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EXAMINE NEW RESULTS OF VOLTERRA FREDHOLM INTEGRO-DIFFERENTIAL EQUATION VIA CAPUTO-HADAMARD TYPE

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Abstract. The objective of this study is to explore a new class of the Caputo-Hadamard fractional Volterra Fredholm integro-differential equation (Caputo-Hadamard fractional-VFIDE). The main contributions include establishing and deriving sufficient conditions for the existence, and uniqueness of solutions for the proposed VFIDE-based Caputo-Hadamard. The study employs the Banach contraction mapping principle and the Krasnoselskii's alternative to ensure the well-posedness of the system and presents a detailed mathematical analysis to discuss the approximate solution of the proposed problem by using the modified Adomian decomposition method (ADM). To enhance the comprehension of the findings, concrete example is provided to showcase the versatility and practical applicability of the VFIDE-based Caputo-Hadamard, highlighting the novelty and potential impact of this research.

1. Introduction

In recent decades, fractional calculus has emerged as a key idea in many areas of mathematica. Fractional order differential equations have been used more often by researchers to obtain important insights in a variety of domains, including control theory, electrodynamics, fluid mechanics, dispersion, and

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porous media, (for further details, refer [11, 17, 18, 20, 23, 24, 29]). The study of the asymptotic behavior of solutions of the fractional integro-differential equation has been investigated by many researchers, see e.g. [2, 6, 7, 8, 9, 16, 21, 26, 27, 31].

The author of [1], considered the following Caputo fractional VFIDE,

$$\begin{cases} {}^{C}\mathfrak{D}^{\partial}_{0^+}\varkappa(\mathfrak{s}) = \zeta(\mathfrak{s}) + \wp_1\varkappa(\mathfrak{s}) + \wp_2\varkappa(\mathfrak{s}),\ \mathfrak{s} \in \mho = [0,1], \\ \varkappa(0) = \varkappa_0 + \xi(\varkappa), \end{cases}$$

where $0 < \partial < 1$, ${}^{C}\mathfrak{D}^{\partial}_{0^+}$ is Caputo fractional derivative of order ∂, $\zeta : \mho \to \mathbb{R}$, $\xi : C(\mho, \mathbb{R}) \to \mathbb{R}$, $\mathcal{J}_1$, $\mathcal{J}_2 : \mho \times \mho \to \mathbb{R}$ are continuous functions and $\chi_1, \chi_2 : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ are Lipschitz continuous functions. To keep our analysis simple, we'll employ the following notations:

$$\wp_1\varkappa(\mathfrak{s}) := \int_0^{\mathfrak{s}} \mathcal{J}_1(\mathfrak{s}, \delta)\chi_1(\varkappa(\delta))d\delta$$

and

$$\wp_2\varkappa(\mathfrak{s}) := \int_0^{1} \mathcal{J}_2(\mathfrak{s}, \delta)\chi_2(\varkappa(\delta))d\delta.$$

Based on the justification provided, in order to evaluate and investigate the existence and uniqueness of the solution, Krasnoselskii and Banach's fixed point theorems (FPTs) were used. Our motivation stems from the arguments presented, which encourage us to assess and look into the prerequisites for the solution of the Caputo-Hadamard VFIDE using the modified Adomian decomposition method (MADM).

$$\begin{cases} {}^{CH}\mathfrak{D}^{\partial}_{a^+}\varkappa(\mathfrak{s}) = \zeta(\mathfrak{s}) + \wp_1\varkappa(\mathfrak{s}) + \wp_2\varkappa(\mathfrak{s}),\ \mathfrak{s} \in \mho = [1,T], \\ \varkappa(0) = \varkappa_0 + \xi(\varkappa), \end{cases} \tag{1.1}$$

where $0 < \partial < 1$, ${}^{CH}\mathfrak{D}^{\partial}_{a^+}$ is Caputo-Hadamard type fractional derivative of order ∂, $\zeta : \mho \to \mathbb{R}$, $\xi : C(\mho, \mathbb{R}) \to \mathbb{R}$, $\mathcal{J}_1$, $\mathcal{J}_2 : \mho \times \mho \to \mathbb{R}$ are continuous functions and $\chi_1, \chi_2 : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ are Lipschitz continuous functions. Briefly, we set up

$$\wp_1\varkappa(\mathfrak{s}) := \int_0^{\mathfrak{s}} \mathcal{J}_1(\mathfrak{s}, \delta)\chi_1(\varkappa(\delta))d\delta$$

and

$$\wp_2\varkappa(\mathfrak{s}) := \int_0^{1} \mathcal{J}_2(\mathfrak{s}, \delta)\chi_2(\varkappa(\delta))d\delta.$$

Furthermore, Adomian [3] introduced the Adomian decomposition method and other numerical methods for solving this kind of equation. For more information, click here. [10, 12, 13, 14, 32]. The ADM offers both style and user-friendly functionality. The approach is presented as a sequence in which each term can be readily computed using Adomian polynomials tailored to the nonlinear terms see [3, 4, 5, 15, 22, 25].

Wazwaz in [30] made a small but effective change to ADM to accelerate the series solution's convergence. Inc, Ergut, and Cherruault (2005) established an effective method for applying the ADM correctly for single boundary value problems (BVPs) and presented a broad framework dealing with the singular BVPs. It has been demonstrated that the decomposition approach offers a recursive procedure to construct the explicit solutions for a large range of non-linear equations, which is a major advantage in most circumstances. For example, when utilizing the MADM approach, we make reference to [19].

The rest of this essay is structured as follows: A review of the notations, definitions, and pertinent lemmas used in fractional calculus that are crucial to our study is given in Section 2. In Section 3 is the core of the work, where the authors utilize the Banach contraction mapping principle and Krasnoselskii alternative to establish sufficient conditions guaranteeing the uniqueness, existence solutions for the main problem (1.1). In Section 4, study the MADM and establish the convergence of the series built by the MADM to the exact solution of our problem (1.1). In Section 5, we provide a numerical illustration.

2. Auxiliary results

Below, we begin with some basic definitions and lemmas.

Definition 2.1. ([20]) Let $\partial > 0$ and $\varkappa \in L^1([0,T],\mathbb{R})$. The left-sided fractional integral of Hadamard of order ∂ is given by

$$ {}^H\mathcal{I}^{\partial}_{a^+}\varkappa(\mathfrak{s}) = \begin{cases} \frac{1}{\Gamma(\partial)}\int_a^{\mathfrak{s}}(\ln\frac{\mathfrak{s}}{\delta})^{\partial-1}\frac{\varkappa(\delta)}{\delta}d\delta, & \partial > 0 \\ \varkappa(\mathfrak{s}), & \partial = 0 \end{cases} $$

provided the right-hand side exists and Γ is the Euler's Gamma function.

Definition 2.2. ([20]) Let $\partial > 0$, $\varkappa \in AC^n([0,T],\mathbb{R})$. The definition of Caputo fractional derivative of order ∂ given by

$$ {}^C\mathfrak{D}^{\partial}_{a^+}\varkappa(\mathfrak{s}) = \mathfrak{D}^{\partial}_{a^+}\left[\varkappa(\mathfrak{s}) - \sum_{k=0}^{n-1}\frac{\varkappa^{(k)}(0)}{k!}\mathfrak{s}^k\right], \quad \mathfrak{s} \in [1,T], \tag{2.1} $$

where $n = [\partial]+1$, $[\partial]$ is the integer part of ∂ and $\mathfrak{D}^{\partial}_{a^+}$ is define by

$$ \begin{aligned} \mathfrak{D}^{\partial}_{a^+}\varkappa(\mathfrak{s}) &= \left(\frac{d}{d\mathfrak{s}}\right)^n \mathcal{I}^{n-\partial}_{0^+}\varkappa(\mathfrak{s}) \\ &= \left(\frac{d}{d\mathfrak{s}}\right)^n \frac{1}{\Gamma(n-\partial)}\int_a^{\mathfrak{s}}(\mathfrak{s}-\delta)^{n-\partial-1}\varkappa(\delta)d\delta, \end{aligned} $$

which is the sense of the Riemann-Liouville fractional derivative of order ∂.

Definition 2.3. ([20]) Let $\partial > 0$, $\varkappa \in AC^n([0,T],\mathbb{R})$. The Hadamard fractional derivative of order ∂ is given by

$$ {}^{H}\mathfrak{D}^{\partial}_{a^+}\varkappa(\mathfrak{s}) = \frac{1}{\Gamma(n-\partial)}\eta^n \int_a^{\mathfrak{s}} (\ln\frac{\mathfrak{s}}{\delta})^{n-\partial-1}\ \frac{\varkappa(\delta)}{\delta}d\delta,\ a \le \mathfrak{s}, $$

that is,

$$ {}^{H}\mathfrak{D}^{\partial}_{a^+}\varkappa(\mathfrak{s}) = \eta^n \left({}^{H}\mathcal{I}^{n-\partial}_{0^+}\varkappa(\mathfrak{s})\right), $$

where $\eta^n = \left(\mathfrak{s}\frac{d}{d\mathfrak{s}}\right)^n$, $n = -[-\partial]$.

Definition 2.4. ([20]) Let $\partial > 0$, $\varkappa \in AC^n([0,T],\mathbb{R})$. The definition of Caputo-Hadamard fractional derivative of order ∂ is given by

$$ {}^{CH}\mathfrak{D}^{\partial}_{a^+}\varkappa(\mathfrak{s}) = {}^{H}\mathfrak{D}^{\partial}_{a^+}\left[\varkappa(\mathfrak{s}) - \sum_{k=0}^{n-1}\frac{\varkappa^{(k)}(0)}{k!}(\ln\frac{\mathfrak{s}}{a})^k\right],\quad \mathfrak{s} \in [1,T], $$

where $n = [\partial] + 1$, and ${}^{H}\mathfrak{D}^{\partial}_{a^+}$ is the Hadamard fractional derivative of order ∂.

Lemma 2.5. ([20]) *If* $\partial, \beta > 0$, *then*

$$ {}^{H}\mathcal{I}^{\partial}_{a^+}\mathfrak{s}^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\partial+\beta)}(\ln\frac{\mathfrak{s}}{a})^{\partial+\beta-1}. \tag{2.2} $$

Our theorems now focus on the existence and uniqueness of solutions for problem (1.1). They rely on Banach's FPT [28] and Krasnoselskii's FPT [28].

3. Main results

For fulfillment the main results, we shall pose the following hypotheses.

(A$_1$): Let $\chi_1(\varkappa(\mathfrak{s})), \chi_2(\varkappa(\mathfrak{s}))$ be continuous functions and there exist constants $\theta_{\chi_i} > 0$ such that

$$ |\chi_i(\varkappa_1(\mathfrak{s})) - \chi_i(\varkappa_2(\mathfrak{s}))| \le \theta_{\chi_i}|\varkappa_1 - \varkappa_2|,\ i = 1,2,\ \forall \varkappa_1, \varkappa_2 \in \mathbb{R}. $$

(A$_2$): The kernels $\mathcal{J}_1(\mathfrak{s},\delta)$ and $\mathcal{J}_1(\mathfrak{s},\delta)$ are continuous on $\mho \times \mho$, and there exist two constants $\mathcal{J}_i^* > 0$ in $\mho \times \mho$ such that

$$ \mathcal{J}_i^* = \sup_{\mathfrak{s}\in\mho}\int_a^{\mathfrak{s}}|\mathcal{J}_i(\mathfrak{s},\delta)|\,d\delta < \infty,\ i = 1,2. $$

(A$_3$): The function $\zeta : \mho \to \mathbb{R}$ is continuous on $\mho$.

(A$_4$): $\xi : C(\mho,\mathbb{R}) \to \mathbb{R}$ is continuous on $C(\mho)$ and there exist constant $0 < \theta_\xi < 1$ such that

$$ |\xi(\varkappa_1(\mathfrak{s})) - \xi(\varkappa_2(\mathfrak{s}))| \le \theta_\xi|\varkappa_1 - \varkappa_2|,\ \forall \varkappa_1, \varkappa_2 \in C(\mho,\mathbb{R}),\ \mathfrak{s} \in \mho. $$

Lemma 3.1. *The function* $\varkappa \in C(\mho, \mathbb{R})$ *is a solution of the problem* (1.1) *if and only if* $\varkappa$ *is a solution of the integral equation*

$$\begin{aligned}\varkappa(\mathfrak{s}) &= \varkappa_0 + \xi(\varkappa) + \frac{1}{\Gamma(\partial)} \int_a^{\mathfrak{s}} (\ln \frac{\mathfrak{s}}{\mathfrak{u}})^{\partial - 1} \frac{\zeta(\mathfrak{u})}{\mathfrak{u}} d\mathfrak{u} \\ & + \frac{1}{\Gamma(\partial)} \int_a^{\mathfrak{s}} \mathfrak{u}^{-1} (\ln \frac{\mathfrak{s}}{\mathfrak{u}})^{\partial - 1} \left\{ \int_0^{\mathfrak{u}} \mathcal{J}_1(\mathfrak{u}, \mathfrak{a}) \chi_1(\varkappa(\mathfrak{a})) d\mathfrak{a} \right. \\ & \left. + \int_0^1 \mathcal{J}_2(\mathfrak{u}, \mathfrak{a}) \chi_2(\varkappa(\mathfrak{a})) d\mathfrak{a} \right\} d\mathfrak{u}.\end{aligned}$$

Proof. The problem stated in equation (1.1) is considered equivalent to the integral equation, as stated in this lemma. So, to avoid boring replication, the proof of this lemma is omitted, as it mirrors certain conventional arguments found in existing literature. □

3.1. **Existence result.** Firstly, we will discuss the existence of the solution of (1.1) by using Krasnoselkii's FPT [28].

Theorem 3.2. *Suppose* $(A_1) - (A_4)$ *hold. Then the problem* (1.1) *has at least one solution on* $\mho$ *If*

$$\Lambda_1 := \left(\theta_\xi + \frac{\sum_{i=1}^2 \theta_{\chi_i} \mathcal{J}_i^*}{\Gamma(\partial + 1)} (\ln \frac{T}{a})^{\partial} \right) < 1. \tag{3.1}$$

Proof. Let $C(\mho, \mathbb{R})$ be a space of continuous functions $\varkappa$ on $\mho$ with the usual norm defined by

$$\|\varkappa\|_\infty = \sup_{\mathfrak{s} \in \mho} |\varkappa(\mathfrak{s})| .$$

Take the ball

$$\Omega_\gamma = \{\varkappa \in C(\mho, \mathbb{R}) : \|\varkappa\|_\infty \leq \gamma\} \subset C(\mho, \mathbb{R}). \tag{3.2}$$

Obviously, Ω_γ is nonempty convex closed subset of $C(\mho, \mathbb{R})$. Choose γ such that $\gamma \geq \frac{\Lambda_2}{1 - \Lambda_1}$, where $\Lambda_1 < 1$,

$$\Lambda_2 := \mu_0 + \frac{\mu_\zeta + \sum_{i=1}^2 \mu_{\chi_i} \mathcal{J}_i^*}{\Gamma(\partial + 1)} (\ln \frac{T}{a})^{\partial} \tag{3.3}$$

for $\mu_\zeta := \sup_{\mathfrak{s} \in [0,1]} |\zeta(\mathfrak{s})|$, $\mu_0 := |\varkappa_0| + \mu_\xi$, $\mu_\xi = |\xi(0)|$, $\mu_{\chi_1} := |\chi_1(0)|$ and $\mu_{\chi_2} := |\chi_2(0)|$.

The equivalent fractional integral equation to the given problem (1.1) can be written as follows based on Lemma 3.1

$$\varkappa = \Upsilon_1 \varkappa + \Upsilon_2 \varkappa, \quad \varkappa \in \Omega_\gamma \subset C(\mho, \mathbb{R}), \tag{3.4}$$

where Υ_1 and Υ_2 are two operators defined on Ω_γ by

$$(\Upsilon_1 \varkappa)(\mathfrak{s}) = \frac{1}{\Gamma(\partial)} \int_a^{\mathfrak{s}} \mathfrak{u}^{-1} (\ln \frac{\mathfrak{s}}{\mathfrak{u}})^{\partial - 1} \Big(\int_0^{\mathfrak{u}} \mathcal{J}_1(\mathfrak{u}, \mathfrak{a}) \chi_1(\varkappa(\mathfrak{a})) d\mathfrak{a} + \int_0^1 \mathcal{J}_2(\mathfrak{u}, \mathfrak{a}) \chi_2(\varkappa(\mathfrak{a})) d\mathfrak{a} \Big) d\mathfrak{u}$$

and

$$(\Upsilon_2 \varkappa)(\mathfrak{s}) = \varkappa_0 + \xi(\varkappa) + \frac{1}{\Gamma(\partial)} \int_a^{\mathfrak{s}} (\ln \frac{\mathfrak{s}}{\mathfrak{u}})^{\partial - 1} \frac{\zeta(\mathfrak{u})}{\mathfrak{u}} d\mathfrak{u}.$$

We split the proof into the following steps:

Step 1: We prove that, $\Upsilon_1 \varkappa + \Upsilon_2 v \in \Omega_\gamma$ for each $\varkappa, v \in \Omega_\gamma$.

Via (A_1) and for any $\varkappa, v \in \Omega_\gamma$, we have

$$\begin{aligned} |\chi_i(\varkappa(\mathfrak{s}))| &\leq |\chi_i(\varkappa(\mathfrak{s})) - \chi_i(0)| + |\chi_i(0)| \\ &\leq \theta_{\chi_i} \|\varkappa\|_\infty + |\chi_i(0)| \\ &\leq \theta_{\chi_i} \gamma + \mu_{\chi_i}, \quad \text{for all } i = 1, 2, \end{aligned}$$

and

$$\begin{aligned} |\xi(v(\mathfrak{s}))| &\leq |\xi(v(\mathfrak{s})) - \xi(0)| + |\xi(0)| \\ &\leq \theta_\xi \|v\|_\infty + |\xi(0)| \\ &\leq \theta_\xi \gamma + \mu_\xi. \end{aligned}$$

Let $\varkappa, v \in \Omega_\gamma$. Then

$$\begin{aligned} &|(\Upsilon_1 \varkappa)(\mathfrak{s}) + (\Upsilon_2 v)(\mathfrak{s})| \\ &\leq \frac{1}{\Gamma(\partial)} \int_a^{\mathfrak{s}} \mathfrak{u}^{-1} (\ln \frac{\mathfrak{s}}{\mathfrak{u}})^{\partial - 1} \Big(\int_0^{\mathfrak{u}} |\mathcal{J}_1(\mathfrak{u}, \mathfrak{a})| \, |\chi_1(\varkappa(\mathfrak{a}))| \, d\mathfrak{a} \\ &\quad + \int_0^1 |\mathcal{J}_2(\mathfrak{u}, \mathfrak{a})| \, |\chi_2(\varkappa(\mathfrak{a}))| \, d\mathfrak{a} \Big) d\mathfrak{u} \\ &\quad + |\varkappa_0| + |\xi(v)| + \frac{1}{\Gamma(\partial)} \int_a^{\mathfrak{s}} \mathfrak{u}^{-1} (\ln \frac{\mathfrak{s}}{\mathfrak{u}})^{\partial - 1} |\zeta(\mathfrak{u})| \, d\mathfrak{u} \\ &\leq \mu_0 + \theta_\xi \gamma + \frac{\mu_\zeta + \sum_{i=1}^2 (\theta_{\chi_i} \gamma + \mu_{\chi_i}) \mathcal{J}_i^*}{\Gamma(\partial + 1)} (\ln \frac{\mathfrak{s}}{a})^\partial, \end{aligned}$$

which implies

$$\begin{aligned} &\|\Upsilon_1 \varkappa + \Upsilon_2 v\|_\infty \\ &\leq \mu_0 + \frac{\mu_\zeta + \sum_{i=1}^2 \mu_{\chi_i} \mathcal{J}_i^*}{\Gamma(\partial + 1)} (\ln \frac{T}{a})^\partial + \left(\theta_\xi + \frac{\sum_{i=1}^2 \theta_{\chi_i} \mathcal{J}_i^*}{\Gamma(\partial + 1)} (\ln \frac{T}{a})^\partial \right) \gamma \\ &\leq \Lambda_2 + \Lambda_1 \gamma \leq \gamma. \end{aligned}$$

Consequently, we have

$$\Upsilon_1 \varkappa + \Upsilon_2 v \in \Omega_\gamma.$$

Step 2: We prove that Υ_2 is contraction on Ω_γ.

Set $\varkappa, \varkappa^* \in \Omega_\gamma$. It follows from (A_4) that

$$\begin{aligned}\|\Upsilon_2 \varkappa - \Upsilon_2 \varkappa^*\|_\infty &= \sup_{\mathfrak{s}\in\mho} |\Upsilon_2 \varkappa(\mathfrak{s}) - \Upsilon_2 \varkappa(\mathfrak{s})| = \sup_{\mathfrak{s}\in\mho} |\xi(\varkappa(\mathfrak{s})) - \xi(\varkappa^*(\mathfrak{s}))| \\ &\le \theta_\xi \|\varkappa - \varkappa^*\|_\infty .\end{aligned}$$

So, Υ_2 is a contraction mapping, because of $\theta_\xi < 1$.

Step 3: We prove that, Υ_1 is completely continuous on Ω_γ.

Stage 1: We prove the continuity of Υ_1. Consider $\{\varkappa_n\}$ be a sequence such that $\varkappa_n \to \varkappa$ in $C(\mho, \mathbb{R})$. So, for $\varkappa_n, \varkappa \in C(\mho, \mathbb{R})$ and for $\mathfrak{s} \in \mho$, we get

$$\begin{aligned}&|(\Upsilon_1 \varkappa_n)(\mathfrak{s}) - (\Upsilon_1 \varkappa)(\mathfrak{s})| \\ &\quad\le \frac{1}{\Gamma(\partial)} \int_a^{\mathfrak{s}} \mathfrak{u}^{-1} (\ln \frac{\mathfrak{s}}{\mathfrak{u}})^{\partial-1} \left(\int_0^{\mathfrak{u}} |\mathcal{J}_1(\mathfrak{u},\mathfrak{a})| \, |\chi_1(\varkappa_n(\mathfrak{a})) - \chi_1(\varkappa(\mathfrak{a}))| \, d\mathfrak{a} \right. \\ &\quad\quad \left. + \int_0^1 |\mathcal{J}_2(\mathfrak{u},\mathfrak{a})| \, |\chi_2(\varkappa_n(\mathfrak{a})) - \chi_2(\varkappa(\mathfrak{a}))| \, d\mathfrak{a} \right) d\mathfrak{u} \\ &\quad\le \frac{\sum_{i=1}^2 \theta_{\chi_i} \mathcal{J}_i^*}{\Gamma(\partial+1)} \ln(\frac{T}{a})^\partial \|\varkappa_n - \varkappa\|_\infty .\end{aligned}$$

Since $\varkappa_n \to \varkappa$ as $n \to \infty$, $\|\Upsilon_1 \varkappa_n - \Upsilon_1 \varkappa\|_\infty \to 0$ as $n \to \infty$. This proves that Υ_1 is continuous on $C(\mho, \mathbb{R})$.

Stage 2: From Step 1, we observe that

$$\begin{aligned}|(\Upsilon_1 \varkappa)(\mathfrak{s})| &\le \frac{1}{\Gamma(\partial)} \int_a^{\mathfrak{s}} \mathfrak{u}^{-1} (\ln \frac{\mathfrak{s}}{\mathfrak{u}})^{\partial-1} \Big(\int_0^{\mathfrak{u}} |\mathcal{J}_1(\mathfrak{u},\mathfrak{a})| \, |\chi_1(\varkappa(\mathfrak{a}))| \, d\mathfrak{a} \\ &\quad + \int_0^1 |\mathcal{J}_2(\mathfrak{u},\mathfrak{a})| \, |\chi_2(\varkappa(\mathfrak{a}))| \, d\mathfrak{a} \Big) d\mathfrak{u} \\ &\le \frac{\sum_{i=1}^2 (\theta_{\chi_i} \gamma + \mu_{\chi_i}) \, \mathcal{J}_i^*}{\Gamma(\partial+1)} \ln \left(\frac{\mathfrak{s}}{a}\right)^\partial .\end{aligned}$$

Thus

$$\|\Upsilon_1 \varkappa\|_\infty \le \frac{\sum_{i=1}^2 (\theta_{\chi_i} \gamma + \mu_{\chi_i}) \, \mathcal{J}_i^*}{\Gamma(\partial+1)} \ln \left(\frac{T}{a}\right)^\partial ,$$

which prove the uniform boundedness of $(\Upsilon_1 \Omega_\gamma)$.

Stage 3: We show that $(\Upsilon_1\Omega_\gamma)$ is equicontinuous. Take $\varkappa \in \Omega_\gamma$. Then for $\mathfrak{s}_1, \mathfrak{s}_2 \in \mho$ with $\mathfrak{s}_1 \leq \mathfrak{s}_2$, we have

$$\begin{aligned}
&|(\Upsilon_1\varkappa)(\mathfrak{s}_2) - (\Upsilon_1\varkappa)(\mathfrak{s}_1)| \\
&= \Bigg| \frac{1}{\Gamma(\partial)} \int_a^{\mathfrak{s}_2} \mathfrak{u}^{-1} (\ln \frac{\mathfrak{s}_2}{\mathfrak{u}})^{\partial-1} \left(\int_0^{\mathfrak{u}} |\mathcal{J}_1(\mathfrak{u},\mathfrak{a})| \, |\chi_1(\varkappa(\mathfrak{a}))| \, d\mathfrak{a} \right. \\
&\quad \left. + \int_0^1 |\mathcal{J}_2(\mathfrak{u},\mathfrak{a})| \, |\chi_2(\varkappa(\mathfrak{a}))| \, d\mathfrak{a} \right) d\mathfrak{u} \\
&\quad - \frac{1}{\Gamma(\partial)} \int_a^{\mathfrak{s}_1} \mathfrak{u}^{-1} (\ln \frac{\mathfrak{s}_1}{\mathfrak{u}})^{\partial-1} \left(\int_0^{\mathfrak{u}} |\mathcal{J}_1(\mathfrak{u},\mathfrak{a})| \, |\chi_1(\varkappa(\mathfrak{a}))| \, d\mathfrak{a} \right. \\
&\quad \left. + \int_0^1 |\mathcal{J}_2(\mathfrak{u},\mathfrak{a})| \, |\chi_2(\varkappa(\mathfrak{a}))| \, d\mathfrak{a} \right) d\mathfrak{u} \Bigg| \\
&\leq \frac{1}{\Gamma(\partial)} \left(\int_{\mathfrak{s}_1}^{\mathfrak{s}_2} \mathfrak{u}^{-1} (\ln \frac{\mathfrak{s}_2}{\mathfrak{u}})^{\partial-1} \int_0^{\mathfrak{u}} |\mathcal{J}_1(\mathfrak{u},\mathfrak{a})| \, |\chi_1(\varkappa(\mathfrak{a}))| \, d\mathfrak{a} d\mathfrak{u} \right. \\
&\quad \left. + \int_0^{\mathfrak{s}_1} \mathfrak{u}^{-1} \left| (\ln \frac{\mathfrak{s}_2}{\mathfrak{u}})^{\partial-1} - (\ln \frac{\mathfrak{s}_1}{\mathfrak{u}})^{\partial-1} \right| \int_0^{\mathfrak{u}} |\mathcal{J}_1(\mathfrak{u},\mathfrak{a})| \, |\chi_1(\varkappa(\mathfrak{a}))| \, d\mathfrak{a} d\mathfrak{u} \right) \\
&\quad + \frac{1}{\Gamma(\partial)} \left(\int_{\mathfrak{s}_1}^{\mathfrak{s}_2} \mathfrak{u}^{-1} (\ln \frac{\mathfrak{s}_2}{\mathfrak{u}})^{\partial-1} \int_0^{\mathfrak{u}} |\mathcal{J}_2(\mathfrak{u},\mathfrak{a})| \, |\chi_2(\varkappa(\mathfrak{a}))| \, d\mathfrak{a} d\mathfrak{u} \right. \\
&\quad \left. + \int_0^{\mathfrak{s}_1} \mathfrak{u}^{-1} \left| (\ln \frac{\mathfrak{s}_2}{\mathfrak{u}})^{\partial-1} - (\ln \frac{\mathfrak{s}_1}{\mathfrak{u}})^{\partial-1} \right| \int_0^{\mathfrak{u}} |\mathcal{J}_2(\mathfrak{u},\mathfrak{a})| \, |\chi_2(\varkappa(\mathfrak{a}))| \, d\mathfrak{a} d\mathfrak{u} \right),
\end{aligned}$$

which implies

$$\begin{aligned}
|(\Upsilon_1\varkappa)(\mathfrak{s}_2) - (\Upsilon_1\varkappa)(\mathfrak{s}_1)| \leq{}& \frac{(\theta_{\chi_1}\gamma + \mu_{\chi_1})\mathcal{J}_1^*}{\Gamma(\partial)} \left(\int_{\mathfrak{s}_1}^{\mathfrak{s}_2} \mathfrak{u}^{-1} (\ln \frac{\mathfrak{s}_2}{\mathfrak{u}})^{\partial-1} d\mathfrak{u} \right. \\
&\left. + \int_a^{\mathfrak{s}_1} \mathfrak{u}^{-1} \left| (\ln \frac{\mathfrak{s}_2}{\mathfrak{u}})^{\partial-1} - (\ln \frac{\mathfrak{s}_1}{\mathfrak{u}})^{\partial-1} \right| d\mathfrak{u} \right) \\
&+ \frac{(\theta_{\chi_2}\gamma + \mu_{\chi_2})\mathcal{J}_2^*}{\Gamma(\partial)} \left(\int_{\mathfrak{s}_1}^{\mathfrak{s}_2} \mathfrak{u}^{-1} (\ln \frac{\mathfrak{s}_2}{\mathfrak{u}})^{\partial-1} d\mathfrak{u} \right. \\
&\left. + \int_a^{\mathfrak{s}_1} \mathfrak{u}^{-1} \left| (\ln \frac{\mathfrak{s}_2}{\mathfrak{u}})^{\partial-1} - (\ln \frac{\mathfrak{s}_1}{\mathfrak{u}})^{\partial-1} \right| d\mathfrak{u} \right),
\end{aligned}$$

which tends to zero as $\mathfrak{s}_2 - \mathfrak{s}_1 \to 0$. So, $(\Upsilon_1\Omega_\gamma)$ is equicontinuous. The Arzela-Ascoli theorem indicates that Υ_1 is relatively compact and thus completely continuous. According to Krasnoselkii's FPT [28], the problem (1.1) has at least one solution in $C(\mho, \mathbb{R})$. □

3.2. **Uniqueness result.** We give the uniqueness of the solution of the problem (1.1).

Theorem 3.3. *Assume* $(A_1)-(A_4)$ *hold. If*

$$\left(\theta_\xi + \frac{\sum_{i=1}^{2}\theta_{\chi_i}\mathcal{J}_i^*}{\Gamma(\partial+1)}\ln(\frac{T}{a})^{\partial}\right) < 1, \qquad a \le \mathfrak{s}, \tag{3.5}$$

then the problem (1.1) *has a unique solution on* $\mho$.

Proof. We transfer the problem (1.1) into a fixed point problem, that is,

$$\varkappa = \Theta\varkappa, \quad \varkappa \in C(\mho,\mathbb{R}),$$

where $\Theta : C(\mho,\mathbb{R}) \to C(\mho,\mathbb{R})$ defined by

$$\begin{aligned}(\Theta\varkappa)(\mathfrak{s}) \quad = \quad & \varkappa_0 + \xi(\varkappa) + \frac{1}{\Gamma(\partial)}\int_a^{\mathfrak{s}} \mathfrak{u}^{-1}(\ln\frac{\mathfrak{s}}{\mathfrak{u}})^{\partial-1}\zeta(\mathfrak{u})d\mathfrak{u} \\ & + \frac{1}{\Gamma(\partial)}\int_a^{\mathfrak{s}} \mathfrak{u}^{-1}(\ln\frac{\mathfrak{s}}{\mathfrak{u}})^{\partial-1}\left(\int_0^{\mathfrak{u}} \mathcal{J}_1(\mathfrak{u},\mathfrak{a})\chi_1(\varkappa(\mathfrak{a}))d\mathfrak{a}\right. \\ & \left. + \int_0^1 \mathcal{J}_2(\mathfrak{u},\mathfrak{a})\chi_2(\varkappa(\mathfrak{a}))d\mathfrak{a}\right) d\mathfrak{u}, \;\; \mathfrak{s}\in\mho.\end{aligned}$$

Let $\varkappa, \varkappa^* \in C(\mho,\mathbb{R})$, then for $\mathfrak{s}\in\mho$, we get

$$\begin{aligned}&|\Theta\varkappa(\mathfrak{s}) - \Theta\varkappa^*(\mathfrak{s})| \\ &\quad \le |\xi(\varkappa(\mathfrak{s})) - \xi(\varkappa^*(\mathfrak{s}))| \\ &\qquad + \frac{1}{\Gamma(\partial)}\int_a^{\mathfrak{s}} \mathfrak{u}^{-1}(\ln\frac{\mathfrak{s}}{\mathfrak{u}})^{\partial-1}\Big(\int_0^{\mathfrak{u}} \mathcal{J}_1(\mathfrak{u},\mathfrak{a})\left|\chi_1(\varkappa(\mathfrak{a})) - \chi_1(\varkappa^*(\mathfrak{a}))\right| d\mathfrak{a}\Big)d\mathfrak{u} \\ &\qquad + \frac{1}{\Gamma(\partial)}\int_a^{\mathfrak{s}} \mathfrak{u}^{-1}(\ln\frac{\mathfrak{s}}{\mathfrak{u}})^{\partial-1}\Big(\int_0^{1} \mathcal{J}_2(\mathfrak{u},\mathfrak{a})\left|\chi_2(\varkappa(\mathfrak{a})) - \chi_2(\varkappa^*(\mathfrak{a}))\right| d\mathfrak{a}\Big)d\mathfrak{u} \\ &\quad \le \theta_\xi \left\|\varkappa - \varkappa^*\right\|_\infty + \frac{1}{\Gamma(\partial)}\int_a^{\mathfrak{s}} \mathfrak{u}^{-1}(\ln\frac{\mathfrak{s}}{\mathfrak{u}})^{\partial-1}\mathcal{J}_1^*\theta_{\chi_1}\left\|\varkappa - \varkappa^*\right\|_\infty d\mathfrak{u} \\ &\qquad + \frac{1}{\Gamma(\partial)}\int_a^{\mathfrak{s}} \mathfrak{u}^{-1}(\ln\frac{\mathfrak{s}}{\mathfrak{u}})^{\partial-1}\mathcal{J}_2^*\theta_{\chi_2}\left\|\varkappa - \varkappa^*\right\|_\infty d\mathfrak{u} \\ &\quad \le \left(\theta_\xi + \frac{\mathcal{J}_1^*\theta_{\chi_1} + \mathcal{J}_2^*\theta_{\chi_2}}{\Gamma(\partial+1)}\ln(\frac{T}{a})^{\partial}\right)\left\|\varkappa - \varkappa^*\right\|_\infty,\end{aligned}$$

which implies

$$\left\|\Theta\varkappa - \Theta\varkappa^*\right\|_\infty \le \left(\theta_\xi + \frac{\sum_{i=1}^{2}\theta_{\chi_i}\mathcal{J}_i^*}{\Gamma(\partial+1)}\ln(\frac{T}{a})^{\partial}\right)\left\|\varkappa - \varkappa^*\right\|_\infty.$$

Consequently, given the conditions outlined in (3.5), we can deduce that Θ functions as a contraction operator. Therefore, in accordance with Banach's

FPT [28], it possesses a unique fixed point. This leads us to conclude that the problem stated in 3.5 has a singular solution. □

4. Approximate solution

First, we recall the classical ADM where the solution of the proposed problem is obtained in the form of a series as

$$\varkappa = \sum_{n=0}^{\infty} \varkappa_n \tag{4.1}$$

and the nonlinear terms χ_1, χ_2 and ξ are decomposed as

$$\chi_1 = \sum_{n=0}^{\infty} \wp_n, \quad \chi_2 = \sum_{n=0}^{\infty} \omega_n, \; \xi = \sum_{n=0}^{\infty} \varpi_n, \tag{4.2}$$

where $\wp_n, \omega_n, \varpi_n$ are Adomian polynomials for all $n \in \mathbb{N}$, and write

$$\varkappa = \varkappa(\lambda) = \sum_{n=0}^{\infty} \lambda^n \varkappa_n = \varkappa_0 + \lambda \varkappa_1 + \lambda^2 \varkappa_2 + \cdots + \lambda^k \varkappa_k + \cdots, \tag{4.3}$$

$$\chi_1 = \chi_1(\lambda) = \sum_{n=0}^{\infty} \lambda^n \wp_n = \wp_0 + \lambda \wp_1 + \lambda^2 \wp_2 + \cdots + \lambda^k \wp_k + \cdots, \tag{4.4}$$

$$\chi_2 = \chi_2(\lambda) = \sum_{n=0}^{\infty} \lambda^n \omega_n = \omega_0 + \lambda \omega_1 + \lambda^2 \omega_2 + \cdots + \lambda^k \omega_k + \cdots, \tag{4.5}$$

$$\xi = \xi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \varpi_n = \varpi_0 + \lambda \varpi_1 + \lambda^2 \varpi_2 + \cdots + \lambda^k \varpi_k + \cdots. \tag{4.6}$$

By utilizing the previous formulas (4.3), (4.4), (4.5) and (4.6), we deduce that

$$\wp_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left(\chi_1 \sum_{i=0}^{\infty} \lambda^i \varkappa_i \right) \right]_{\lambda=0},$$

$$\omega_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left(\chi_2 \sum_{i=0}^{\infty} \lambda^i \varkappa_i \right) \right]_{\lambda=0},$$

$$\varpi_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left(\xi \sum_{i=0}^{\infty} \lambda^i \varkappa_i \right) \right]_{\lambda=0},$$

where $\varkappa_0$, $\varkappa_1$, $\varkappa_2, ...$ are repeatedly specified by

$$\begin{cases} \varkappa_0(\mathfrak{s}) = \varkappa_0 + {}^{H}\mathcal{I}^{\partial}_{a^+}\left(\zeta(\mathfrak{s})\right), \\ \varkappa_{k+1}(\mathfrak{s}) = \varpi_k + {}^{H}\mathcal{I}^{\partial}_{a^+}\left(\int_0^{\mathfrak{s}} \mathcal{J}_1(\mathfrak{s},\delta)\wp_k d\delta\right) \\ \qquad\qquad + {}^{H}\mathcal{I}^{\partial}_{a^+}\left(\int_0^1 \mathcal{J}_2(\mathfrak{s},\delta)\omega_k d\delta\right), \quad k \geq 1. \end{cases} \tag{4.7}$$

Here, we use the MADM. Therefore, the scheme (4.7) gives

$$\begin{cases} \varkappa_0(\mathfrak{s}) = \varkappa_0 + R_1(\mathfrak{s}), \\ \varkappa_1(\mathfrak{s}) = R_2(\mathfrak{s}) + \varpi_0 + {}^{H}\mathcal{I}^{\partial}_{a^+}\left(\int_0^{\mathfrak{s}} \mathcal{J}_1(\mathfrak{s},\delta)\wp_0 d\delta\right) \\ \qquad\qquad + {}^{H}\mathcal{I}^{\partial}_{a^+}\left(\int_0^1 \mathcal{J}_2(\mathfrak{s},\delta)\omega_0 d\delta\right), \\ \varkappa_{k+1}(\mathfrak{s}) = \varpi_k + {}^{H}\mathcal{I}^{\partial}_{a^+}\left(\int_0^{\mathfrak{s}} \mathcal{J}_1(\mathfrak{s},\delta)\wp_k d\delta\right) \\ \qquad\qquad + {}^{H}\mathcal{I}^{\partial}_{a^+}\left(\int_0^1 \mathcal{J}_2(\mathfrak{s},\delta)\omega_k d\delta\right), \quad k \geq 1. \end{cases} \tag{4.8}$$

Now, we will study the convergence theorem of the solution based on the MADM.

Theorem 4.1. *Assume that* $(A_1)-(A_4)$ *and* (3.1) *are satisfied, if the solution* $\varkappa(\mathfrak{s}) = \sum_{i=0}^{\infty} \varkappa_i(\mathfrak{s})$ *and* $\|\varkappa\|_\infty < \infty$ *is convergent, then it converges to the exact solution of the problem* (1.1).

Proof. We omit the proof because it resembles some works in the literature [10]. □

Example 4.2. Consider an integro-differential equation with Caputo-Hadamard fractional derivative

$$\begin{cases} {}^{CH}\mathfrak{D}^{\frac{1}{3}}_{1^+}\varkappa(\mathfrak{s}) = \frac{2}{\sqrt{\pi}}\left(\frac{4\mathfrak{s}^{\frac{3}{2}}}{\Gamma(6)} + \mathfrak{s}^{\frac{1}{2}}\right) + \frac{\mathfrak{s}^3}{\Gamma(7)} + \frac{\mathfrak{s}}{\Gamma(8)} \\ +\frac{1}{4}\int_0^{\mathfrak{s}}(1+\mathfrak{s}-\mathfrak{u})\varkappa(\mathfrak{u})d\mathfrak{u} + \frac{5}{18}\int_0^1 e^{\mathfrak{u}-\mathfrak{s}}\varkappa^2(\mathfrak{u})d\mathfrak{u}, \end{cases}, \quad \mathfrak{s} \in [1,e] \tag{4.9}$$

with the nonlocal condition

$$\varkappa(0) = \frac{1}{5}\varkappa(\frac{1}{4}), \tag{4.10}$$

where

$$\begin{aligned} \partial &= \frac{1}{3},\ \varkappa_0 = 0,\ \xi(\varkappa(\mathfrak{s})) = \frac{1}{5}\varkappa(\frac{1}{4}), \\ \zeta(\mathfrak{s}) &= \frac{2}{\sqrt{\pi}}\left(\frac{4\mathfrak{s}^{\frac{3}{2}}}{\Gamma(6)} + \mathfrak{s}^{\frac{1}{2}}\right) + \frac{\mathfrak{s}^3}{\Gamma(7)} + \frac{\mathfrak{s}}{\Gamma(8)}, \\ \mathcal{J}_1(\mathfrak{s},\delta) &= \frac{1}{4}(1+\mathfrak{s}-\delta),\ \mathcal{J}_2(\mathfrak{s},\delta) = \frac{5}{18}e^{\delta-\mathfrak{s}}. \end{aligned}$$

Then clearly, $\theta_{\chi_1} = \theta_{\chi_2} = 1$, $\theta_\xi = \frac{1}{5}$.

$$\begin{aligned}\mu_\zeta : &= \sup_{\mathfrak{s}\in[0,1]} |\zeta(\mathfrak{s})| = \|\zeta\|_\infty \\ &= \frac{2}{\sqrt{\pi}}\left(\frac{4e^{\frac{3}{2}}}{\Gamma(6)} + e^{\frac{1}{2}}\right) + \frac{e^3}{\Gamma(7)} + \frac{e}{\Gamma(8)} \\ &= 2.0574,\end{aligned}$$

$$\begin{aligned}\mathcal{J}_1^* &= \frac{1}{4}\sup_{\mathfrak{s}\in\mho}\int_0^{\mathfrak{s}} |1+\mathfrak{s}-\delta|\, d\delta = \frac{1}{8}. \\ \mathcal{J}_2^* &= \frac{5}{18}\sup_{\mathfrak{s}\in\mho}\int_0^{\mathfrak{s}} \left|e^{\delta-\mathfrak{s}}\right| d\delta = \frac{5}{18}\sup_{\mathfrak{s}\in\mho} e^{-\mathfrak{s}}\int_0^{\mathfrak{s}} \left|e^{\delta}\right| d\delta \\ &= \frac{5}{18}(1-\frac{1}{e}).\end{aligned}$$

Hence,

$$\Lambda_1 := \left(\theta_\xi + \frac{\sum_{i=1}^2 \theta_{\chi_i}\mathcal{J}_i^*}{\Gamma(\partial+1)}\ln(e)\right) \approx 0.22752 < 1.$$

As consequence of Theorem 3.3, the problem (4.9)-(4.10) has a unique solution on $[1, e]$.

Applying the operator $\mathcal{I}_{0^+}^{\frac{1}{2}}$ to both sides of equation (4.9), we get

$$\begin{aligned}\varkappa(\mathfrak{s}) &= \frac{1}{5}\varkappa(\frac{1}{4}) + {}^H\mathcal{I}_{1^+}^{\frac{1}{3}}\left(\frac{2}{\sqrt{\pi}}\left(\frac{4\mathfrak{s}^{\frac{3}{2}}}{\Gamma(6)} + \mathfrak{s}^{\frac{1}{2}}\right) + \frac{\mathfrak{s}^3}{\Gamma(7)} + \frac{\mathfrak{s}}{\Gamma(8)}\right) \\ &\quad + {}^H\mathcal{I}_{1^+}^{\frac{1}{3}}\left(\frac{1}{4}\int_0^{\mathfrak{s}}(1+\mathfrak{s}-\mathfrak{u})\varkappa(\mathfrak{u})d\mathfrak{u}\right) + {}^H\mathcal{I}_{1^+}^{\frac{1}{3}}\left(\frac{5}{18}\int_0^1 e^{\mathfrak{u}-\mathfrak{s}}\varkappa^2(\mathfrak{u})d\mathfrak{u}\right).\end{aligned}$$

Suppose

$$\begin{aligned}R(\mathfrak{s}) = &\ {}^H\mathcal{I}_{1^+}^{\frac{1}{3}}\left(\frac{2}{\sqrt{\pi}}\left(\frac{4\mathfrak{s}^{\frac{3}{2}}}{\Gamma(6)} + \mathfrak{s}^{\frac{1}{2}}\right) + \frac{\mathfrak{s}^3}{\Gamma(7)} + \frac{\mathfrak{s}}{\Gamma(8)}\right) \\ = &\ \frac{2}{\sqrt{\pi}}\frac{4}{\Gamma(6)}\left({}^H\mathcal{I}_{1^+}^{\frac{1}{3}}\mathfrak{u}^{\frac{3}{2}}\right)(\mathfrak{s}) + \frac{2}{\sqrt{\pi}}\left({}^H\mathcal{I}_{1^+}^{\frac{1}{3}}\mathfrak{u}^{\frac{1}{2}}\right)(\mathfrak{u}) \\ &+ \frac{1}{\Gamma(7)}\left({}^H\mathcal{I}_{1^+}^{\frac{1}{3}}\mathfrak{u}^3\right)(\mathfrak{u}) + \frac{1}{\Gamma(8)}\left({}^H\mathcal{I}_{1^+}^{\frac{1}{3}}\mathfrak{u}\right)(\mathfrak{s}) \\ = &\ \frac{8\Gamma(\frac{5}{2})}{\sqrt{\pi}\Gamma(6)\Gamma(\frac{17}{6})}(\ln\mathfrak{s})^{\frac{11}{6}} + \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi}\Gamma(\frac{11}{6})}(\ln\mathfrak{s})^{\frac{5}{6}} \\ &+ \frac{\Gamma(4)}{\Gamma(7)\Gamma(\frac{13}{3})}(\ln\mathfrak{s})^{\frac{10}{3}} + \frac{\Gamma(2)}{\Gamma(\frac{7}{3})}(\ln\mathfrak{s})^{\frac{4}{3}}.\end{aligned}$$

Now, we apply the modified ADM,

$$R(\mathfrak{s}) = R_1(\mathfrak{s}) + R_2(\mathfrak{s}),$$

where

$$R_1(\mathfrak{s}) = \frac{8\Gamma(\frac{5}{2})}{\sqrt{\pi}\Gamma(6)\Gamma(\frac{17}{6})}(\ln \mathfrak{s})^{\frac{11}{6}}$$

and

$$R_2(\mathfrak{s}) = \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi}\Gamma(\frac{11}{6})}(\ln \mathfrak{s})^{\frac{5}{6}} + \frac{\Gamma(4)}{\Gamma(7)\Gamma(\frac{13}{3})}(\ln \mathfrak{s})^{\frac{10}{3}} + \frac{\Gamma(2)}{\Gamma(\frac{7}{3})}(\ln \mathfrak{s})^{\frac{4}{3}}.$$

The modified recursive relation

$$\varkappa_0(\mathfrak{s}) = R_1(\mathfrak{s}) = \frac{8\Gamma(\frac{5}{2})}{\sqrt{\pi}\Gamma(6)\Gamma(\frac{17}{6})}(\ln \mathfrak{s})^{\frac{11}{6}},$$

$$\begin{aligned}
\varkappa_1(\mathfrak{s}) &= R_2(\mathfrak{s}) + {}^{H}\mathcal{I}^{\frac{1}{3}}_{1^+}\left(\frac{1}{4}\int_0^{\mathfrak{s}}(1+\mathfrak{s}-\mathfrak{u})\wp_0(\mathfrak{u})d\mathfrak{u}\right) \\
&\quad + {}^{H}\mathcal{I}^{\frac{1}{3}}_{1^+}\left(\frac{5}{18}\int_0^{1}e^{\mathfrak{u}-\mathfrak{s}}\omega_0(\mathfrak{u})d\mathfrak{u}\right) + \varpi_0(\mathfrak{s}) \\
&= \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi}\Gamma(\frac{11}{6})}(\ln \mathfrak{s})^{\frac{5}{6}} + \frac{\Gamma(4)}{\Gamma(7)\Gamma(\frac{13}{3})}(\ln \mathfrak{s})^{\frac{10}{3}} + \frac{\Gamma(2)}{\Gamma(\frac{7}{3})}(\ln \mathfrak{s})^{\frac{4}{3}} \\
&\quad + {}^{H}\mathcal{I}^{\frac{1}{3}}_{1^+}\left(\frac{1}{4}\int_0^{\mathfrak{s}}(1+\mathfrak{s}-\mathfrak{u})\varkappa_0(\mathfrak{u})d\mathfrak{u}\right) \\
&\quad + {}^{H}\mathcal{I}^{\frac{1}{3}}_{1^+}\left(\frac{5}{18}\int_0^{1}e^{\mathfrak{u}-\mathfrak{s}}\varkappa_0(\mathfrak{u})d\mathfrak{u}\right) + \frac{1}{5}\varkappa_0(\frac{1}{4}) \\
&= \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi}\Gamma(\frac{11}{6})}(\ln \mathfrak{s})^{\frac{5}{6}} + \frac{\Gamma(4)}{\Gamma(7)\Gamma(\frac{13}{3})}(\ln \mathfrak{s})^{\frac{10}{3}} + \frac{\Gamma(2)}{\Gamma(\frac{7}{3})}(\ln \mathfrak{s})^{\frac{4}{3}} \\
&\quad + {}^{H}\mathcal{I}^{\frac{1}{3}}_{1^+}\left(\frac{1}{4}\int_0^{\mathfrak{s}}(1+\mathfrak{s}-\mathfrak{u})\frac{8\Gamma(\frac{5}{2})}{\sqrt{\pi}\Gamma(6)\Gamma(\frac{17}{6})}(\ln \mathfrak{u})^{\frac{11}{6}}d\mathfrak{u}\right) \\
&\quad + {}^{H}\mathcal{I}^{\frac{1}{3}}_{1^+}\left(\frac{5}{18}\int_0^{1}e^{\mathfrak{u}-\mathfrak{s}}\frac{8\Gamma(\frac{5}{2})}{\sqrt{\pi}\Gamma(6)\Gamma(\frac{17}{6})}(\ln \mathfrak{u})^{\frac{11}{6}}d\mathfrak{u}\right) \\
&\quad + \frac{1}{5}\frac{8\Gamma(\frac{5}{2})}{\sqrt{\pi}\Gamma(6)\Gamma(\frac{17}{6})}(\ln\frac{1}{3})^{\frac{11}{6}} \\
&= 0,
\end{aligned}$$

$$\varkappa_2(\mathfrak{s}) = 0,$$

$$\vdots$$

$$\varkappa_n(\mathfrak{s}) = 0.$$

Therefore, the obtained solution is

$$\varkappa(\mathfrak{s}) = \sum_{i=0}^{\infty} \varkappa_i(\mathfrak{s}) = \frac{8\Gamma(\frac{5}{2})}{\sqrt{\pi}\Gamma(6)\Gamma(\frac{17}{6})}(\ln \mathfrak{s})^{\frac{11}{6}}.$$

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