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COMMON FIXED POINT THEOREMS FOR MULTI-VALUED MAPPINGS IN CONE METRIC SPACES

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Abstract. It is proved that the distance between two disjoint closed sets are not equal to zero where at least one set is bounded in cone metric space. Some common fixed point theorems for multi-valued mappings are obtained in cone metric space. It is proved that multi-valued mappings have common fixed point which generalized the C. T. Aage and J. N. Salunke's result in cone metric space.

1. Introduction

In 2007, Huang and Zhang [1] generalized the concept of metric space and introduced the concept of cone metric space. After that some fixed point theorems have been given in [2-4]. Many authors also investigated the fixed points for multi-valued mappings in metric spaces. In [8] Wardowski proved some results about multi-valued mappings in cone metric space. Furthermore, Rezapour and Hamlbarani respectively in [5] and [6] used the Hausdroff cone metric and gave some common fixed point theorems.

Definition 1.1. ([1]) Let E be a real Banach space and P be a subset of E. Then P is called a cone, if it satisfies following conditions:
(1) P is closed, non-empty and $P \neq \{0\}$;
(2) $ax + by \in P$, for all $x, y \in P$ and a, b are non-negative real numbers;
(3) $P \cap (-P) = \{0\}$.

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Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We say $x \ll y$ if and only if $y - x \in intP$.

Definition 1.2. ([1]) Let X be a non-empty set. Suppose that the mapping $d : X \times X \to E$ satisfies:
(1) $0 \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a cone on X and (X, d) is called a cone metric space.

Definition 1.3. ([5]) (1) Let X be a cone metric space and $B \subset X$. A point b in B is called an interior point of B, whenever there exists a point c, $0 \ll c$, such that $N(b, c) \subset B$, where

$$N(b, c) = \{y \in X : d(y, b) \ll c\}.$$

(2) A subset $A \subset X$ is called open, if each element of A is an interior point.

The family $\beta = \{N(x, c) : x \in X, 0 \ll c\}$ is a sub-basis for a topology on X. We denotes this cone topology by τ_c which is $Hausdroff$ and $first\ countable$.

Lemma 1.4. ([5]) *Let (x, d) be a complete cone metric space, P be a normal cone constant $M = 1$ and A be a compact set in (x, τ_c). Then for every $x \in X$, there exists $a_0 \in A$ such that*

$$\|d(x, a_0)\| = \inf_{a \in A} \|d(x, a)\|.$$

Lemma 1.5. ([5]) *Let (X, d) be a cone metric space, P be a normal cone with normal constant $M = 1$ and A, B be two compact sets in (x, τ_c). Then*

$$\sup_{x \in B} d^{'}(x, A) < \infty,$$

where $d^{'}(x, A) = \inf_{a \in A} \|d(x, a)\|$, for each x in X.

If A is a single point set, then $A = \{y\}$, so $d^{'}(x, y) = \|d(x, y)\|$.

Definition 1.6. ([6]) Let X be a cone metric space, P be a normal cone with normal constant $M = 1$. H_c denotes all compact sets of (x, τ_c), $A \in H_c(X)$. Define

$$h_A : H_c(X) \to [0, \infty), d_H : H_c(X) \times H_c(X) \to [0, \infty),$$

$$h_A(B) = \sup_{x \in A} d^{'}(x, B), d_H(A, B) = \max\{h_A(B), h_B(A)\}.$$

Lemma 1.7. ([11]) *Let (X,d) be a cone metric space, P be a normal cone with normal constant one and $T: X \to H_c(X)$. Then*

$$\|d(x,T_x)\| = \|\inf_{y\in Tx} d(x,y)\| = \inf_{y\in Tx} \|d(x,y)\|.$$

In this paper, we study the problem between two closed sets and the common fixed points of multi-valued mappings.

2. Main results

Definition 2.1. (X,d) is a cone metric space, $A \subset X$. If for any $x, y \in A$, there exists $M > 0$ such that $\|d(x,y)\| \leq M$, then we say A is norm bounded.

Lemma 2.2. *If X is a cone metric space, $B \in H_c(X)$, $\inf_{b\in B} \|d(x,b)\| = 0$, then $x \in B$.*

Proof. By $\inf_{b\in B} \|d(x,b)\| = 0$, there exists $\{b_n\} \subset B$ such that $\|d(x,b_n)\| < \frac{1}{n}$. So $\lim\limits_{n\to\infty} d(x,b_n) = 0$. Thus $b_n \to x$ as $n \to \infty$. Since B is closed, $x \in B$. □

Theorem 2.3. *If $A, B \in H_c(X)$, $d_H(A,B) = 0$, we have $A = B$.*

Proof.

$$\begin{aligned} d_H(A,B) &= \max\{h_B(A), h_A(B)\} \\ &= \max\{\sup_{x\in A}\inf_{b\in B} \|d(x,b)\|, \sup_{x\in B}\inf_{a\in A} \|d(x,a)\|\} \\ &= 0. \end{aligned}$$

So

$$\sup_{x\in A}\inf_{b\in B} \|d(x,b)\| = 0; \sup_{x\in B}\inf_{a\in A} \|d(x,a)\| = 0.$$

Since

$$\|d(x,b)\| \geq 0 (\forall x \in A, b \in B); \|d(x,a)\| \geq 0 (\forall x \in B, a \in A),$$

it follows that

$$\inf_{b\in B} \|d(x,b)\| = 0; \inf_{a\in A} \|d(x,a)\| = 0.$$

By lemma 2.2, for any $x \in A$, we have $x \in B$. So $A \subset B$. On the other hand, $B \subset A$. This implies that $A = B$. □

Theorem 2.4. *If X is a cone metric space, $A, B \in H_c(X)$, then there exists $b_0 \in B$ such that*

$$\|d(a_0,b_0)\| \leq d_H(A,B) + \epsilon,$$

for each $a_0 \in A$ and $\epsilon > 0$.

Proof. For any $a_0 \in A$, we have $d_H(A,B) \geq \inf_{b\in B} \|d(a_0,b)\|$. So $\forall \epsilon > 0$, we have $d_H(A,B) + \epsilon \geq \|d(a_0,b_0)\|$. □

Theorem 2.5. *Let F_1 and F_2 are two disjoint closed sets in cone metric space and P is a normal cone with normal constant one. Assume that there are at least one sets norm bounded, then $d(F_1,F_2) \neq \theta$, where*

$$d(F_1,F_2) = \inf\{d(a,b), a \in F_1, b \in F_2\}.$$

Proof. Without loss of generality, we set F_1 is a norm bounded set. For any $x \in F_1$, $y \in F_2$, we have $\|d(x,y)\| > 0$.

At first, we will show that $\inf\{\|d(a,b)\|, a \in F_1, b \in F_2\} > 0$. Suppose on the contrary $\inf\{\|d(a,b)\|, a \in F_1, b \in F_2\} = 0$. Then there exist $\{x_n\} \subset F_1$, $\{y_n\} \subset F_2$, such that

$$\|d(x_n,y_n)\| \leq \frac{1}{n}.$$

Let $E = \{x_n\} \bigcup \{y_n\}$. Claim that E is a infinite bounded set. In fact, since F_1 is norm bounded, then there exists $M_1 > 0$, such that for any $x_n, x_m \in E$, we have $\|d(x_n,x_m)\| \leq M_1$. Consequently,

$$\|d(x_n,y_m)\| \leq \|d(x_n,x_m)\| + \|d(x_m,y_m)\| \leq M_1 + \frac{1}{m} \leq M_1 + 1,$$

$$\|d(y_n,y_m))\| \leq \|d(x_n,y_m)\| + \|d(x_n,y_n)\| \leq M_1 + \frac{1}{m} + \frac{1}{n} \leq M_1 + 2.$$

Let $M = M_1 + 2$. Then for any $x, y \in E$, we have $\|d(x,y)\| \leq M$. So E has a accumulation point x_0. let $z_{n_k} \to x_0$.

Case1. $\{z_{n_k}\}$ contains infinite points $\{x_{n_k}\}$ of $\{x_n\}$. Let $\{x_{n_k}\} \subset \{x_n\}$, $x_{n_k} \to x_0$, so

$$\|d(y_{n_k},x_0)\| \leq \|d(y_{n_k},x_{n_k})\| + \|d(x_{n_k},x_0)\| \to 0.$$

We get $\{y_{n_k}\} \subset \{y_n\}$ and $y_{n_k} \to x_0$.

Case2. $\{z_{n_k}\}$ contains infinite points $\{y_{n_k}\}$ of $\{y_n\}$. One can get $\{x_{n_k}\} \subset \{x_n\}$, $x_{n_k} \to x_0$.

Case3. $\{z_{n_k}\}$ contains infinite points $\{x_{n_k}\}$ of $\{x_n\}$ and infinite points $\{y_{n_k}\}$ of $\{y_n\}$.

From the cases we have $\{x_{n_k}\} \subset \{x_n\}$ and $x_{n_k} \to x_0$ as $n \to \infty$. $\{y_{n_k}\} \subset \{y_n\}$ and $y_{n_k} \to x_0$ as $n \to \infty$. So x_0 is a accumulation point of F_1 and F_2, this is a contradiction. So

$$\inf\{\|d(a,b)\|, a \in F_1, b \in F_2\} > 0.$$

According to Lemma 1.7, we have $\|\inf\{d(a,b), a \in F_1, b \in F_2\}\| > 0$. So $inf\{d(a,b),\ a \in F_1, b \in F_2\} \neq \theta$. It follows that $d(F_1,F_2) \neq \theta$. □

Theorem 2.6. *Let X be a complete cone metric space with normal constant $M = 1$, $T_1, T_2 : X \to H_c(X)$ be two multi-valued maps satisfying the condition*

$$\alpha d_H(T_1(x), T_2(y)) + \beta d^{'}(x, T_1(x)) + \gamma d^{'}(y, T_2(y)) \leq \delta d^{'}(x, y)$$

for all $x, y \in X$ and $\alpha, \beta, \gamma > 0$ where $\beta < \delta$, $\gamma < \delta$, $\delta < \alpha + \beta + \gamma$. Then T_1 and T_2 have common fixed point. It means there exists $x \in X$ such that $x \in T_1(x)$ and $x \in T_2(x)$.

Proof. Let $x_0 \in X$, by lemma 1.4, there exists $x_1 \in T_1 x_0$ satisfying

$$d^{'}(x_0, T_1 x_0) = \|d(x_0, x_1)\|.$$

We also have $x_2 \in T_2 x_1$,

$$d^{'}(x_1, T_2 x_1) = \|d(x_1, x_2)\|.$$

By this way, there is a sequence $\{x_n\}_{n\geq 1}$ in X such that

$$x_{2n-1} \in T_1 x_{2n-2}, x_{2n} \in T_2 x_{2n-1}.$$

So

$$d^{'}(x_{2n-2}, T_1 x_{2n-2}) = \|d(x_{2n-2}, x_{2n-1})\|,$$
$$d^{'}(x_{2n-1}, T_2 x_{2n-1}) = \|d(x_{2n-1}, x_{2n})\|.$$

For all $n \geq 1$, therefore

$$\begin{aligned} & \|d(x_{2n+1}, x_{2n})\| \\ = \; & d^{'}(x_{2n}, T_1 x_{2n}) \leq h_{T_2 x_{2n-1}}(T_1 x_{2n}) \leq d_H(T_2 x_{2n-1}, T_1 x_{2n}) \\ \leq \; & \frac{\delta}{\alpha}\|d(x_{2n-1}, x_{2n})\| - \frac{\beta}{\alpha} d^{'}(x_{2n}, T_1 x_{2n}) - \frac{\gamma}{\alpha} d^{'}(x_{2n-1}, T_2 x_{2n-1}) \\ \leq \; & \frac{\delta}{\alpha}\|d(x_{2n-1}, x_{2n})\| - \frac{\beta}{\alpha}\|d(x_{2n+1}, x_{2n})\| - \frac{\gamma}{\alpha}\|d(x_{2n-1}, x_{2n})\|. \end{aligned}$$

It follows that

$$(1 + \frac{\beta}{\alpha})\|d(x_{2n+1}, x_{2n})\| \leq (\frac{\delta}{\alpha} - \frac{\gamma}{\alpha})\|d(x_{2n}, x_{2n-1})\|.$$

Let $h_1 = \frac{\delta-\gamma}{\alpha+\beta}$, so

$$\|d(x_{2n+1}, x_{2n})\| \leq h_1 \|d(x_{2n}, x_{2n-1})\|.$$

Similarly,

$$\|d(x_{2n+2}, x_{2n+1})\| \leq \frac{\delta - \beta}{\alpha + \gamma}\|d(x_{2n+1}, x_{2n})\|.$$

Let $h_2 = \frac{\delta-\beta}{\alpha+\gamma}$, consequently, we have

$$\|d(x_{2n+2}, x_{2n+1})\| \leq h_2 \|d(x_{2n+1}, x_{2n})\|.$$

So

$$\|d(x_{2n+2}, x_{2n+1})\| \leq (h_2 h_1)^n \|d(x_2, x_1)\|;$$

$$\|d(x_{2n+1}, x_{2n})\| \leq (h_2h_1)^n\|d(x_1, x_0)\|.$$

For $m > n$, then

$$\begin{aligned}
&\|d(x_{2n}, x_{2m})\| \\
\leq\ & \|d(x_{2n+1}, x_{2n})\| + \|d(x_{2n+2}, x_{2n+1})\| + \cdots + \|d(x_{2m}, x_{2m-1})\| \\
\leq\ & (h_2h_1)^n\|d(x_1, x_0)\| + (h_2h_1)^n\|d(x_2, x_1)\| + (h_2h_1)^{n+1}\|d(x_1, x_0)\| \\
& + \cdots + (h_2h_1)^{m-1}\|d(x_1, x_0)\| + (h_2h_1)^{m-1}\|d(x_2, x_1)\| \\
\leq\ & \frac{(h_2h_1)^n}{1 - h_2h_1}(\|d(x_2, x_1)\| + \|d(x_1, x_0)\|).
\end{aligned}$$

$\|d(x_{2m}, x_{2n})\| \to 0$ as $n \to \infty$. By the same way, we get $\|d(x_{2n+1}, x_{2m+1})\| \to 0$ and $\|d(x_{2n}, x_{2m+1})\| \to 0$ as $n \to \infty$. It is easy to verify that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in X. There exists $x^* \in X$ such that $x_n \to x^*$.

$$\begin{aligned}
d^{'}(x^*, T_1x^*) &\leq d^{'}(x,^*, T_2x_{2n-1}) + d_H(T_2x_{2n-1}, T_1x^*) \\
&\leq \|d^{'}(x^*, x_{2n})\| + \frac{\beta}{\alpha}\|d(x^*, x_{2n-1})\| \\
&\quad -\frac{\gamma}{\alpha}\|d(x_{2n-1}, x_{2n})\| - \frac{\beta}{\alpha}\|d^{'}(x^*, T_1x^*).
\end{aligned}$$

So

$$\begin{aligned}
d^{'}(x^*, T_1x^*) \leq\ & \frac{\alpha}{\alpha+\beta}\|d(x^*, x_{2n})\| \\
& + \frac{\delta}{\alpha+\beta}\|d(x^*, x_{2n-1})\| - \frac{\gamma}{\alpha+\beta}\|d(x_{2n-1}, x_{2n})\|.
\end{aligned}$$

Let $n \to \infty$, clearly, $d^{'}(x^*, T_1x^*) = 0$. It implies $x^* \in T_1x^*$. By the same way, one can get $x^* \in T_2x^*$. Therefore x^* is a common fixed point of T_1 and T_2.

In[10], T_1 and T_2 are single-valued mappings, so in this paper we extended Theorem 2.1 to multi-valued mappings. □

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