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SOME SANDWICH THEOREMS FOR MEROMORPHIC UNIVALENT FUNCTIONS DEFINED BY A NEW HADAMARD PRODUCT OPERATOR

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Abstract. The present study develops differential subordination and superordination results for meromorphic univalent functions defined by a novel Hadamard product operator within a punctured open unit disk.

1. Introduction

Let D be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}^*$ denotes the class of analytic functions of the form:

$$f(z) = z^{-1} + \sum_{k=0}^{\infty} a_k z^k, \quad (z \in D^* = D \setminus \{0\}), \tag{1.1}$$

that are meromorphic and univalent in the punctured open unit disk

$$D^* = \{z : z \in \mathbb{C},\ 0 < |z| < 1\}.$$

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Let $\mathcal{H}$ be the class of all analytic functions in D. For a positive integer number n such that $a \in \mathbb{C}$, we let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}, \quad (a \in \mathbb{C}).$$

The class of functions $\mathcal{H}[a,n]$ is denoted by $\mathcal{A}$ when $a = 0$, $n = 1$, and $a_1 = 1$. However, for f and g as analytic functions in $\mathcal{H}$, it is said that f is subordinate to g in D, or g is superordinate to f in D, and we write $f(z) \prec g(z)$, if there exists a Schwarz function ω in D, such that $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in D)$, where

$$f(z) = g(\omega(z)).$$

Moreover, if the function g is univalent in D, we have the following equivalence relationship (cf., e.g., [14, 21, 22]):

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(D) \subset g(D), \quad (z \in D).$$

Definition 1.1. ([21]) Let the functions $p, h \in \mathcal{A}$ and $\Phi(r,s,t;z) : \mathbb{C}^3 \times D \to \mathbb{C}$. When p and $\Phi(p(z), zp'(z), z^2p''(z); z)$ are both univalent functions within the domain D, and p fulfills:

$$h(z) \prec \Phi(p(z), zp'(z), z^2p''(z); z), \tag{1.2}$$

then, if p satisfies the differential subordination (1.2), it is referred to as a solution.

An analytic function $q(z)$, which is also univalent, is considered to be the dominant solution of the differential subordination (1.2), alternatively dominant if $p(z) \prec q(z)$ for every $p(z)$ fulfilling (1.2).

An univalent dominant $\tilde{q}(z)$ which meets the condition $\tilde{q}(z) \prec q(z)$ for every dominant $q(z)$ in equation (1.2) is referred to as the best subordinate, with it being unique except for a relation on D.

Definition 1.2. ([19]) Let the function $\Phi : \mathbb{C}^3 \times D \to \mathbb{C}$ and consider a function $h(z)$ to be a univalent function within a domain D. Suppose $p(z)$ is an analytic function within the region D and satisfies the condition of being subordinate to a second-order differential equation:

$$\Phi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.3}$$

then $p(z)$ is said to satisfy the differential subordination in (1.3), and it is referred to as a solution.

The function $q(z)$, which is univalent, is considered to be the dominant solution of the differential subordination (1.3), alternatively dominant when $p(z) \prec q(z)$ for every $p(z)$ fulfilling (1.3).

A univalent dominant $\tilde{q}(z)$ that meets the condition $\tilde{q}(z) \prec q(z)$ for every dominant $q(z)$ in equation (1.3) is referred to as a best dominant, and it is unique except for a relation on D.

Several authors [11, 16, 21, 26] have derived necessary conditions on the functions h, p, and Φ for which the following implication holds:

$$h(z) \prec \Phi(p(z), zp'(z), z^2p''(z); z),$$

then

$$q(z) \prec p(z). \tag{1.4}$$

Using the results (see [1, 3, 4, 5, 6, 7, 12, 13, 22, 24, 27]), adequate conditions have been established for a normalized analytic function to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in D and $q_1(0) = q_2(0) = 1$.

Additionally, several scholars (see [2, 9, 10, 15, 17, 18, 20, 23]) have established differential subordination and superordination conclusions using sandwich theorems.

If $f \in \mathcal{H}^*$ is defined by (1.1) and $g \in \mathcal{H}^*$ is given by

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k b_k z^k = (g * f)(z), \quad (z \in D^*).$$

A linear operator $I_{c,r,1(n,\lambda)} : \mathcal{H}^* \to \mathcal{H}^*$ (see [8]) is defined as

$$I_{c,r,1(n,\lambda)} f(z) = z^{-1} + \sum_{k=0}^{\infty} \left(\frac{r}{1+k+r}\right)^c \left(\frac{k+\lambda}{\lambda-1}\right)^n a_k z^k, \tag{1.5}$$

where $\lambda > 1$, $c \in \mathbb{C}$, $r \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and $z \in D^*$.

Liu et al. [19] defined the operator D^α for a function $f \in \mathcal{H}^*$ as follows:

$$D^\alpha : \mathcal{H}^* \to \mathcal{H}^*,$$

where

$$D^\alpha f(z) = z^{-1} + \sum_{k=0}^{\infty} (k+2)^\alpha a_k z^k \tag{1.6}$$

with $\alpha \in \mathbb{N}$ and $z \in D^*$.

Define the convolution (or Hadamard product) $S^{\alpha}_{c,r,1,n,\lambda}f(z)$ of the operators $I_{c,r,1(n,\lambda)}f(z)$ and $D^{\alpha}f(z)$ to get a new Hadamard product operator as follows:

$$S^{\alpha}_{c,r,1,n,\lambda}f(z) = z^{-1} + \sum_{k=0}^{\infty} \left(\frac{r}{1+k+r}\right)^{c} \left(\frac{k+\lambda}{\lambda-1}\right)^{n} (k+2)^{\alpha} a_k^2 z^k, \tag{1.7}$$

where $z \in \mathbb{D}^*$.

We note from (1.7) that

$$z\left(S^{\alpha}_{c,r,1,n,\lambda}f(z)\right)' = rS^{\alpha}_{c-1,r,1,n,\lambda}f(z) - (1+r)S^{\alpha}_{c,r,1,n,\lambda}f(z). \tag{1.8}$$

The primary objective of this work is to create appropriate criteria for a certain normalized analytic function f to fulfill specific requirements:

$$q_1(z) \prec \left[\frac{(1-\sigma)zS^{\alpha}_{c-1,r,1,n,\lambda}f(z) + 2\sigma zS^{\alpha}_{c,r,1,n,\lambda}f(z)}{\sigma}\right]^{\delta} \prec q_2(z)$$

and

$$q_1(z) \prec \left[zS^{\alpha}_{c,r,1,n,\lambda}f(z)\right]^{\delta} \prec q_2(z),$$

where q_1 and q_2 are given as univalent functions within D such that $q_1(0) = q_2(0) = 1$.

This work presents a solution for several sandwich theorems that include the operator $S^{\alpha}_{c,r,1,n,\lambda}f(z)$.

2. Preliminaries

We need the following definitions and lemmas to prove our results:

Definition 2.1. ([21]) Consider that $\mathcal{Q}$ represents a collection of any functions q that are both analytic and injective onto $\overline{D}\setminus E(q)$, where $\overline{D} = D\cup\{z \in \partial D\}$, and

$$E(q) = \{\epsilon \in \partial D : \lim_{z\to\epsilon} q(z) = \infty\},$$

and have the property that $q'(\epsilon) \neq 0$ for $\epsilon \in \partial D \setminus E(q)$.

Additionally, we can represent the subclass of $\mathcal{Q}$ where $q(0) = a$ as $\mathcal{Q}(a)$, with $\mathcal{Q}(0) = \mathcal{Q}_0$ and $\mathcal{Q}(1) = \mathcal{Q}_1 = \{q \in \mathcal{Q} : q(0) = 1\}$.

Lemma 2.2. ([22]) *Consider the function q to be a convex univalent function within D, with $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}\setminus\{0\}$, and $q(0) = 1$. Suppose that*

$$\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\{0, -\mathrm{Re}(\alpha/\beta)\}.$$

If p is analytic within D and satisfies the condition

$$\alpha p(z) + \beta zp'(z) \prec \alpha q(z) + \beta zq'(z), \tag{2.1}$$

then $p \prec q$*, where* q *is the best dominant of equation* (2.1)*.*

Lemma 2.3. ([6]) *Let* q *be a convex univalent function in* D*, and suppose that* Θ *and* ϕ *are analytic within a domain* D *comprising* $q(D)$*, with* $\phi(w) \neq 0$ *for* $w \in q(D)$*. Define*

$$\mathcal{Q}(z) = zq'(z)\phi(q(z)) \quad \textit{and} \quad h(z) = \Theta(q(z)) + \mathcal{Q}(z).$$

Assume the following conditions hold:

(a) $\mathcal{Q}(z)$ *is starlike univalent within* D*,*

(b) $\operatorname{Re}\left\{\frac{zh'(z)}{\mathcal{Q}(z)}\right\} > 0 \quad$ *for* $z \in D$*.*

If p *is analytic within* D*, with* $p(0) = q(0)$ *and* $p(D) \subset q(D)$*, and satisfies the condition*

$$\Theta(p(z)) + zp'(z)\phi(p(z)) \prec \Theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.2}$$

then $p \prec q$*, where* q *is the best dominant of the equation* (2.2)*.*

Lemma 2.4. ([25]) *Consider a function* q *that is convex univalent within* D*, and let* $\gamma_1, \gamma_2 \in \mathbb{C}$ *such that* $\gamma_2 \neq 0$*. Assume that*

$$\operatorname{Re}\left\{\frac{\gamma_1}{\gamma_2}\right\} > 0.$$

If $p \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$ *and* $\gamma_1 p(z) + \gamma_2 p'(z)$ *is univalent in* D*, then*

$$\gamma_1 q(z) + \gamma_2 zq'(z) \prec \gamma_1 p(z) + \gamma_2 p'(z), \tag{2.3}$$

which implies that $q \prec p$*, where* q *is the best subordinant.*

Lemma 2.5. ([22]) *Consider a function* q *that is univalent within* D*, with* Θ *and* Φ *being analytic within a domain* D *comprising* $q(D)$*. Assume the following conditions:*

(i) $\operatorname{Re}\left\{\frac{\Theta'(q(z))}{\Phi(q(z))}\right\} > 0 \quad (z \in D)$,

(ii) $\mathcal{Q}(z) = zq'(z)\Phi(q(z))$ *is starlike and univalent within* D*.*

If $p \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$*,* $p(D) \subset q(D)$*, and* $\Theta(p(z)) + zp'(z)\Phi(p(z))$ *is univalent within* D*, with*

$$\Theta(q(z)) + zq'(z)\Phi(q(z)) \prec \Theta(p(z)) + zp'(z)\Phi(p(z)), \tag{2.4}$$

then $q \prec p$*, where* q *is the best subordinant of the equation* (2.4)*.*

3. Differential subordination results

Here, we introduce some differential subordination results by using Hadamard product operator.

Theorem 3.1. *Consider a function q that is univalent within the unit disk D and $q(0) = 1$ such that $q(z) \neq 0$ for every $z \in D$. Let $\delta, \sigma \in \mathbb{C} \setminus \{0\}$, $t \in \mathbb{C}$, $\epsilon > 0$, and $f \in \mathcal{H}^*$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent within D, and f, q satisfy the following conditions:*

$$\frac{(1-\sigma)zS^{\alpha}_{c-1,r,1,n,\lambda}f(z) + 2\sigma z S^{\alpha}_{c,r,1,n,\lambda}f(z)}{\sigma} \neq 0 \tag{3.1}$$

and

$$\operatorname{Re}\left\{1 + \frac{2tq(z)^2}{\epsilon} - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0. \tag{3.2}$$

If

$$R(z) \prec 1 + t\,(q(z))^2 + \epsilon z\frac{q'(z)}{q(z)}, \tag{3.3}$$

where

$$R(z) = \left[1 + t\left(\frac{\begin{array}{c}(1-\sigma)zS^{\alpha}_{c-1,r,1,n,\lambda}f(z)\\ + 2\sigma z S^{\alpha}_{c,r,1,n,\lambda}f(z)\end{array}}{\sigma}\right)^{\delta}\right]^2 + \epsilon\delta\left[\frac{\begin{array}{c}(r-r\sigma)S^{\alpha}_{c-2,r,1,n,\lambda}f(z)\\ -(r-3r\sigma)S^{\alpha}_{c-1,r,1,n,\lambda}f(z)\\ -2r\sigma S^{\alpha}_{c,r,1,n,\lambda}f(z)\end{array}}{\begin{array}{c}(1-\sigma)S^{\alpha}_{c-1,r,1,n,\lambda}f(z)\\ +2\sigma S^{\alpha}_{c,r,1,n,\lambda}f(z)\end{array}}\right], \tag{3.4}$$

then

$$\left(\frac{(1-\sigma)zS^{\alpha}_{c-1,r,1,n,\lambda}f(z) + 2\sigma z S^{\alpha}_{c,r,1,n,\lambda}f(z)}{\sigma}\right)^{\delta} \prec q(z), \tag{3.5}$$

and q is the best dominant of (3.3).

Proof. Define the function p as follows:

$$p(z) = \left(\frac{(1-\sigma)zS^{\alpha}_{c-1,r,1,n,\lambda}f(z) + 2\sigma z S^{\alpha}_{c,r,1,n,\lambda}f(z)}{\sigma}\right)^{\delta}. \tag{3.6}$$

Since that function $p(z)$ is analytic within D such that $p(0) = 1$, we can differentiate equation (3.6) with respect to z, we have

$$\frac{zp'(z)}{p(z)} = \delta\left[\frac{\begin{array}{c}(1-\sigma)z\left(S^{\alpha}_{c-1,r,1,n,\lambda}f(z)\right)' - (1-\sigma)S^{\alpha}_{c-1,r,1,n,\lambda}f(z)\\ +2\sigma z\left(S^{\alpha}_{c,r,1,n,\lambda}f(z)\right)' + 2\sigma S^{\alpha}_{c,r,1,n,\lambda}f(z)\end{array}}{(1-\sigma)S^{\alpha}_{c-1,r,1,n,\lambda}f(z) + 2\sigma S^{\alpha}_{c,r,1,n,\lambda}f(z)}\right]. \tag{3.7}$$

Applying identity (1.8) in (3.7), we get

$$\frac{zp'(z)}{p(z)} = \delta\left[\frac{(r-r\sigma)S^{\alpha}_{c-2,r,1,n,\lambda}f(z) - (r-3r\sigma)S^{\alpha}_{c-1,r,1,n,\lambda}f(z) - 2r\sigma S^{\alpha}_{c,r,1,n,\lambda}f(z)}{(1-\sigma)S^{\alpha}_{c-1,r,1,n,\lambda}f(z) + 2\sigma S^{\alpha}_{c,r,1,n,\lambda}f(z)}\right].$$

By setting

$$\Theta(\omega) = 1 + t\omega^2 \quad \text{with } \phi(\omega) = \frac{\epsilon}{\omega},\ \omega \neq 0,$$

it is seen that $\Theta(\omega)$ is analytic within $\mathbb{C}$, with $\phi(\omega)$ analytic in $\mathbb{C}\setminus\{0\}$ and $\phi(\omega) \neq 0$ for $\omega \in \mathbb{C}\setminus\{0\}$. Additionally, we get

$$\mathcal{Q}(z) = zq'(z)\phi(q(z)) = \epsilon z\frac{q'(z)}{q(z)}$$

and

$$h(z) = \Theta(q(z)) + \mathcal{Q}(z) = 1 + t[q(z)]^2 + \epsilon z\frac{q'(z)}{q(z)}.$$

It is seen that $\mathcal{Q}(z)$ is starlike univalent in D. We get

$$\operatorname{Re}\left\{\frac{zh'(z)}{\mathcal{Q}(z)}\right\} = \operatorname{Re}\left\{1 + \frac{2t[q(z)]^2}{\epsilon} - z\frac{q'(z)}{q(z)} + z\frac{q''(z)}{q'(z)}\right\} > 0.$$

Therefore, according to Lemma 2.3, we have $p(z) \prec q(z)$. By using equation (3.6), we obtain the result. $\square$

By substituting $q(z) = e^{\tau z}$, $|\tau| \leq 1$ into Theorem 3.1, we deduce the subsequent corollary:

Corollary 3.2. *Consider a function $f \in \mathcal{H}^*$ such that $|\tau| \leq 1$, and also the condition* (3.2) *is satisfied. If*

$$R(z) \prec 1 + te^{2\tau z} + \tau\epsilon z, \tag{3.8}$$

where $R(z)$ is given by (3.4), *then*

$$\left[\frac{(1-\sigma)zS^{\alpha}_{c-1,r,1,n,\lambda}f(z) + 2\sigma zS^{\alpha}_{c,r,1,n,\lambda}f(z)}{\sigma}\right]^{\delta} \prec e^{\tau z},$$

and $e^{\tau z}$ is the best dominant.

Therefore, when $\tau = \sigma = 1$, the following result is obtained.

Corollary 3.3. *Consider a function $f \in \mathcal{H}^*$ that fulfills the subordination*

$$1 + t\left[\left(zS^{\alpha}_{c,r,1,n,\lambda}f(z)\right)^{\delta}\right]^2 + \epsilon r\delta\left[\frac{S^{\alpha}_{c-1,r,1,n,\lambda}f(z)}{S^{\alpha}_{c,r,1,n,\lambda}f(z)} - 1\right] \prec 1 + te^{2z} + \epsilon z,$$

then

$$\left[2zS^{\alpha}_{c,r,1,n,\lambda}f(z)\right]^{\delta} \prec e^{z}$$

with $q(z) = e^z$ being the best dominant.

Theorem 3.4. *Consider a function q that is convex univalent in the unit disk D such that $q(0) = 1$. Let $\epsilon > 0$, $\delta \in \mathbb{C} \setminus \{0\}$, $t \in \mathbb{C}$, $f \in \mathcal{H}^*$, and assume that f and q fulfill the following conditions:*

$$zS^{\alpha}_{c,r,1,n,\lambda}f(z) \neq 0$$

and

$$Re\left\{1 + \frac{1}{\epsilon} + z\frac{q''(z)}{q'(z)}\right\} > 0. \tag{3.9}$$

If

$$\psi(z) \prec t + q(z) + \epsilon zq'(z), \tag{3.10}$$

where

$$\psi(z) = t + \left(zS^{\alpha}_{c,r,1,n,\lambda}f(z)\right)^{\delta} + \epsilon r\delta\left(zS^{\alpha}_{c,r,1,n,\lambda}f(z)\right)^{\delta}\left[\frac{S^{\alpha}_{c-1,r,1,n,\lambda}f(z)}{S^{\alpha}_{c,r,1,n,\lambda}f(z)} - 1\right], \tag{3.11}$$

then

$$\left(zS^{\alpha}_{c,r,1,n,\lambda}f(z)\right)^{\delta} \prec q(z) \tag{3.12}$$

with q being the best dominant of (3.10).

Proof. Specify the function p as follows:

$$p(z) = \left(zS^{\alpha}_{c,r,1,n,\lambda}f(z)\right)^{\delta}. \tag{3.13}$$

Then the function $p(z)$ is analytic in D such that $p(0) = 1$. A simple computation shows that

$$\begin{aligned}\psi(z) &= t + \left(zS^{\alpha}_{c,r,1,n,\lambda}f(z)\right)^{\delta} + \epsilon r\delta\left(zS^{\alpha}_{c,r,1,n,\lambda}f(z)\right)^{\delta}\left[\frac{S^{\alpha}_{c-1,r,1,n,\lambda}f(z)}{S^{\alpha}_{c,r,1,n,\lambda}f(z)} - 1\right] \\ &= t + p(z) + \epsilon zp'(z).\end{aligned} \tag{3.14}$$

To prove our result, use Lemma 2.3. Consider in this lemma $\Theta(w) = t + w$ and $\Phi(w) = \epsilon$, where Θ is analytic in $\mathbb{C}$ and Φ is analytic in $\mathbb{C} \setminus \{0\}$. Also, we get

$$\mathcal{Q}(z) = zq'(z)\Phi(q(z)) = \epsilon zq'(z)$$

and

$$h(z) = \Theta(q(z)) + \mathcal{Q}(z) = t + q(z) + \epsilon z q'(z).$$

We see that $\mathcal{Q}(z)$ is starlike univalent in D, and we have

$$\operatorname{Re}\left\{\frac{zh'(z)}{\mathcal{Q}(z)}\right\} = \operatorname{Re}\left\{1 + \frac{1}{\epsilon} + z\frac{q''(z)}{q'(z)}\right\} > 0.$$

Thus, using Lemma 2.3, we obtain $p(z) \prec q(z)$. By applying equation (3.13), we obtain the result. □

Theorem 3.5. *Assume that q is a univalent function within D such that $q(0) = 1$, $\delta, \eta \in \mathbb{C} \setminus \{0\}$, and $\sigma \in \mathbb{R}^+$. Furthermore, assume that q satisfies the inequality:*

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\operatorname{Re}\frac{\delta}{\eta}\right\}. \tag{3.15}$$

If $f \in \mathcal{H}^*$ satisfies the subordination condition:

$$G(z) \prec q(z) + \frac{\eta}{\delta} z q'(z), \tag{3.16}$$

where

$$\begin{aligned} G(z) = &\left(\frac{(1-\sigma)zS^{\alpha}_{c-1,r,1,n,\lambda}f(z) + 2\sigma z S^{\alpha}_{c,r,1,n,\lambda}f(z)}{\sigma}\right)^{\delta} \\ &\times \left[1 + \eta \frac{(r - r\sigma)S^{\alpha}_{c-2,r,1,n,\lambda}f(z) - (r - 3r\sigma)S^{\alpha}_{c-1,r,1,n,\lambda}f(z) - 2r\sigma S^{\alpha}_{c,r,1,n,\lambda}f(z)}{(1-\sigma)S^{\alpha}_{c-1,r,1,n,\lambda}f(z) + 2\sigma S^{\alpha}_{c,r,1,n,\lambda}f(z)}\right], \end{aligned} \tag{3.17}$$

then the subordination

$$\left(\frac{(1-\sigma)zS^{\alpha}_{c-1,r,1,n,\lambda}f(z) + 2\sigma z S^{\alpha}_{c,r,1,n,\lambda}f(z)}{\sigma}\right)^{\delta} \prec q(z), \tag{3.18}$$

holds, with $q(z)$ being the best dominant of (3.16)*.*

Proof. Suppose that $p(z)$ is defined by (3.6). Further calculations show that

$$G(z) = p(z) + \frac{\eta}{\delta} z p'(z),$$

where $G(z)$ is given by (3.17). Therefore, the subordination (3.16) is equivalent to

$$p(z) + \frac{\eta}{\delta} z p'(z) \prec q(z) + \frac{\eta}{\delta} z q'(z).$$

By applying Lemma 2.2 with $\beta = \frac{\eta}{\delta}$ and $\alpha = 1$, we obtain (3.18). □

4. Differential superordination results

Theorem 4.1. *Consider a function q which is a convex univalent function within D such that $q(0) = 1$. Let $t \in \mathbb{C}$, $\delta, \epsilon \in \mathbb{C} \setminus \{0\}$, and $z \in D^*$. Suppose that*

$$Re\left\{\frac{q'(z)}{\epsilon}\right\} > 0, \tag{4.1}$$

and f fulfills the following conditions:

$$zS^{\alpha}_{c,r,1,n,\lambda} f(z) \neq 0,$$

and

$$\left[zS^{\alpha}_{c,r,1,n,\lambda} f(z)\right]^{\delta} \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}.$$

Additionally, if the function $\psi(z)$ described by (3.11) is univalent in D, then the subsequent superordination condition

$$t + q(z) + \epsilon z q'(z) \prec \psi(z), \tag{4.2}$$

holds. Then,

$$q(z) \prec \left[zS^{\alpha}_{c,r,1,n,\lambda} f(z)\right]^{\delta}, \tag{4.3}$$

with q being the best subordinant.

Proof. Let the function p be defined as follows:

$$p(z) = \left[zS^{\alpha}_{c,r,1,n,\lambda} f(z)\right]^{\delta}, \tag{4.4}$$

following the process of computation, we obtain

$$t + p(z) + \epsilon z p'(z) = \psi(z),$$

where $\psi(z)$ is given by (3.11). This implies

$$t + q(z) + \epsilon z q'(z) \prec t + p(z) + \epsilon z p'(z).$$

Putting

$$\Theta(\omega) = t + \omega \quad \text{and} \quad \varphi(\omega) = \epsilon,$$

then it is clear that $\Theta(\omega)$ is analytic in $\mathbb{C}$, and $\varphi(\omega) \neq 0$ is analytic in $\mathbb{C} \setminus \{0\}$.

Additionally, we have

$$\operatorname{Re}\left(\frac{\Theta'(q(z))}{\varphi(q(z))}\right) = \operatorname{Re}\left(\frac{q'(z)}{\epsilon}\right) > 0.$$

Thus, according to Lemma 2.5, we can conclude that

$$q(z) \prec \left[zS^{\alpha}_{c,r,1,n,\lambda} f(z)\right]^{\delta}.$$

□

Now, by using Lemma 2.4, it is simple to prove the next theorem.

Theorem 4.2. *Consider a function q is a convex univalent function in D such that $q(0) = 1$, $\delta, \eta \in \mathbb{C} \setminus \{0\}$, $\sigma \in \mathbb{R}^+$, and $Re\left\{\frac{\delta}{\eta}\right\} > 0$. Let $f \in \mathcal{H}^*$ such that*

$$\frac{(1-\sigma)zS^{\alpha}_{c-1,r,1,n,\lambda}f(z) + 2\sigma zS^{\alpha}_{c,r,1,n,\lambda}f(z)}{\sigma} \neq 0$$

and

$$\left[\frac{(1-\sigma)zS^{\alpha}_{c-1,r,1,n,\lambda}f(z) + 2\sigma zS^{\alpha}_{c,r,1,n,\lambda}f(z)}{\sigma}\right]^{\delta} \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}.$$

If the function $G(z)$ as defined by equation (3.17) *is univalent in D and fulfills the given superordination condition*

$$q(z) + \frac{\eta}{\delta} zq'(z) \prec G(z) \tag{4.5}$$

holds, then

$$q(z) \prec \left[\frac{(1-\sigma)zS^{\alpha}_{c-1,r,1,n,\lambda}f(z) + 2\sigma zS^{\alpha}_{c,r,1,n,\lambda}f(z)}{\sigma}\right]^{\delta}, \tag{4.6}$$

with q being the best subordinant of (4.1).

5. Sandwich results

By applying Theorem 3.4 with Theorem 4.1 and Theorem 3.5 with Theorem 4.2, we get, respectively, the following two sandwich results:

Theorem 5.1. *Consider q_1 to be a convex univalent function within D such that $q_1(0) = 1$, and fulfills condition* (4.1). *Additionally, let q_2 be univalent in D such that $q_2(0) = 1$ and fulfills* (3.9). *Assume that $\varepsilon > 0$, $\delta \in \mathbb{C} \setminus \{0\}$, $t \in \mathbb{C}$,*

$$S^{\alpha}_{c,r,1,n,\lambda}f(z) \neq 0$$

and

$$\left[S^{\alpha}_{c,r,1,n,\lambda}f(z)\right]^{\delta} \in \mathcal{H}[1, 1] \cap \mathcal{Q}.$$

If the function $\psi(z)$ defined by (3.11) *is univalent in D and*

$$t + q_1(z) + \varepsilon zq_1'(z) \prec \psi(z) \prec t + q_2(z) + \varepsilon zq_2'(z),$$

then

$$q_1(z) \prec \left[S^{\alpha}_{c,r,1,n,\lambda}f(z)\right]^{\delta} \prec q_2(z),$$

and q_1 and q_2 are respectively, the best subordinant and the best dominant.

Theorem 5.2. *Consider q_1 to be a convex univalent function in D such that $q_1(0) = 1$, and let q_2 be univalent in D. Suppose that $\Re\left\{\frac{\delta}{\eta}\right\} > 0$, $\delta, \eta \in \mathbb{C}\backslash\{0\}$, $\sigma \in \mathbb{R}^+$, and q_2 satisfies (3.15). Let*

$$\left[\frac{(1-\sigma)zS^{\alpha}_{c-1,r,1,n,\lambda}f(z) + 2\sigma zS^{\alpha}_{c,r,1,n,\lambda}f(z)}{\sigma}\right]^{\delta} \in \mathcal{H}[q(0),1] \cap \mathcal{Q},$$

and the function $G(z)$ defined by (3.17) is univalent in D. If $f \in \mathcal{H}^*$ fulfills

$$q_1(z) + \frac{\eta}{\delta}zq_1'(z) \prec G(z) \prec q_2(z) + \frac{\eta}{\delta}zq_2'(z),$$

then

$$q_1(z) \prec \left[\frac{(1-\sigma)zS^{\alpha}_{c-1,r,1,n,\lambda}f(z) + 2\sigma zS^{\alpha}_{c,r,1,n,\lambda}f(z)}{\sigma}\right]^{\delta} \prec q_2(z),$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

6. Conclusions

This study introduces significant advancements in the theory of differential subordination and superordination for meromorphic univalent functions via a novel Hadamard product operator. By deriving new sandwich theorems, the research connects dominant and subordinant functions under defined geometric constraints, providing a unified framework for analyzing such functions in the punctured unit disk. The findings enhance the understanding of convexity and starlikeness properties in this context, offering a foundation for broader applications in analytic function theory. Future work may extend these results to higher-order equations and explore new operators for further theoretical and practical advancements.

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