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EXISTENCE OF RANDOM FIXED POINT IN CONE RANDOM METRIC SPACES

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Abstract. The purpose of this paper is to prove existence of common random fixed point in the setting of cone random metric space under weak contractive condition. Our result generalizes the corresponding recent result of Dhagat et al. [8] (Advances in Fixed Point Theory, **2(3)** (2012), 357-363) and some others.

1. Introduction

Random nonlinear analysis is an important mathematical discipline which is mainly concerned with the study of random nonlinear operators and their properties and is needed for the study of various classes of random equations. The study of random fixed point theory was initiated by the Prague school of Probabilities in the 1950s [9, 10, 22]. Common random fixed point theorems are stochastic generalization of classical common fixed point theorems. The machinery of random fixed point theory provides a convenient way of modeling many problems arising from economic theory(see e.g. [17]) and references mentioned therein. Random methods have revolutionized the financial markets. The survey article by Bharucha-Reid [7] attracted the attention of several mathematicians and gave wings to the theory. Itoh [14] extended Spacek's and Hans's theorem to multivalued contraction mappings. Now this theory has become the full fledged research area and various ideas associated with random fixed point theory are used to obtain the solution of nonlinear random system (see [4, 5, 6, 11, 20]). Papageorgiou [15, 16], Beg [2, 3] studied common random

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fixed points and random coincidence points of a pair of compatible random operators and proved fixed point theorems for contractive random operators in Polish spaces.

In 2007, Huang and Zhang [12] introduced the concept of cone metric space and establish some fixed point theorems for contractive mappings in normal cone metric spaces. Subsequently, several other authors [1, 13, 19, 21] studied the existence of fixed points and common fixed points of mappings satisfying contractive type condition on a normal cone metric space.

In 2008, Rezapour and Hamlbarani [19] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space. In this paper we prove existence of common random fixed point in the setting of cone random metric spaces under weak contractive condition. Our result generalizes the corresponding recent result of Dhagat et al. [8]

Recently, Dhagat et al. [8] introduced the concept of cone random metric space and proved an existence of random fixed point under weak contraction condition in the setting of cone random metric spaces. The purpose of this paper is to extends the result of [8] to the case of more general class of weak contraction condition.

2. Preliminaries

Definition 2.1. ([8]) Let (E, τ) be a topological vector space. A subset P of E is called a cone whenever the following conditions hold:

($\mathbf{c_1}$) P is closed, nonempty and $P \neq \{0\}$;
($\mathbf{c_2}$) $a, b \in R$, $a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;
($\mathbf{c_3}$) If $x \in P$ and $-x \in P$ implies $x = 0$.

For a given cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in P^0$, where P^0 stands for the interior of P.

Definition 2.2. ([12, 23]) Let X be a nonempty set. Suppose that the mapping $d\colon X \times X \to E$ satisfies:

($\mathbf{d_1}$) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
($\mathbf{d_2}$) $d(x, y) = d(y, x)$ for all $x, y \in X$;
($\mathbf{d_3}$) $d(x, y) \leq d(x, z) + d(z, y)$ $x, y, z \in X$.

Then d is called a cone metric [12] or K-metric [23] on X and (X, d) is called a cone metric space [12].

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Example 2.3. ([12]) Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, $X = \mathbb{R}$ and $d\colon X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space with normal cone P where $K = 1$.

Example 2.4. ([18]) Let $E = \ell^2$, $P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0, \text{for all } n\}$, (X, ρ) a metric space, and $d\colon X \times X \to E$ defined by $d(x, y) = \{\rho(x, y)/2^n\}_{n\geq 1}$. Then (X, d) is a cone metric space.

Clearly, the above examples show that class of cone metric spaces contains the class of metric spaces.

Definition 2.5. (See [12]) Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:

(i) a Cauchy sequence if for every ε in E with $0 \ll \varepsilon$, then there is an N such that for all $n, m > N$, $d(x_n, x_m) \ll \varepsilon$;

(ii) a convergent sequence if for every ε in E with $0 \ll \varepsilon$, then there is an N such that for all $n > N$, $d(x_n, x) \ll \varepsilon$ for some fixed x in X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

In the following (X, d) will stands for a cone metric space with respect to a cone P with $P^0 \neq \emptyset$ in a real Banach space E and $\leq$ is partial ordering in E with respect to P.

Definition 2.6. ([8], **Measurable function**) Let (Ω, Σ) be a measurable space with Σ-a sigma algebra of subsets of Ω and M be a nonempty subset of a metric space $X = (X, d)$. Let 2^M be the family of nonempty subsets of M and $C(M)$ the family of all nonempty closed subsets of M. A mapping $G\colon \Omega \to 2^M$ is called measurable if for each open subset U of M, $G^{-1}(U) \in \Sigma$, where $G^{-1}(U) = \{\omega \in \Omega : G(\omega) \cap U \neq \emptyset\}$.

Definition 2.7. ([8], **Measurable selector**) A mapping $\xi\colon \Omega \to M$ is called a measurable selector of a measurable mapping $G\colon \Omega \to 2^M$ if ξ is measurable and $\xi(\omega) \in G(\omega)$ for each $\omega \in \Omega$.

Definition 2.8. ([8], **Random operator**) The mapping $T\colon \Omega \times M \to X$ is said to be a random operator if and only if for each fixed $x \in M$, the mapping $T(., x)\colon \Omega \to X$ is measurable.

Definition 2.9. ([8], **Continuous random operator**) A random operator $T\colon \Omega \times M \to X$ is said to be continuous random operator if for each fixed $x \in M$ and $\omega \in \Omega$, the mapping $T(\omega, .)\colon X \to X$ is continuous.

Definition 2.10. ([8], **Random fixed point**) A measurable mapping $\xi\colon \Omega \to M$ is a random fixed point of a random operator $T\colon \Omega \times M \to X$ if and only if $T(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$.

Definition 2.11. ([8], **Cone Random Metric Space**) Let M be a nonempty set and the mapping $d\colon \Omega \times M \to X$, $P \subset X$ be a cone, $\omega \in \Omega$ be a selector, satisfy the following conditions:

(**2.11.1**) $d(x(\omega), y(\omega)) > 0$ for all $x(\omega),\ y(\omega) \in \Omega \times X$ if and only if $x(\omega) = y(\omega)$,

(**2.11.2**) $d(x(\omega), y(\omega)) = d(y(\omega), x(\omega))$ for all $x, y \in X$, $\omega \in \Omega$ and $x(\omega),\ y(\omega) \in \Omega \times X$,

(**2.11.3**) $d(x(\omega), y(\omega)) = d(x(\omega), z(\omega)) + d(z(\omega), y(\omega))$ for all $x, y \in X$ and $\omega \in \Omega$ be a selector,

(**2.11.4**) for any $x, y \in X$, $\omega \in \Omega$, $x(\omega)$, $y(\omega)$ is non-increasing and left continuous in α.

Then d is called cone random metric on M and (M, d) is called a cone random metric space.

Definition 2.12. (**Implicit Relation**) Let Φ be the class of real valued continuous functions $\varphi\colon (R^+)^4 \to R^+$ non-decreasing in the first argument and satisfying the following conditions:

$$x \le \varphi\Big(y,\, x,\, y,\, \frac{y+x}{2}\Big) \quad \text{or} \quad x \le \varphi\Big(y,\, x,\, y,\, \frac{x}{2}\Big) \quad \text{or} \quad x \le \varphi(x,\, y,\, y,\, x),$$

there exists a real number $0 < h < 1$ such that $x \le h\, y$.

3. Main Results

In this section we shall prove a common fixed point theorem under weak contractive condition in the setting of cone random metric spaces.

Theorem 3.1. *Let (X, d) be a complete cone random metric space with respect to a cone P and let M be a nonempty separable closed subset of X. Let S and T be two continuous random operators defined on M such that for $\omega \in \Omega$, $S(\omega, .)$, $T(\omega, .)\colon \Omega \times M \to M$ satisfying the condition:*

$$\begin{aligned} d(S(x(\omega)), T(y(\omega))) \;\le\; & \varphi\Big(d(x(\omega), y(\omega)),\, d(x(\omega), S(x(\omega)),\, d(y(\omega), T(y(\omega)),\\ & \frac{d(x(\omega), T(y(\omega)) + d(y(\omega), S(x(\omega))}{2}\Big) \end{aligned} \tag{3.1}$$

for all $x, y \in X$ and $\omega \in \Omega$. Then S and T have a unique common random fixed point in X.

Proof. For each $x_0(\omega) \in \Omega \times X$ and $n = 0, 1, 2, \ldots$, we choose $x_1(\omega), x_2(\omega) \in \Omega \times X$ such that $x_1(\omega) = S(x_0(\omega))$ and $x_2(\omega) = S(x_1(\omega))$. In general we define sequence of elements of X such that $x_{2n+1}(\omega) = S(x_{2n}(\omega)) = S^{2n+1}(x_0(\omega))$ and $x_{2n+2}(\omega) = T(x_{2n+1}(\omega)) = T^{2n+2}(x_0(\omega))$. Then

$$\begin{aligned}
& d(x_{2n+1}(\omega), x_{2n}(\omega)) \\
& = d(S(x_{2n}(\omega)), T(x_{2n-1}(\omega)) \\
& \leq \varphi\Big(d(x_{2n}(\omega), x_{2n-1}(\omega)), d(x_{2n}(\omega), S(x_{2n}(\omega)), d(x_{2n-1}(\omega), T(x_{2n-1}(\omega)), \\
& \qquad \frac{d(x_{2n}(\omega), T(x_{2n-1}(\omega)) + d(x_{2n-1}(\omega), S(x_{2n}(\omega))}{2}\Big) \\
& \leq \varphi\Big(d(x_{2n}(\omega), x_{2n-1}(\omega)), d(x_{2n}(\omega), x_{2n+1}(\omega)), d(x_{2n-1}(\omega), x_{2n}(\omega)), \\
& \qquad \frac{d(x_{2n}(\omega), x_{2n}(\omega)) + d(x_{2n-1}(\omega), x_{2n+1}(\omega))}{2}\Big) \\
& \leq \varphi\Big(d(x_{2n}(\omega), x_{2n-1}(\omega)), d(x_{2n}(\omega), x_{2n+1}(\omega)), d(x_{2n-1}(\omega), x_{2n}(\omega)), \\
& \qquad \frac{d(x_{2n-1}(\omega), x_{2n}(\omega)) + d(x_{2n}(\omega), x_{2n+1}(\omega))}{2}\Big).
\end{aligned}$$

Hence from definition 2.12, we have

$$d(x_{2n+1}(\omega), x_{2n}(\omega)) \leq h\, d(x_{2n}(\omega), x_{2n-1}(\omega)).$$

Similarly, we obtain

$$d(x_{2n}(\omega), x_{2n-1}(\omega)) \leq h\, d(x_{2n-1}(\omega), x_{2n-2}(\omega)).$$

Hence

$$d(x_{2n+1}(\omega), x_{2n}(\omega)) \leq h^2\, d(x_{2n-1}(\omega), x_{2n-2}(\omega)).$$

On continuing in this process, we get

$$d(x_{2n+1}(\omega), x_{2n}(\omega)) \leq h^{2n}\, d(x_1(\omega), x_0(\omega)).$$

Also for $n > m$, we have

$$\begin{aligned}
d(x_n(\omega), x_m(\omega)) & \leq d(x_n(\omega), x_{n-1}(\omega)) + d(x_{n-1}(\omega), x_{n-2}(\omega)) + \cdots \\
& \quad + d(x_{m+1}(\omega), x_m(\omega)) \\
& \leq (h^{n-1} + h^{n-2} + \cdots + h^m)\, d(x_1(\omega), x_0(\omega)) \\
& \leq \Big(\frac{h^m}{1-h}\Big)\, d(x_1(\omega), x_0(\omega)).
\end{aligned}$$

Let $0 \ll c$ be given. Choose a natural number N such that $\left(\frac{h^m}{1-h}\right) d(x_1(\omega), x_0(\omega)) \ll c$ for every $m \geq N$. Thus

$$d(x_n(\omega), x_m(\omega)) \leq \left(\frac{h^m}{1-h}\right) d(x_1(\omega), x_0(\omega)) \ll c,$$

for every $n > m \geq N$. This shows that the sequence $\{x_n(\omega)\}$ is a Cauchy sequence in $\Omega \times X$. Since (X, d) is complete, there exists $z(\omega) \in \Omega \times X$ such that $x_n(\omega) \to z(\omega)$ as $n \to \infty$. Hence, we have

$$\begin{aligned}
&d(z(\omega), S(z(\omega))) \\
&\leq d(z(\omega), x_{2n+2}(\omega)) + d(x_{2n+2}(\omega), S(z(\omega)) \\
&= d(z(\omega), x_{2n+2}(\omega)) + d(S(z(\omega), T(x_{2n+1}(\omega)) \\
&\leq d(z(\omega), x_{2n+2}(\omega)) \\
&\quad + \varphi\Big(d(z(\omega), x_{2n+1}(\omega)), d(z(\omega), S(z(\omega))), d(x_{2n+1}(\omega), T(x_{2n+1}(\omega))), \\
&\qquad \frac{d(z(\omega), T(x_{2n+1}(\omega))) + d(x_{2n+1}(\omega), S(z(\omega)))}{2}\Big) \\
&\leq d(z(\omega), x_{2n+2}(\omega)) \\
&\quad + \varphi\Big(d(z(\omega), x_{2n+1}(\omega)), d(z(\omega), S(z(\omega))), d(x_{2n+1}(\omega), x_{2n+2}(\omega)), \\
&\qquad \frac{d(z(\omega), x_{2n+2}(\omega)) + d(x_{2n+1}(\omega), S(z(\omega)))}{2}\Big).
\end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$\begin{aligned}
d(z(\omega), S(z(\omega))) &\leq 0 + \varphi\Big(0, d(z(\omega), S(z(\omega))), 0, \frac{d(z(\omega), S(z(\omega)))}{2}\Big) \\
&\leq 0.
\end{aligned}$$

Thus $-\big(d(z(\omega), S(z(\omega)))\big) \in P$. But $d(z(\omega), S(z(\omega))) \in P$. Therefore by definition 2.1($\mathbf{c_3}$) $d(z(\omega), S(z(\omega)) = 0$ and so $S(z(\omega)) = z(\omega)$.

In an exactly similar way we can prove that for all $\omega \in \Omega$, $T(z(\omega)) = z(\omega)$. Hence $S(z(\omega)) = T(z(\omega)) = z(\omega)$. This shows that $z(\omega)$ is a common random fixed point of S and T.

Uniqueness: Let $v(\omega)$ be another random fixed point common to S and T, that is, for $\omega \in \Omega$, $S(v(\omega)) = T(v(\omega)) = v(\omega)$. Then for $\omega \in \Omega$, we have

$$\begin{aligned}d(z(\omega),v(\omega)) &= d(S(z(\omega)),T(v(\omega)))\\ &\leq \varphi\Big(d(z(\omega),v(\omega)),d(z(\omega),S(z(\omega))),d(v(\omega),T(v(\omega))),\\ &\qquad \frac{d(z(\omega),T(v(\omega))+d(v(\omega),S(z(\omega)))}{2}\Big)\\ &\leq \varphi\Big(d(z(\omega),v(\omega)),d(z(\omega),z(\omega)),d(v(\omega),v(\omega)),\\ &\qquad \frac{d(z(\omega),v(\omega))+d(v(\omega),z(\omega))}{2}\Big)\\ &\leq \varphi\Big(d(z(\omega),v(\omega)),0,0,d(z(\omega),v(\omega))\Big)\end{aligned}$$

which gives

$$d(z(\omega),v(\omega)) \leq 0.$$

Hence $d(z(\omega),v(\omega)) = 0$, it follows that $z(\omega) = v(\omega)$ and so $z(\omega)$ is a unique common random fixed point of S and T. This completes the proof. □

Remark 3.2. Our result extends the corresponding result of Dhagat et al. [8] (Advances in Fixed Point Theory, **2(3)** (2012), 357–363).

Example 3.3. Let $M = R$ and $P = \{x \in M : x \geq 0\}$, also $\Omega = [0,1]$ and Σ be the sigma algebra of Lebesgue's measurable subset of $[0,1]$. Let $X = [0,\infty)$ and define a mapping $d\colon (\Omega\times X)\times(\Omega\times X) \to M$ by $d(x(\omega),y(\omega)) = |x(\omega) - y(\omega)|$. Then (X,d) is a cone random metric space. Define random operator T form $(\Omega\times X)$ to X as $T(\omega,x) = \frac{1-\omega^2+x}{2}$. Also sequence of mapping $\xi_n\colon \Omega \to X$ is defined by $\xi_n(\omega) = (1-\omega^2)^{1+(1/n)}$ for every $\omega \in \Omega$ and $n \in N$. Define measurable mapping $\xi\colon \Omega \to X$ as $\xi(\omega) = (1-\omega^2)$ for every $\omega \in \Omega$. Hence $(1-\omega^2)$ is the random fixed point of the random operator T.

Example 3.4. Let $M = R$ and $P = \{x \in M : x \geq 0\}$, also $\Omega = [0,1]$ and Σ be the sigma algebra of Lebesgue's measurable subset of $[0,1]$. Let $X = [0,\infty)$ and define a mapping $d\colon (\Omega\times X)\times(\Omega\times X) \to M$ by $d(x(\omega),y(\omega)) = |x(\omega) - y(\omega)|$. Then (X,d) is a cone random metric space. Define random operators S and T form $(\Omega\times X)$ to X as $S(\omega,x) = \frac{1-\omega^2+2x}{3}$ and $T(\omega,x) = \frac{1-\omega^2+3x}{4}$. Also sequence of mapping $\xi_n\colon \Omega \to X$ is defined by $\xi_n(\omega) = (1-\omega^2)^{1+(1/n)}$ for every $\omega \in \Omega$ and $n \in N$. Define measurable mapping $\xi\colon \Omega \to X$ as $\xi(\omega) = (1-\omega^2)$ for every $\omega \in \Omega$. Hence $(1-\omega^2)$ is a common random fixed point of the random operators S and T.

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References

[1] M. Abbas and G. Jungck, *Common fixed point results for non commuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl., **341** (2008), 416–420.

[2] I. Beg, *Random fixed points of random operators satisfying semicontractivity conditions*, Mathematics Japonica, **46(1)** (1997), 151–155.

[3] I. Beg, *Approximation of random fixed points in normed spaces*, Nonlinear Analysis, **51(8)** (2002), 1363–1372.

[4] I. Beg and M. Abbas, *Equivalence and stability of random fixed point iterative procedures*, Journal of Applied Mathematics and Stochastic Analysis, 2006 (2006), Article ID 23297, 19 pages.

[5] I. Beg and M. Abbas, *Iterative procedures for solutions of random operator equations in Banach spaces*, Journal of Mathematical Analysis and Applications, **315(1)** (2006), 181–201.

[6] A.T. Bharucha-Reid, *Random Integral equations*, Mathematics in Science and Engineering, vol. 96, Academic Press, New York(1972).

[7] A.T. Bharucha-Reid, *Fixed point theorems in Probabilistic analysis*, Bulletin of the American Mathematical Society, **82(5)** (1976), 641–657.

[8] V.B. Dhagat, R. Shrivastav and V. Patel, *Fixed point theorems in cone random metric spaces*, Advances in Fixed Point Theory, **2(3)** (2012), 357–363.

[9] O. Hanš, *Reduzierende zufallige transformationen*, Czechoslovak Mathematical Journal, **7(82)** (1957), 154–158.

[10] O. Hanš, *Random operator equations*, Proceeding of the 4th Berkeley Symposium on Mathematical Statistics and Probability, Vol. II, University of California Press, California, 1961, 185–202.

[11] C.J. Himmelberg, *Measurable relations*, Fundamenta Mathematicae, **87** (1975), 53–72.

[12] L.-G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332(2)** (2007), 1468–1476.

[13] D. Ilic and V. Rakocevic, *Common fixed points for maps on cone metric space*, J. Math. Anal. Appl., **341** (2008), 876–882.

[14] S. Itoh, *Random fixed point theorems with an application to random differential equations in Banach spaces*, Journal of Mathematical Analysis and Applications, **67(2)** (1979), 261–273.

[15] N.S. Papageorgiou, *Random fixed point theorems for measurable multifunctions in Banach spaces*, Proceedings of the American Mathematical Society, **97(3)** (1986), 507–514.

[16] N.S. Papageorgiou, *On measurable multifunctions with stochastic domain*, Journal of Australian Mathematical Society. Series A, **45(2)** (1988), 204–216.

[17] R. Penaloza, *A characterization of renegotiation proof contracts via random fixed points in Banach spaces*, working paper 269, Department of Economics, University of Brasilia, Brasilia, December (2002).

[18] Sh. Rezapour, *A review on topological properties of cone metric spaces*, in Proceedings of the International Conference on Analysis, Topology and Appl. (ATA 08), Vrinjacka Banja, Serbia, May-June (2008).

[19] Sh. Rezapour and R. Hamlbarani, *Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings"*, J. Math. Anal. Appl., **345(2)** (2008), 719–724.

[20] V.M. Sehgal and S.P. Singh, *On random approximations and a random fixed point theorem for set valued mappings*, Proceedings of the American Mathematical Society, **95(1)** (1985), 91–94.

[21] P. Vetro, *Common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo, (2) **56(3)** (2007), 464–468.
[22] D.H. Wagner, *Survey of measurable selection theorem*, SIAM Journal on Control and Optimization, **15(5)** (1977), 859–903.
[23] P.P. Zabrejko, *K-metric and K-normed linear spaces: survey*, Collectanea Mathematica, **48(4-6)** (1997), 825–859.