab)+i(a/t+bs)}{(s^2+a^2)(1/t+b^2 t)}.$$

Example 5.4. If $f(x,y) = (xy)^n$, then

$$ {}_qL_xS_y\{(xy)^n\} = \frac{1}{(1-q)^2 t}\int_0^\infty\int_0^\infty (xy)^n e_q^{-sx-y/t} d_qx d_qy $$
$$ = \frac{1}{(1-q)^2 t}\int_0^\infty e_q^{-sx} x^n d_qx \cdot \int_0^\infty e_q^{-y/t} y^n d_qy. $$

Putting $sx = m$, then

$$ {}_qL_xS_y\{(xy)^n\} = \frac{1}{(1-q)^2 t}\int_0^\infty (m/s)^n e_q^{-m} \frac{d_qm}{s} \cdot \int_0^\infty e_q^{-y/t} y^n d_qy $$
$$ = \frac{1}{(1-q)^2 t}\frac{\Gamma_q(n+1)}{s^{n+1}} \cdot \int_0^\infty e_q^{-y/t} y^n d_qy. $$

Similarly by putting $y/t = p$, we get

$$ {}_qL_xS_y\{(xy)^n\} = \frac{1}{(1-q)^2} \cdot \frac{\Gamma_q(n+1)}{s^{n+1}} \cdot \Gamma_q(n+1) t^{n+1}. $$

Example 5.5. If $f(x,y) = cos_q(ax+by)$, then

$$ {}_qL_xS_y\{cos_q(ax+by)\} = \frac{1}{2(1-q)^2} \cdot \frac{(st-ab)}{(s^2+a^2)(t^2+b^2)}. $$

In fact, we have

$$ cos_q(ax+by) = \frac{e_q^{i(ax+by)} + e_q^{-i(ax+by)}}{2}. $$

Therefore,

$$ {}_qL_xS_y\{cos_q(ax+by)\} $$
$$ = \frac{1}{2}\left[{}_qL_xS_y\{e_q^{i(ax+by)}\} +_q L_xS_y\{e_q^{-i(ax+by)}\}\right] $$
$$ = \frac{1}{2}\left[\int_0^\infty\int_0^\infty e_q^{i(ax+by)} e_q^{-sx-y/t} d_qx d_qy + \int_0^\infty\int_0^\infty e_q^{-i(ax+by)} e_q^{-sx-y/t} d_qx d_qy\right], $$

which with the help of Example 5.2 and Example 5.3, we have

$$ {}_qL_xS_y\{cos_q(ax+by)\} $$
$$ = \frac{1}{2(2-q)^4} \cdot \frac{(s/t-ab)^2 - (a/t+bs)^2}{(s^2+a^2)(s^2-a^2)(1/t+b^2t)(1/t-b^2t)}. $$

Similarly, we have $sin_q(ax+by), cosh_q(ax+by)$ and $sinh_q(ax+by)$.

6. Application

In this section, we give some applications of $L_{\tilde{q}}$-Laplace-Sumudu transform in Heat, Wave and Space-time telegraphic equations.

Heat diffusion equation: Consider the following q-diffusion equation [11].

$$\frac{\delta_q u}{\delta_q t}(x,t) = k\frac{\delta_q^2 u}{\delta_q x^2}(x,t), \qquad x \in (-\infty,\infty), \quad t \in \mathbb{R}_+ \tag{6.1}$$

with initial conditions:

$$u(x,0) = 0, \text{ for } 0 < x < \infty,$$
$$\frac{\delta_q u}{\delta_q x}(0,t) = 0, \; u(0,t) = f(t).$$

Applying q-Laplace-Sumudu transform to equation (6.1) both sides, where s and r are transform variables, we have

$${}_qL_xS_y\left[\frac{\delta_q u}{\delta_q t}(x,t)\right](s,r) = k \cdot_q L_xS_y\left[\frac{\delta_q^2 u}{\delta_q x^2}(x,t)\right](s,r).$$

Now with the help of results by Lokenath and Bhatta [8, pp. 275], we have

$$\frac{1}{r}{}_qL_xS_y[u(x,t)](s,r) - \frac{1}{r}L_q[u(x,0)](s)$$
$$= ks^2_qL_xS_y[u(x,t)](s,r) - ksS_q[u(0,t)](r) - kS_q\left[\frac{\delta_q u}{\delta_q x}(0,t)\right](r).$$

This implies that

$$(1/r - ks^2)_qL_xS_y[u(x,t)](s,r)$$
$$= \frac{1}{r}L_q[u(x,0)](s) - ks_qS_t[u(0,t)](r) - kS_q\left[\frac{\delta_q u}{\delta_q x}(0,t)\right](r).$$

Hence, we have

$$(1/r - ks^2)_qL_xS_y[u(x,t)](s,r) = 0 - ksS_q\{f(t)\} - 0.$$

That is,

$${}_qL_xS_y[u(x,t)](s,r) = \frac{-krsS_q\{f(t)\}}{(1-ks^2r)}$$

or

$$u(x,t) = ({}_qL_xS_y)^{-1}\left[\frac{-ksrS_q\{f(t)\}}{(1-ks^2r)}\right](x,t)$$

provided the inverse transform exists for each term in R.H.S.

Generalized wave equation: The generalized wave equation in [5] is defined as

$$\frac{\delta_q^2 u}{\delta_q t^2}(x,t) - c^2 \frac{\delta_q^2 u}{\delta_q x^2}(x,t) = 0 \tag{6.2}$$

with initial conditions:

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\delta_q u}{\delta_q t}(x,0) = g(x), \quad x > 0,$$

$$u(0,t) = 0 \quad \text{and} \quad \frac{\delta_q u}{\delta_q x}(0,t) = 0.$$

Apply q-double Laplace transform to equation (6.2), we have

$$_qL_xS_y \left\{ \frac{\delta_q^2 u}{\delta_q t^2}(x,t) - c^2 \frac{\delta_q^2 u}{\delta_q x^2}(x,t) \right\} = 0,$$

it implies

$$\frac{1}{r^2}{}_qL_xS_y[u(x,t)](s,r) - \frac{1}{r^2}L_q[u(x,0)](s) - \frac{1}{r}L_q\left[\frac{\delta_q u}{\delta_q t}(x,0)\right](s)$$
$$- c^2\left\{ s_q^2 L_xS_y[u(x,t)](s,r) - sS_q[u(0,t)](r) - S_q\left[\frac{\delta_q u}{\delta_q x}(0,t)\right](r) \right\} = 0.$$

Therefore, we have

$$\begin{aligned} \frac{1}{r^2}{}_qL_xS_y[u(x,t)](s,r) &- c^2 s_q^2 L_xS_y[u(x,t)](s,r) \\ &= \frac{1}{r^2}L_q[u(x,0)](s) + \frac{1}{r}L_q\left[\frac{\delta_q u}{\delta_q t}(x,0)\right](s) \\ &\quad + c^2 sS_q[u(0,t)](r) + c^2 S_q\left[\frac{\delta_q u}{\delta_q x}(0,t)\right](r). \end{aligned}$$

Hence,

$$_qL_xS_y[u(x,t)](s,r)\{1/r^2 - c^2s^2\} = \frac{1}{r^2}L_q\{f(x)\}(s) + \frac{1}{r}L_q\{g(x)\}(s) + 0 + 0.$$

That is,

$$_qL_xS_y[u(x,t)](s,r) = \frac{L_q\{f(x)\} + rL_q\{g(x)\}}{(1 - c^2s^2r^2)}$$

or

$$u(x,t) = ({}_qL_xS_y)^{-1}\left[\frac{L_q\{f(x)\} + rL_q\{g(x)\}}{(1 - c^2s^2r^2)}\right](x,t)$$

provided $\left(\frac{L_q\{f(x)\} + rL_q\{g(x)\}}{(1 - c^2s^2r^2)}\right)^{-1}$ exists for each term in R.H.S.

Space-time telegraphic equation: Consider the following generalized space-time telegraphic equation

$$c^2\frac{\delta_q^2 u}{\delta_q x^2}(x,t)-\frac{\delta_q^2 u}{\delta_q t^2}(x,t)-(\alpha+\beta)\frac{\delta_q u}{\delta_q t}(x,t)-\alpha\beta u(x,t)=[c^2-(\alpha+1)(\beta+1)]\exp(x+t) \tag{6.3}$$

with initial conditions:

$$u(0,t)=e_q(t) \text{ and } \frac{\delta_q u}{\delta_q x}(0,t)=e_q(t),$$

$$u(x,0)=e_q(x) \text{ and } \frac{\delta_q u}{\delta_q t}(x,0)=e_q(x).$$

Applying q-Laplace-Sumudu transform on equation (6.3), we get

$$\begin{aligned}&{}_qL_xS_y\left[c^2\frac{\delta_q^2 u}{\delta_q x^2}(x,t)\right]-_q L_xS_y\left[\frac{\delta_q^2 u}{\delta_q t^2}(x,t)\right]\\ &\quad-(\alpha+\beta)_qL_xS_y\left[\frac{\delta_q u}{\delta_q t}(x,t)\right]-\alpha\beta_qL_xS_y\,[u(x,t)]\\ &=_q L_xS_y[c^2-(\alpha+1)(\beta+1)]\exp_q(x+t),\end{aligned}$$

this implies,

$$\begin{aligned}&c^2\left\{s_q^2L_xS_y[u(x,t)](s,t)-sS_q[u(0,t)](r)-S_q[u_x(0,t)](r)\right\}\\ &\quad-\{\frac{1}{r^2}{}_qL_xS_y[u(x,t)](s,t)-\frac{1}{r^2}L_q[u(x,0)](s)-\frac{1}{r}L_q[u_t(x,0)](s)\}\\ &\quad-(\alpha{+}beta)\{\frac{1}{r}{}_qL_xS_y[u(x,t)](s,r){+}\frac{1}{r}L_q[u(x,0)](s)\}\\ &\quad-\alpha\beta_qL_xS_y[u(x,t)](s,r)\\ &=[c^2-(\alpha+1)(\beta+1)]_qL_xS_y\{\exp_q(x+t)\}(s,r).\end{aligned}$$

That is,

$$\begin{aligned}&c^2s^2{}_qL_xS_y[u(x,t)](s,r)-c^2sS_q\{e_q(t)\}-c^2S_q[e_q(t)](r)\\ &\quad-\frac{1}{r^2}{}_qL_xS_y[u(x,t)](s,t)-\frac{1}{r^2}L_q[e_q(x)](s)\\ &\quad-\frac{1}{r}L_q[e_q(x)](s)-(\alpha+\beta)\{\frac{1}{r}{}_qL_xS_y[u(x,t)](s,r)+\frac{1}{r}L_q[e_q(x)](s)\}\\ &\quad-\alpha\beta_qL_xS_y[u(x,t)](s,r)\\ &=[c^2-(\alpha+1)(\beta+1)]_qL_xS_y\{exp_q(x+t)\}(s,r).\end{aligned}$$

Hence, we get

$$\left(c^2s^2-\frac{1}{r^2}-\frac{(\alpha+\beta)}{r}-\alpha\beta\right){}_qL_xS_y[u(x,t)](s,r)$$
$$=c^2sS_q\{e_q(t)\}+c^2S_q[e_q(t)](r)+\frac{1}{r^2}L_q[e_q(x)](s)$$
$$+\frac{1}{r}L_q[e_q(x)](s)+\frac{(\alpha+\beta)}{r}\{L_q\{e_q(x)\}$$
$$+[c^2-(\alpha+1)(\beta+1)]_qL_xS_y\{exp_q(x+t)\}.$$

We have the help of above examples,

$${}_qL_xS_y\{\exp_q(x+t)\}=\frac{1}{(1-q)^2(2-q)^2(s-1)(1/r-1)},$$

$$L_q\{e_q(x)\}=\frac{1}{(1-q)(r-1)}$$

and

$$S_q\{e_q(t)\}=\frac{1}{(1-q)(2-q)(1-s)}.$$

Substitute these results in above equation, we have

$$\left(c^2s^2-\frac{1}{r^2}-\frac{(\alpha+\beta)}{r}-\alpha\beta\right){}_qL_xS_y[u(x,t)](s,r)$$
$$=\frac{c^2s^2}{(1-q)(2-q)(1-s)}+\frac{c^2}{(1-q)(2-q)(1-s)}$$
$$+\frac{1}{r^2(1-q)(r-1)}+\frac{1}{r(1-q)(r-1)}+\frac{(\alpha+\beta)}{r(1-q)(r-1)}$$
$$+\frac{c^2-(\alpha+1)(\beta+1)}{(1-q)^2(2-q)^2(s-1)(1/r-1)}$$

or

$$u(x,t)=({}_qL_xS_y)^{-1}\Big\{\frac{r^2}{(c^2s^2r^2-(\alpha+\beta)r-\alpha\beta r^2-1)}\Big[\frac{c^2(s^2+1)}{(1-q)(2-q)(1-s)}$$
$$+\frac{1+(\alpha+\beta)r}{r^2(1-q)(r-1)}+\frac{c^2-(\alpha+1)(\beta+1)}{(1-q)^2(2-q)^2(s-1)(1/r-1)}\Big]\Big\}$$

provided the inverse transform exists for each term in R.H.S.

7. Conclusion

In this study, a new double integral transform called the double q-Laplace-Sumudu transform were presented. Several properties and theorems related to the linearity, existence and the double convolution theorem were introduced. The results are developed and tested with the help of examples. We find that this research focuses on the same direction with the possibility of using partial differential equations in explaining physical phenomenon. Therefore, we recommend that this study is continued using the applications of this method.

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