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STUDY OF IMPULSIVE PROBLEM WITH CAPUTO FRACTIONAL DERIVATIVE INVOLVING NONLOCAL CONDITIONS USING FIXED POINT THEORY

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Abstract. In this article, we study the existence of solutions for an impulsive Caupto fractional differential equations with a class of initial value problem dependence on the Lipschitz first derivative conditions. Our main tool is a Banach's fixed point theorem and Leray-Schauder fixed point theorem. We also investigate the existence of fractional Derivative with non-local conditions. An numerical example is given to clarify the results.

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1. Introduction

During the last decade, fractional differential equations (FDE) have a major role in many fields of study accomplished by mathematicians, physician, physicists, and engineers. They have used to evolve mathematical modeling, many physical applications, and engineering disciplines such as higher education information power method [4], hand foot disease identified method [6], neural network design [30], medical science [19, 32, 33], Dynamics analysis of Romeo and Juliet love affairs [2], Maxwell nanofluid [15], hybrid nanofluid [24], chemical kinetics [13], sensitivity analysis of pine wilt disease [14], chaotic system [3]. Examine the stability findings for both infected and non-infected equilibrium points in relation to HIV (human immunodeficiency virus) infection and tumor growth model with fractional order [27].

Differential equations of fractional order are a better mathematical tool for describing certain real systems. Engineering applications in fractional order [23] numerous academics have recently used a variety of fixed point theorems to prove some intriguing results regarding the existence of solutions for FDEs. Next, new primitive differential conditions with various promotions, such as Riemann-Liouville, Caputo, Hadamard, Hilfer-Hadamard, and Grunwald-Letnikov, will be implemented (see [5, 10, 11, 22]). The nonlinear problem of FDEs with indispensable limit conditions, which uses Banach's constriction standard and Leray Schaefer's option fixed point hypothesis as support, is one of the notable models. For impulsive fractional equations with nonlocal circumstances, Zhang et al. provided the existence and uniqueness of mild solutions in [31]. In [1], Guida et al. established that a class of impulsive Hilfer fractional coupled systems have mild solutions. In a non-compact semigroup, Hilal et al. [8] investigated whether impulsive fractional integro-differential equations exist. In [12, 25], Pandiyammal et al. studied existence of fractional order problem using Atangana Baleanu derivative with dependence on the Lipschitz first derivative.

Motivated by the above mentioned works and Wahash et al [29], this article examines the existence of solutions for the following impulsive Caputo FDEs with a class of initial value problems in Banach space.

$$\begin{cases} [\hat{D}_C]^{\mu}\bar{G}(a) = \chi(a, \bar{G}(a), \bar{G}'(a, \bar{G}(a))), \quad a \in I = [0, T],\ a \neq a_k, \\ \Delta \bar{G}\big|_{a=a_k} = I_k(\bar{G}(a_k^-)), \\ \bar{G}(0) = \bar{G}_0, \end{cases} \tag{1.1}$$

where, $[\hat{D}_C]^{\mu}\bar{G}(a)$ is the Caputo derivative, $\chi : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $I_k : \mathbb{R} \to \mathbb{R}$, $0 < \mu \leq 1$, $k \in \mathbb{N}_m$, $\mathbb{N}_m = \{1, 2, \ldots, m\}$, $\bar{G}_0 \in \mathbb{R}$, $0 = a_0 = a_1 < \cdots < a_m < a_{m+1} = T$, $\Delta \bar{G}\big|_{a=a_k} = \bar{G}(a_k^+) - \bar{G}(a_k^-)$, $\bar{G}(a_k^+) = \lim_{h \to 0^+} \bar{G}(a_k + h)$ and

$\bar{G}(a_k^-) = \lim_{h\to 0^-} \bar{G}(a_k + h)$, $\mathbb{R}), \chi(a, \bar{G}(a), \bar{G}^{'}(a, \bar{G}(a))) = 0$.
Consider $\bar{G}^{'}(a, \bar{G}(a)) = \hat{d}\bar{G}(a)$. Then (1.1) becomes

$$\begin{cases} [\hat{D}_C]^{\mu}\bar{G}(a) = \chi(a, \bar{G}(a), \hat{d}\bar{G}(a)) \quad a \in I = [0, T],\ a \neq a_k, \\ \Delta\bar{G}\big|_{a=a_k} = I_k(\bar{G}(a_k^-)), \\ \bar{G}(0) = \bar{G}_0. \end{cases} \tag{1.2}$$

Initially, this paper aims to investigate the possibility that the equation (1.2) has solutions by using various fixed point theorems. In addition, it discusses the issue with nonlocal situations.

This paper is organized as five parts. Section 2 reviews a few foundational concepts and lemmas from fractional calculus that are necessary for the following section. The existence of solutions for initial value problem (IVP) for a fractional derivative (FD) results relying on various fixed point theorems for the problem (1.2) are proved in Section 3. The presence of nonlocal impulsive FDEs in section 4. We conclude by providing an example.

2. Facts

This section familiarises with some notations, definitions, fundamental lemmas, and theorems here. These are employed in the following section of this paper. $C(I, \mathbb{R})$ means set of all continuous functions from I into $\mathbb{R}$ in the Banach space with the norm

$$\|\bar{G}\|_{\infty} = \sup_{a\in I} \left\{|\bar{G}(a)|\right\}. \tag{2.1}$$

Definition 2.1. ([18, 26]) The integral of the function $\chi \in L^1(L, \mathbb{R}_+)$ with fractional order $\mu \in \mathbb{R}_+$ is defined by

$$I_{\alpha}^{\mu}\chi(t) = \frac{1}{\Phi(\mu)} \int_{\alpha}^{a} (a-b)^{\mu-1}\chi(b)db, \tag{2.2}$$

where, Φ represents Gamma function, when $\alpha = 0$, we get $I^{\mu}\chi(a) = [\chi * \varphi_{\mu}](a)$, where $\varphi_{\mu}(a) = \frac{a^{\mu-1}}{\Phi(\mu)}$ for $a > 0$, and $\varphi_{\mu}(a) = 0$ for $a \leq 0$, and φ_{μ} tends to $\delta(a)$, the value taken by the delta function at a, as $\mu \to 0$.

Definition 2.2. ([18, 26]) For a given function $\chi \in [\alpha, \beta] = L$, the μth R-L fractional order derivative of χ, is given as

$$(D_{\alpha+}^{\mu}\chi)(a) = \frac{1}{\Phi(n-\mu)} \left(\frac{d}{da}\right)^{n} \int_{\alpha}^{a} (a-b)^{n-\mu-1}\chi(b)db, \tag{2.3}$$

where, $n = [\mu] + 1$ and $[\mu]$ is the integer part of μ.

Definition 2.3. ([17]) For a function $\chi \in L$, the Caputo FD of order μ of χ, is defined as

$$([\hat{D}_C]^{\mu}_{\alpha+}\chi)(a) = \frac{1}{\Phi(n-\mu)}\int_{\alpha}^{a}(a-b)^{n-\mu-1}\chi^n(b)db, \tag{2.4}$$

where $n = [\mu] + 1$.

Proposition 2.4. ([20, 21]) *$\chi'(\bar{G}) \in D$ satisfy the Lipschitz condition. That is, there exists a constant η such that*

$$\|\chi'(\bar{G}) - \chi'(\bar{H})\| \leq \eta\ (\|\bar{G} - \bar{H}\|), \quad \forall\ \bar{G},\ \bar{H} \in D. \tag{2.5}$$

3. Main Results

This section aims to solve problems referred in (1.2). It first brings here the space.

$$\begin{aligned} C_P(I,\mathbb{R}) \quad = \quad & \big\{\bar{G} : I \to \mathbb{R} : \bar{G} \in C((a_k, a_{k+1}], \mathbb{R}), k \in \mathbb{W}_m, \text{ and there exist} \\ & \bar{G}(a_k^-) \text{ and } \bar{G}(a_k^+) \text{ with } \bar{G}(a_k^-) = \bar{G}(a_k)\big\}, \end{aligned} \tag{3.1}$$

where $\mathbb{W}_m = \{0, 1, \ldots, m\}$. The indicated set is a Banach space beside the norm

$$\|\bar{G}\|_{PC} = \sup_{a\in I} |\bar{G}(a)|. \tag{3.2}$$

Set

$$I^{'} = [0, T] \setminus \{a_1, a_2, \ldots, a_m\}.$$

Definition 3.1. $\bar{G} \in C_P(I,\mathbb{R})$ is a solution of (1.2) if its μ-derivative exists on $I^{'}$ and it satisfies following equations:

$$[\hat{D}_C]^{\mu}\bar{G}(a) = \chi(a, \bar{G}(a), \hat{d}\bar{G}(a)) \text{ on } I^{'}, \tag{3.3}$$

$$\begin{aligned} \Delta\bar{G}\big|_{a=a_k} &= I_k(\bar{G}(a_k^-)),\ k \in \mathbb{N}_m, \\ \bar{G}(0) &= \bar{G}_0. \end{aligned} \tag{3.4}$$

The next two lemmas offer evidence to support the conclusion that the equation (1.2) is solvable.

Lemma 3.2. ([16]) *Let $\mu > 0$. Then the differential equation*

$$[\hat{D}_C]^{\mu}\chi(a) = 0 \tag{3.5}$$

has solution $\chi(a) = a_0 + a_1 a + a_2 a^2 + \cdots + a_{n-1}a^{n-1}$, $a_i \in \mathbb{R}, i \in \mathbb{W}_{n-1}, n = [\mu] + 1$.

Lemma 3.3. ([16]) *Let* $\mu > 0$. *Then*

$$I^{\mu}\ {}^{c}D^{\mu}h(a) = h(a) + a_0 + a_1 a + a_2 a^2 + \cdots + a_{n-1}a^{n-1}$$

for some $a_i \in \mathbb{R}, i \in \mathbb{W}_{n-1}, n = [\mu] + 1$.

Lemma 3.4. *Let* $0 < \mu \leq 1$, $k \in \mathbb{N}_m$ *and let* $\chi : I \to \mathbb{R}$. *A function* $\bar{G}$ *is a solution of the fractional integral equation*

$$\bar{G}(a) = \begin{cases} \bar{G}_0 + \frac{1}{\Phi(\mu)} \int_0^a (a-b)^{\mu-1}\chi(b)db, & \text{if } a \in [0, a_1], \\ \bar{G}_0 + \frac{1}{\Phi(\mu)} \sum_{i=1}^{g} \int_{a_{i-1}}^{a_i} (a_i - b)^{\mu-1}\chi(b)db & \\ + \frac{1}{\Phi(\mu)} \int_{a_k}^{a} (a-b)^{\mu-1}\chi(b)db + \sum_{i=1}^{k} I_i(\bar{G}(a_i^-)), & \text{if } a \in (a_k, a_{k+1}] \end{cases} \tag{3.6}$$

if and ony if the fractional IVP solution is $\bar{G}$,

$$\begin{cases} [\hat{D}_C]^{\mu}\bar{G}(a) = \chi(a), \quad a \in I', \\ \Delta\bar{G}|_{a=a_k} = I_k(\bar{G}(a_k^-)), \quad k \in \mathbb{N}_m, \\ \bar{G}(0) = \bar{G}_0. \end{cases} \tag{3.7}$$

Proof. Suppose $\bar{G}$ satisfies (3.7). If $a \in [0, a_1]$, then

$$[\hat{D}_C]^{\mu}\bar{G}(a) = \chi(a).$$

From Lemma 3.3,

$$\bar{G}(a) = \bar{G}_0 + \frac{1}{\Phi(\mu)} \int_0^a (a-b)^{\mu-1}\chi(b)db.$$

For $a \in (a_1, a_2]$,

$$\begin{aligned} \bar{G}(a) &= \bar{G}(a_1^+) + \frac{1}{\Phi(\mu)} \int_{a_1}^{a} (a-b)^{\mu-1}\chi(b)db \\ &= \Delta\bar{G}|_{a=a_1} + \bar{G}(a_1^-) \frac{1}{\Phi(\mu)} \int_{a_1}^{a} (a-b)^{\mu-1}\chi(b)db \\ &= I_1(\bar{G}(a_1^-)) + \bar{G}_0 + \frac{1}{\Phi(\mu)} \int_0^{a_1} (a_1 - b)^{\mu-1}\chi(b)db \\ &\quad + \frac{1}{\Phi(\mu)} \int_{a_1}^{a} (a-b)^{\mu-1}\chi(b)db. \end{aligned}$$

Suppose $a \in (a_2, a_3]$. Then,

$$\begin{aligned}\bar{G}(a) &= \bar{G}(a_2^+) + \frac{1}{\Phi(\mu)} \int_{a_2}^{a} (a-b)^{\mu-1}\chi(b)db \\ &= \Delta\bar{G}|_{t=a_2} + \bar{G}(a_2^-)\frac{1}{\Phi(\mu)} \int_{a_2}^{a} (a-b)^{\mu-1}\chi(b)db \\ &= I_2(\bar{G}(a_2^-)) + I_1(\bar{G}(a_1^-)) + \bar{G}_0 + \frac{1}{\Phi(\mu)} \int_{0}^{a_1} (a_1-b)^{\mu-1}\chi(b)db \\ &\quad + \frac{1}{\Phi(\mu)} \int_{a_1}^{a_2} (a_2-b)^{\mu-1}\chi(b)db + \frac{1}{\Phi(\mu)} \int_{a_2}^{a} (a-b)^{\mu-1}\chi(b)db.\end{aligned}$$

Now $a \in (a_k, a_{k+1}]$. Furthermore, from Lemma 3.3, we arrive at the equation (3.6).

Conversely, suppose $\bar{G}$ satisfies (3.6). Choose $a \in [0, a_1]$. Since the left inverse of I^μ is $[\hat{D}_C]^\mu$ and $\bar{G}_0 = \bar{G}(0)$, it can be derived that

$$[D_C]^\mu \bar{G}(a) = \chi(a), \ \forall \ a \in [0, a_1].$$

Let $a \in [a_k, a_{k+1})$, $k \in \mathbb{N}_m$. Then,

$$[\hat{D}_C]^\mu \bar{G}(a) = \chi(a), \quad \forall \ a \in [a_k, a_{k+1}).$$

Thus we can get

$$\Delta\bar{G}|_{a=a_k} = I_k(\bar{G}(a_k^-)), \ \in \mathbb{N}_m.$$

□

The following assumptions are required to demonstrate the Banach fixed point theorem.

$\mathbf{A_1}$. Suppose $\bar{G} \in C[I, \mathbb{R}]$ and $\chi : C(L) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R})$ is continuous and there exist constants $\mathfrak{M}_1 > 0, \mathfrak{M}_2 > 0$ and $\mathfrak{M} > 0$ such that

$$\|\chi(a, \bar{G}_1, \bar{H}_1) - \chi(a, \bar{G}_2, \bar{H}_2)\| \le \mathfrak{M}_1(\|\bar{G}_1 - \bar{G}_2\| + \|\bar{H}_1 - \bar{H}_2\|) \qquad (3.8)$$

for all $\bar{G}_1, \bar{H}_1, \bar{G}_2, \bar{H}_2 \in Y$, where $Y = C[\mathbb{R}, X]$ is the set of all continuous functions defined from $\mathbb{R}$ to the Banach spaces X. $\mathfrak{M}_2 = \max_{a\in\mathbb{R}} \|\chi(a, 0, 0)\|$ and $\mathfrak{B} = \max\{\mathfrak{M}_1, \mathfrak{M}_2\}$.

$\mathbf{A_2}$. Let $\bar{G}' \in C(L)$ satisfy the Lipschitz condition, that is, there exist constants $\mathfrak{N}_1, \mathfrak{N}_2$ and $\mathfrak{N}$ such that

$$\|\hat{d}(a, \bar{G}) - \hat{d}(a, \bar{H})\| \le \mathfrak{N}_1(\|\bar{G} - \bar{H}\|) \qquad (3.9)$$

for all $\bar{G}, \bar{H}$ in Y, $\mathfrak{N}_2 = \max_{a\in D} \|\hat{d}(a, 0)\|$ and $\mathfrak{N} = \max\{\mathfrak{N}_1, \mathfrak{N}_2\}$.

$\mathbf{A_3}$. There exists a fixed real number $\varpi > 0$ satisfying the condition

$$|I_k(\bar{G}) - I_k(\bar{H})| \le \varpi|\bar{G} - \bar{H}|$$

for all $\bar{G}, \bar{H} \in \mathbb{R}$ and $k \in \mathbb{N}_m$.

Lemma 3.5. *Assume* $\mathbf{A_1}$ *and* $\mathbf{A_2}$. *Then, for all* $a \in \mathbb{R}$ *and* $\bar{G}, \bar{H} \in Y$,

$$\|\hat{d}\bar{G}(a)\| \leq a(\mathfrak{N}_1\|\bar{G}\| + \mathfrak{N}_2), \quad \|\hat{d}\bar{G}(a) - \hat{d}\bar{H}(a)\| \leq \mathfrak{N}a\|\bar{G} - \bar{H}\|.$$

Theorem 3.6. *Suppose the conditions* $\mathbf{A_1}$, $\mathbf{A_2}$ *and* $\mathbf{A_3}$ *are satisfied by* $\bar{G}(a) \in C(L)$ *and* $\chi \in C(L \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. *If* $\chi(a, \bar{G}(a), \hat{d}\bar{G}(a)) = 0$ *and*

$$\Big[\frac{T^\mu \varpi(m+1)}{\Phi(\mu+1)} + m\varpi_1\Big] < 1, \tag{3.10}$$

then there is a unique solution for (1.2).

Proof. Let the operator $\Pi : C_P(I, \mathbb{R}) \to C_P(I, \mathbb{R})$ and we define

$$\begin{aligned}\Pi(\bar{G})(a) = \bar{G}_0 &+ \frac{1}{\Phi(\mu)} \sum_{0<a_k<a} \int_{a_{k-1}}^{a_k} (a_k - b)^{\mu-1} \chi(b, u(b), \hat{d}p(b))db \\ &+ \frac{1}{\Phi(\mu)} \int_{a_k}^{a} (a-b)^{\mu-1} \chi(b, \bar{G}(b), \hat{d}\bar{G}(b))db \\ &+ \sum_{0<a_k<a} I_k(\bar{G}(a_k^-)).\end{aligned} \tag{3.11}$$

It is clear that the fixed points of Π provide solutions to the problem. The Banach contraction principle can be used to show that Π has a fixed point. Therefore, it is to be demonstrated that Π is a contraction. Let $\bar{G}, \bar{H} \in C_P(I, \mathbb{R})$ and $a \in I$. Then, we have

$$\begin{aligned}&|\Pi(\bar{G})(b) - \Pi(\bar{H})(b)| \\ &\leq \frac{1}{\Phi(\mu)} \sum_{0<a_g<a} \int_{a_{k-1}}^{a_k} (a_k - b)^{\mu-1} |\chi(b, \bar{G}(b), \hat{d}\bar{G}(b)) - \chi(b, \bar{H}(b), \hat{d}\bar{H}(b))|db \\ &\quad + \frac{1}{\Phi(\mu)} \int_{a_k}^{a} (a-b)^{\mu-1} |\chi(b, \bar{G}(b), \hat{d}\bar{G}(b)) - \chi(b, \bar{H}(b), \hat{d}\bar{H}(b))|db \\ &\quad + \sum_{0<a_k<a} |I_k(\bar{G}(a_k^-)) - I_k(\bar{G}(a_k^-))| \\ &\leq \frac{1}{\Phi(\mu)} \sum_{0<a_k<a} \int_{a_{k-1}}^{a_k} (a_k - b)^{\mu-1} (\mathfrak{M}\|\bar{G} - \bar{H}\| + \mathfrak{N}a\|\bar{G} - \bar{H}\|)db \\ &\quad + \frac{1}{\Phi(\mu)} \int_{a_k}^{a} (a-b)^{\mu-1} (\mathfrak{M}\|\bar{G} - \bar{H}\| + \mathfrak{N}a\|\bar{G} - \bar{H}\|)db \\ &\quad + \sum_{0<a_k<a} \varpi_1\|\bar{G} - \bar{H}\|\end{aligned}$$

$$\leq \frac{\varpi}{\Phi(\mu)} \sum_{0<a_k<a} \int_{a_{k-1}}^{a_k} (a_k - b)^{\mu-1} \|\bar{G} - \bar{H}\|_\infty db$$
$$+ \frac{\varpi}{\Phi(\mu)} \int_{a_k}^{a} (a-b)^{\mu-1} \|\bar{G} - \bar{H}\|_\infty db + \sum_{0<a_k<a} \varpi_1 \|\bar{G} - \bar{H}\|_\infty$$
$$\leq \frac{m\varpi T^\mu}{\Phi(\mu+1)} \|\bar{G} - \bar{H}\|_\infty + \frac{T^\mu \varpi}{\Phi(\mu+1)} \|\bar{G} - \bar{H}\|_\infty + m\varpi_1 \|\bar{G} - \bar{H}\|_\infty.$$

Therefore,

$$\|\Pi(\bar{G}) - \Pi(\bar{H})\| \leq \Big[\frac{T^\mu \varpi (m+1)}{\Phi(\mu+1)} + m\varpi_1\Big] \|\bar{G} - \bar{H}\|_\infty,$$

Π is a contraction according to the equation (3.10). Therefore, we infer that Π is a solution of (1.2) and it possesses a fixed point from the Banach fixed point theorem. □

Theorem 3.7. *Assume that*

A_4. *Function $\chi \in (I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is continuous.*

A_5. *There exists a constant ℓ such that*

$$|\chi(a, \bar{G}(a), \hat{d}\bar{G}(a))| \leq \ell, \quad \forall\ a \in I,\ \bar{G} \in \mathbb{R}. \tag{3.12}$$

A_6. *The function $I_k : \mathbb{R} \to \mathbb{R}$ are continuous and there exists a constant $\ell^* > 0$ such that*

$$|I_k(\bar{G})| \leq \ell^*, \quad \forall\ \bar{G} \in \mathbb{R},\ k \in \mathbb{N}_m. \tag{3.13}$$

Then the problem (1.2) *has at least one solution on J.*

Proof. We will use Schaefer's fixed point theorem to demonstrate that Π has a fixed point. This proof consists four steps.

Step 1 : Π is continuous. Let $\{\bar{G}_n\}$ be a sequence such that $\bar{G}_n \to \bar{G}$ in $PC(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\ n = 1, 2, 3.....$ Then, for each $a \in I$ with $\lim_{n\to\bar{G}} \|\bar{G}_n - \bar{G}\| = 0$, we get $\lim_{n\to\bar{G}} \bar{G}_n(a) = \bar{G}(a)$, for $a \in I$. Thus by $\mathbf{A_1}$, we have $\lim_{n\to\infty} \chi(a, \bar{G}_n(a), \hat{d}\bar{G}_n(a)) = \chi(a, \bar{G}(a), \hat{d}\bar{G}(a))$ for $a \in \ell$. We get that

$$|\Pi(\bar{G}_n)(a) - \Pi(\bar{G})(a)|$$
$$\leq \frac{1}{\Phi(\mu)} \sum_{0<a_k<a} \int_{a_{k-1}}^{a_k} (a_k - b)^{\mu-1} |\chi(b, \bar{G}_n(b), \hat{d}\bar{G}_n(b)) - \chi(b, \bar{G}(b), \hat{d}\bar{G}(b))| db$$
$$+ \frac{1}{\Phi(\mu)} \int_{a_k}^{a} (a-b)^{\mu-1} |\chi(b, \bar{G}_n(b), \hat{d}\bar{G}_n(b)) - \chi(b, \bar{G}(b), \hat{d}\bar{G}(b))| db$$
$$+ \sum_{0<a_k<a} |I_k(\bar{G}_n(a_k^-)) - I_k(\bar{G}(a_k^-))|.$$

Here, χ and I_k, $k \in \mathbb{N}_m$ are continuous and we have

$$\|\Pi(\bar{G}_n) - \Pi(\bar{G})\|_\infty \to 0 \text{ as } n \to \infty. \tag{3.14}$$

Step 2 : Since Π maps $I \times \mathbb{R} \times \mathbb{R}$ to bounded sets, and since $\mathbf{A_6}$ and $\mathbf{A_7}$ hold, it suffices to demonstrate that for any $\sigma^* > 0$ and there exists a constant $\wedge > 0$ satisfying the condition $\|\Pi(\bar{G})\|_\infty \leq \wedge$, for each $\bar{G} \in B_{\sigma^*} = \{\bar{G} \in C_P(I, \mathbb{R}) : \|\bar{G}\|_\infty \leq \sigma^*\}$ and for each $a \in I$,

$$\begin{aligned}
|\Pi(\bar{G})(a)| &\leq |\bar{G}_0| + \frac{1}{\Phi(\mu)} \sum_{0<a_k<a} \int_{a_k-1}^{a_k} (a_k - b)^{\mu-1} \chi(b, \bar{G}(b), \hat{d}\bar{G}(b))db \\
&\quad + \frac{1}{\Phi(\mu)} \int_{a_k}^{a} (a-b)^{\mu-1} \chi(b, \bar{G}(b), \hat{d}\bar{G}(b))db + \sum_{0<a_k<a} I_k(\bar{G}(a_k^-)) \\
&\leq |\bar{G}_0| + \frac{\ell T^\mu (m+1)}{\Phi(\mu+1)} + m\ell^* = \wedge.
\end{aligned}$$

Step 3 : Here the operator Π maps into equicontinuous sets of PC($I \times \mathbb{R} \times \mathbb{R}, \mathbb{R}$). Consider a bounded set $\varepsilon_1, \varepsilon_2 \in I, \varepsilon_1 < \varepsilon_2, B_{\sigma^*}$ of $PC(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, as in Step 2. Let $\bar{G} \in B_{\sigma^*}$. Then,

$$\begin{aligned}
&|\Pi(\bar{G})(\varepsilon_2) - \Pi(\bar{G})(\varepsilon_2)| \\
&\leq \frac{1}{\Phi(\mu)} \int_0^{\varepsilon_1} |(\varepsilon_2 - b)^{\mu-1} - (\varepsilon_1 - b)^{\mu-1}| \chi(b, \bar{G}(b), \hat{d}p(b))|db \\
&\quad + \frac{1}{\Phi(\mu)} \int_{\varepsilon_1}^{\varepsilon_2} |(\varepsilon_2 - s)^{\mu-1}| \chi(b, \bar{G}(b), \hat{d}\bar{G}(b))|db + \sum_{0<a_k<\varepsilon_2-\varepsilon_1} |I_k(\bar{G}(a_k^-))|. \\
&\leq \frac{\ell}{\Phi(\mu-1)} [2(\varepsilon_2 - \varepsilon_1)^\mu + \varepsilon_2^\mu - \varepsilon_1^\mu] + \sum_{0<a_k<\varepsilon_2-\varepsilon_1} |I_k(\bar{G}(a_k^-))|.
\end{aligned}$$

As $\varepsilon_2 \to \varepsilon_1$, the RHS of the equation tends to 0. Next, we draw the conclusion that the operator Π is completely continuous which is proved with Steps 1 through 3 as well as the Arzelá-Ascoli theorem.

Step 4 : Now, claim

$$\delta = \{\bar{G} \in C_P(I, \mathbb{R}) : \bar{G} = \omega\Pi(\bar{G}) \text{ for some } \ 0 < \omega < 1\} \text{ is bounded.}$$

Let $\bar{G} \in \delta$, for each $a \in I$. Then $\bar{G} = \omega\Pi(\bar{G})$ for some $0 < \omega < 1$, we have

$$\begin{aligned}
\bar{G}(a) &= \omega\bar{G}_0 + \frac{\omega}{\Phi(\mu)} \sum_{0<a_k<a} \int_{a_{k-1}}^{a_k} (a_k - b)^{\mu-1} \chi(b, \bar{G}(b), \hat{d}\bar{G}(b))db \\
&\quad + \frac{\omega}{\Phi(\mu)} \int_{a_k}^{a} (a-b)^{\mu-1} \chi(b, \bar{G}(b), \hat{d}\bar{G}(b))db + \omega \sum_{0<a_k<a} I_k(u(I_k^-)).
\end{aligned}$$

This equality together with $\mathbf{A_6}$ and $\mathbf{A_7}$ (from Step 2) imply that for every $a \in I$,

$$|\bar{G}(a)| \leq |\bar{G}_0| + \frac{\ell T^\mu m}{\Phi(\mu+1)} + \frac{\ell T^\mu}{\Phi(\mu+1)} + m\ell^* = \wedge.$$

Thus, δ is bounded. With the use of Schaefer's fixed point theorem, it can be concluded that Π has a fixed point and this point is a solution of the problem (1.2). □

Using the theorem, and by applying non-linear alternative of Leray-Schauder type, the fixed points can be found.

Theorem 3.8. *If* $\mathbf{A_4}$ *and the following conditions hold*

$\mathbf{A_7}$. *There exist* $\theta_\chi \in C(I, \mathbb{R}^+), \xi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ *and* $\Psi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ *continuous and non-decreasing such that*

$$|\chi(a, x(a), \hat{d}x(a))| \leq \theta_\chi(a)\xi(|x|)\Psi(|\hat{d}x|), \quad \forall\ a\ \in I, x \in \mathbb{R}. \tag{3.15}$$

$\mathbf{A_8}$. *There exist* $\xi^* : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ *continuous and non-decreasing such that*

$$|I_k(x)| \leq \xi^*(|x|),\ \forall\ x \in \mathbb{R}. \tag{3.16}$$

$\mathbf{A_9}$. *There exists a number* $\overline{N} > 0$ *such that*

$$\frac{\overline{N}}{|\bar{G}_0| + \mathfrak{N}_1\xi(\overline{N})\Psi(\overline{N})\frac{(m+1)T^\mu\theta_\chi^0\Psi^0}{\Phi(\mu+1)} + \mathfrak{N}_2\xi(\overline{N})\frac{(m+1)T^\mu\theta_\chi^0\Psi^0}{\Phi(\mu+1)} + m\xi^*(\overline{N})} > 1, \tag{3.17}$$

where $\theta_\chi^0 = \sup\{\theta_\chi(a) : a \in I\}$ *and* $\Psi^0 = \sup\{\Psi(a) : a \in I\}$.

Then (1.2) *has at least one solution on* I.

Proof. Let us take the operator Π defined in Theorem 3.6. We shall prove that Π is continuous & completely continuous. For $\lambda \in [0, 1]$, let $\bar{G}(a) = \lambda(\Pi(\bar{G}))(a)$ for all $a \in I$ satisfied for each $a\ \in I$. Then from $\mathbf{A_8}$ and $\mathbf{A_9}$ hold. We have

$$
\begin{aligned}
|\bar{G}(a)| &\leq |\bar{G}_0| + \xi(\|\bar{G}\|_\infty)\Psi(a(\mathfrak{N}_1\|\bar{G}\|_\infty + \mathfrak{N}_2))\frac{mT^\mu\theta_\chi^0}{\Phi(\mu+1)} \\
&\quad + \xi(\|\bar{G}\|_\infty)\Psi(a(\mathfrak{N}_1\|\bar{G}\|_\infty + \mathfrak{N}_2))\frac{T^\mu\theta_\chi^0}{\Phi(\mu+1)} + m\xi^*(\|\bar{G}\|_\infty) \\
&\leq |\bar{G}_0| + \xi(\|\bar{G}\|_\infty)(\mathfrak{N}_1\Psi(a)\Psi(\|\bar{G}\|_\infty) + \mathfrak{N}_2\Psi(a))\frac{mT^\mu\theta_\chi^0}{\Phi(\mu+1)} \\
&\quad + \xi(\|\bar{G}\|_\infty)(\mathfrak{N}_1\Psi(a)\Psi(\|\bar{G}\|_\infty) + \mathfrak{N}_2\Psi(a))\frac{T^\mu\theta_\chi^0}{\Phi(\mu+1)} + m\xi^*(\|\bar{G}\|_\infty) \\
&\leq |\bar{G}_0| + \mathfrak{N}_1\xi(\|\bar{G}\|_\infty)\Psi(\|\bar{G}\|_\infty)\frac{mT^\mu\theta_\chi^0\Psi^0}{\Phi(\mu+1)} + \mathfrak{N}_2\xi(\|\bar{G}\|_\infty)\frac{mT^\mu\theta_\chi^0\Psi^0}{\Phi(\mu+1)} \\
&\quad + \mathfrak{N}_1\xi(\|\bar{G}\|_\infty)\Psi(\|\bar{G}\|_\infty)\frac{T^\mu\theta_\chi^0\Psi^0}{\Phi(\mu+1)} + \mathfrak{N}_2\xi(\|\bar{G}\|_\infty)\frac{T^\mu\theta_\chi^0\Psi^0}{\Phi(\mu+1)} \\
&\quad + m\xi^*(\|\bar{G}\|_\infty).
\end{aligned}
$$

Therefore,

$$
\frac{\|\bar{G}\|_\infty}{|\bar{G}_0| + \mathfrak{N}_1\xi(\|\bar{G}\|_\infty)\Psi(\|\bar{G}\|_\infty)\frac{(m+1)T^\mu\theta_\chi^0\Psi^0}{\Phi(\mu+1)} + \mathfrak{N}_2\xi(\|\bar{G}\|_\infty)\frac{(m+1)T^\mu\theta_\chi^0\Psi^0}{\Phi(\mu+1)} + m\xi^*(\|\bar{G}\|_\infty)} \leq 1. \tag{3.18}
$$

From (3.17) it follows that there exists $\overline{N}$ such that $\|\bar{G}\|_\infty \neq \overline{N}$, assumption (3.18) is violated. Let

$$
W = \{\bar{G} \in C_P(I, \mathbb{R}) : \|\bar{G}\|_\infty < \overline{N}\}. \tag{3.19}
$$

The operator $\Pi : \overline{W} \to C_P(I, \mathbb{R})$ is completely continuous. The choice of W, there is no $\bar{G} \in \partial W$ such that $\bar{G} = \lambda\Pi(\bar{G})$ for some $\lambda \in (0, 1)$. As a outcome of the theorem [22], we conclude that Π is a solution of (1.2) and it has a fixed point u in $\overline{W}$. □

4. Nonlocal conditions

In this section, we discuss the hypothesis of the result what we discussed in the previous section to nonlocal impulsive FDEs. Exactly here we will mount existence of the following nonlocal problem

$$
\begin{cases}
([\hat{D}_C]^\mu)(\bar{G})(a) = \chi(a, \bar{G}(a), \hat{d}\bar{G}(a)) \text{ for all } a \in I = [0,1],\ a \neq a_k, \\
\Delta\bar{G}|_{a=a_k} = I_k(\bar{G}(a_k^-)), \\
\bar{G}(0) + h(\bar{G}) = \bar{G}_0,
\end{cases} \tag{4.1}
$$

where $k \in \mathbb{N}_m,\ 0 < \mu \leq 1$. Here, we consider χ, I_k assumptions as in the previous section and $h : C_P(I, \mathbb{R}) \to \mathbb{R}$ which is a continuous function. Zhang et al. [31] studied nonlocal existence of mild solutions of impulsive fractional equations. For example, h may be defined by

$$h(\bar{G}) = \sum_{i=1}^{r} c_i \bar{G}(\epsilon_i),$$

where $c_i,\ i \in \mathbb{N}_r$ are given constants $0 < \epsilon_1 < \epsilon_2 < \cdots < \epsilon_r \leq T$.

Now consider the following assumptions:

$\mathbf{A_{10}}$. There exists a constants $N_1^* > 0$ such that

$$|h(x)| \leq N_1^*, \quad \forall\, x \in C_P(I, \mathbb{R}). \tag{4.2}$$

$\mathbf{A_{11}}$. There exists a constant ϖ_2 such that

$$|h(x) - h(y)| \leq \varpi_2 |x - y|, \quad \forall\, x, y\ \in C_P(I, \mathbb{R}). \tag{4.3}$$

$\mathbf{A_{12}}$. There exist $\xi^* : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ nondecreasing continuous such that

$$|h(x)| \leq \xi^*(|x|), \quad \forall\, x \in C_P(I, \mathbb{R}). \tag{4.4}$$

$\mathbf{A_{13}}$. There exists a number $\overline{N^*}$ such that

$$\frac{\overline{N^*}}{|\bar{G}_0| + \xi^*(\overline{N^*}) + \mathfrak{N}_1 \xi^*(\overline{N^*}) \Psi(\overline{N^*}) \frac{(m+1)T^\mu \theta_\chi^0 \Psi^0}{\Phi(\mu+1)} + \mathfrak{N}_2 \xi(\overline{N^*}) \frac{(m+1)T^\mu \theta_\chi^0 \Psi^0}{\Phi(\mu+1)} + m\xi^*(\overline{N^*})} > 1. \tag{4.5}$$

Theorem 4.1. *If* $\mathbf{A_1}$-$\mathbf{A_3}$ *and* $\mathbf{A_{11}}$ *hold and*

$$\Big[\frac{T^\mu \varpi(m+1)}{\Phi(\mu+1)} + m\varpi_1 + \varpi_2\Big] < 1. \tag{4.6}$$

Then (4.1) *has one solution on* J.

Proof. The problem (4.1) can be seen as a fixed point theorem. Define $\Pi_1 : C_P(I, \mathbb{R}) \to C_P(I, \mathbb{R})$ as

$$(\Pi_1 \bar{G})(a) = \bar{G}_0 - h(\bar{G}) + \frac{1}{\Phi(\mu)} \sum_{0 < a_k < a} \int_{a_k - 1}^{a_k} (a_k - b)^{\mu - 1} \chi(b, \bar{G}(b), \hat{d}\bar{G}(b)) db$$

$$+ \frac{1}{\Phi(\mu)} \int_{a_k}^{a} (a - b)^{\mu - 1} \chi(b, \bar{G}(b), \hat{d}\bar{G}(b)) db + \sum_{0 < a_k < a} I_k(u(a_k^-)).$$

Thus the operator Π_1 is the solution of the problem (4.1). Then, it can be proved that the operator Π_1 is a contraction. □

Theorem 4.2. *If* $\mathbf{A_4}$-$\mathbf{A_6}$ *and* $\mathbf{A_{10}}$ *are hold, then the problem* (4.1) *has one or more solutions on* I.

Theorem 4.3. *If* $\mathbf{A_7}$-$\mathbf{A_8}$, $\mathbf{A_{12}}$-$\mathbf{A_{13}}$ *are hold, then the problem* (4.1) *has one or more solutions on* I.

5. Test problem

A problem which is connected to the main result is considered here. Consider the following impulsive FDE

$$\begin{cases} ([\hat{D}_C]^{\mu}\bar{G})(a) = \frac{a}{3\sqrt{\pi}}sin(\bar{G}(a)+\bar{G}'(a)),\ a\in[0,1],\ a\neq\frac{1}{3},\ 0<\mu\leq 1, \\ \Delta\bar{G}|_{a=\frac{1}{3}} = \frac{\left|\frac{1}{3}\right|\left|\bar{G}\left(\frac{1}{3}\right)\right|}{2\sqrt{\pi}(1+\left|\bar{G}\left(\frac{1}{3}\right)\right|)}, \\ \bar{G}(0)=\bar{G}_0, \end{cases} \tag{5.1}$$

$$\chi(a,\bar{G}(a),\hat{d}\bar{G}(a)) = \frac{a}{3\sqrt{\pi}}sin(\bar{G}(a)+\bar{G}'(a)),$$

$$I_g(\bar{G}) = \frac{a\bar{G}(a)}{2\sqrt{\pi}(1+\bar{G}(a))}, \bar{G}\in\mathbb{R}^+\cup\{0\},$$

$$\begin{aligned} |\chi(a,\bar{G}_1,\bar{H}_1)-\chi(a,\bar{G}_2,\bar{H}_2)| &\leq \frac{a}{3\sqrt{\pi}}\Big(|\bar{G}_1-\bar{G}_2|+|\bar{H}_1-\bar{H}_2|\Big) \\ &\leq \frac{1}{3\sqrt{\pi}}|\bar{G}-\bar{H}| \\ &\leq \frac{1}{5}|\bar{G}-\bar{H}|. \end{aligned}$$

Hence from the assumption $\zeta=\frac{1}{5}$. Let $\bar{G},\bar{H}\in\mathbb{R}^+\cup\{0\}$. Then

$$|I_g(\bar{G})-I_g(\bar{H})| = \Big|\frac{a\bar{G}}{2\sqrt{\pi}(1+\bar{G}(a))}-\frac{a\bar{H}}{2\sqrt{\pi}(1+\bar{H}(a))}\Big| \leq \frac{1}{3}|\bar{G}-\bar{H}|.$$

Thus, the condition $\mathbf{A_4}$ holds, $\zeta_1=\frac{1}{3}$. Also, the condition (4.6) is satisfied, $T=1$ and $m=1$.

$$\Big[\frac{T^{\mu}\varpi(m+1)}{\Phi(\mu+1)}+m\varpi_1\Big]<1 \iff \Phi(\mu+1)>\frac{3}{5} \text{ for some } \mu\in(0,1].$$

By Theorem 3.6, the problem (5.1) has a fixed point on $a\in[0,1]$.

6. Concluding points

We have established some new results for the existence of solutions to an impulsive Caputo fractional differential equations with a class of initial value problem depends on the Lipschitz first derivative conditions. Sufficient results have been proved for the existence and uniqueness of solution to the mentioned problem. Also some results for nonlocal conditions have been discussed. A proper example in this regard has been given. The given problem can extended to various fractional order derivative techniques with fixed point theory.

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