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FIXED POINT RESULTS IN A MULTIPLICATIVE b_2-METRIC SPACE

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Abstract. The main purpose of this paper is to provide a generalization of the concept of a multiplicative metric space. We prove some fixed point results that satisfy some generalized contraction mapping related to the multiplicative b_2-metric space. Furthermore, we provide examples to justify the generalization.

1. Introduction

Grossman et. al, introduced a new kind of calculus called multiplicative calculus by interchanging the roles of subtraction and addition with the role of division and multiplication, respectively, [10]. Prompted by this idea Bashirov et. al. introduced the concept of a multiplicative metric space, [5]. Ozavsar et. al. proved properties of multiplicative metric space and proved some fixed point results in a multiplicative metric space, [13]. In [8], authors proved the existence of fixed point of contractions of rational type multiplicative metric

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spaces. Other results on fixed point theory in a multiplicative metric space and other types of contractions can be found in [2, 3, 6, 9, 11, 12, 14, 15].

In this paper, we prove some fixed point results in a multiplicative metric space inspired by the concept of a b_2-metric and provide an example to justify the generalization.

Definition 1.1. ([4]) Let X be a nonempty set and $s \geq 1$ be a real number. A mapping $d : X \times X \to [1, \infty)$ is a multiplicative b-metric if it satisfies the following conditions:

(i) $d(x, y) > 1$ for all $x, y \in X$ and $x \neq y$.
(ii) $d(x, y) = 1$ if and only if $x = y$.
(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(iv) $d(x, z) \leq [d(x, y)d(y, z)]^s$.

Then (X, d) is a multiplicative b-metric space.

In the special case $s = 1$, we get a multiplicative metric introduced in [5].

Example 1.2. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to [1, \infty)$ by

$$d(x, y) = e^{|x-y|^\xi}$$

for $x, y \in X$ and $\xi > 1$. Then, (X, d) is a multiplicative b-metric space.

Properties (i)-(iii) of Definition 1.1 can be easily verified. We shall verify property (iv) of Definition 1.1. For x, y, z, we get

$$\begin{aligned} |x-y|^\xi &= |x-z+z-y|^\xi \\ &\leq \left| 2\left[\frac{1}{2}(x-z) + \frac{1}{2}(z-y)\right] \right|^\xi \\ &\leq 2^{\xi-1}\left[|(x-z)|^\xi + |(z-y)|^\xi\right]. \end{aligned}$$

Since the exponential function is an increasing function, we get

$$\begin{aligned} d(x, y) &= e^{|x-y|^\xi} \\ &\leq \left[e^{|x-z|^\xi} e^{|z-y|^\xi}\right]^{2^{\xi-1}} \\ &= [d(x, z)d(z, y)]^{2^{\xi-1}}, \end{aligned}$$

where $s = 2^{\xi-1}$. Thus d is a multiplicative b-metric and (X, d) is a multiplicative b-metric space.

2. Preliminary

The following definition was inspired by the definition of a b_2-metric.

Definition 2.1. Let X be a nonempty set and $s \geq 1$ be a real number. A mapping $d : X \times X \times X \to [1, \infty)$ is a multiplicative b_2-metric if it satisfies the following conditions:

(i) $d(x, y, z) > 1$ for all $x, y, z \in X$ and $x \neq y \neq z$.
(ii) $d(x, y, z) = 1$ if and only if $x = y = z$.
(iii) $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$ for all $x, y, z \in X$.
(iv) $d(x, y, z) \leq [d(x, y, t)d(y, z, t)d(z, x, t)]^s$ for all $x, y, z, t \in X$.

Then (X, d) is a multiplicative b_2-metric space.

In the special case $s = 1$, we obtain a multiplicative 2-metric. The following example justifies the concept of a multiplicative b_2-metric.

Example 2.2. Let $X = \left[1, \eta^{\frac{1}{\xi(\eta-1)}}\right)$ and define a mapping $d : X \times X \times X \to [1, \infty)$ by

$$d(x, y, z) = e^{\left||x-y|^\xi + |y-z|^\xi + |z-x|^\xi\right|^\eta}$$

for $x, y, z \in X$ and real number $\xi, \eta > 1$. Then (X, d) is a multiplicative b_2-metric space.

Properties (i)-(iii) of Definition 2.1 can be easily verified. We shall show property (iv) of Definition 2.1. For x, y, z, we get

$$\begin{aligned}\left||x-y|^\xi + |y-z|^\xi + |z-x|^\xi\right|^\eta &= 3^\eta \left|\frac{1}{3}|x-y|^\xi + \frac{1}{3}|y-z|^\xi + \frac{1}{3}|z-x|^\xi\right|^\eta \\ &\leq 3^{\eta-1}\left[|x-y|^{\xi\eta} + |y-z|^{\xi\eta} + |z-x|^{\xi\eta}\right].\end{aligned}$$

For the term $|x-y|^{\xi\eta}$, we get

$$|x-y|^{\xi\eta} \leq |x-y|^{\xi\eta} + |y-t|^{\xi\eta} + |t-x|^{\xi\eta}$$

for $x, y, t \in X$. Similar result can be obtained for the remaining terms. Thus, we get

$$\begin{aligned}\left||x-y|^\xi + |y-z|^\xi + |z-x|^\xi\right|^\eta \leq 3^{\eta-1}\Big[&|x-y|^{\xi\eta} + |y-t|^{\xi\eta} + |t-x|^{\xi\eta} \\ &+ |y-z|^{\xi\eta} + |z-t|^{\xi\eta} + |t-y|^{\xi\eta} \\ &+ |z-x|^{\xi\eta} + |x-t|^{\xi\eta} + |t-z|^{\xi\eta}\Big].\end{aligned}$$

Since the exponential function $(\cdot) \to e^{(\cdot)}$ is an increasing function and $x^{\xi\eta} \leq \eta x^{\xi}$ for $1 \leq x \leq \eta^{\frac{1}{\xi(\eta-1)}}$, we get

$$
\begin{aligned}
&d(x,y,z)\\
&= e^{\left||x-y|^{\xi}+|y-z|^{\xi}+|z-x|^{\xi}\right|^{\eta}}\\
&\leq e^{3^{\eta-1}\left[|x-y|^{\xi\eta}+|y-t|^{\xi\eta}+|t-x|^{\xi\eta}+|y-z|^{\xi\eta}+|z-t|^{\xi\eta}+|t-y|^{\xi\eta}+|z-x|^{\xi\eta}+|x-t|^{\xi\eta}+|t-z|^{\xi\eta}\right]}\\
&= e^{3^{\eta-1}\left[|x-y|^{\xi\eta}+|y-t|^{\xi\eta}+|t-x|^{\xi\eta}\right]} e^{3^{\eta-1}\left[|y-z|^{\xi\eta}+|z-t|^{\xi\eta}+|t-y|^{\xi\eta}\right]}\\
&\quad \times e^{3^{\eta-1}\left[|z-x|^{\xi\eta}+|x-t|^{\xi\eta}+|t-z|^{\xi\eta}\right]}\\
&\leq \left[\left(e^{|x-y|^{\xi}+|y-t|^{\xi}+|t-x|^{\xi}}\right)^{\eta}\left(e^{|y-z|^{\xi}+|z-t|^{\xi}+|t-y|^{\xi}}\right)^{\eta}\left(e^{|z-x|^{\xi}+|x-t|^{\xi}+|t-z|^{\xi}}\right)^{\eta}\right]^{3^{\eta-1}}\\
&= \left[d(x,y,t)d(y,z,t)d(z,x,t)\right]^{3^{\eta-1}},
\end{aligned}
$$

where $s = 3^{\eta-1}$. Hence, (X,d) is a multiplicative b_2-metric space.

Example 2.3. Let $X = [1,\infty)$ and define a mapping $d : X \times X \times X \to [1,\infty)$ by

$$d(x,y,z) = \left(e^{|x-y|^{\xi}+|y-z|^{\xi}+|z-x|^{\xi}}\right)^{\eta}$$

for $x,y,z \in X$ and $\xi,\eta \geq 1$. It can easily be shown that d is a multiplicative 2-metric.

Definition 2.4. Let (X,d) be a multiplicative b_2-metric space, $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$. The sequence is convergent to x if $d(x_n,x,z) \to 1$ as $n \to \infty$ for all $z \in X$.

Definition 2.5. Let (X,d) be a multiplicative b_2-metric space, $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$. The sequence is Cauchy if $d(x_n,x_m,z) \to 1$ as $n,m \to \infty$ for all $z \in X$.

Definition 2.6. The multiplicative b_2-metric space (X,d) is complete if every Cauchy sequence is convergent to some $x \in X$.

In the definition to follow we have extended the concept of a multiplicative contraction mapping to three dimensions found in [13].

Definition 2.7. Let (X,d) be a multiplicative b_2-metric space. A mapping $T : X \to X$ is a multiplicative contraction mapping if there exists a real number $0 < \alpha < 1$ such that

$$d(Tx,Ty,z) \leq [d(x,y,z)]^{\alpha} \tag{2.1}$$

for all $x,y,z \in X$.

Definition 2.8. Let (X,d) be a multiplicative b_2-metric space and for $x,y\in X$ and $\varepsilon>1$ define $B_\varepsilon(x,y)=\{z\in X : d(x,y,z)<\varepsilon\}$ which is the multiplicative open balls of radius ε with center (x,y).

Definition 2.9. Let (X,d) be a multiplicative b_2-metric space and $A\subset X$ then $x\in A$ is an interior point of A if there exists $\varepsilon>1$ and $y\in A$ such that $B_\varepsilon(x,y)\subset A$.

Every multiplicative metric space is a topological space based on the set of open balls.

Theorem 2.10. *Let (X,d) be a multiplicative b_2-metric space. Every multiplicative convergent sequence in X is a multiplicative Cauchy sequence in X.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary convergent sequence in X. Then there exists an $x\in X$ such that $d(x_n,x,z)\to 1$ for any $z\in X$. Hence for $\varepsilon>1$ there exist $n_1\in\mathbb{N}$ such that $d(x_n,x,z)<(\varepsilon)^{\frac{1}{3s}}$ for $n\geq n_1$ and there exist $n_2\in\mathbb{N}$ such that $d(x_m,x,z)<(\varepsilon)^{\frac{1}{3s}}$. For $n,m\geq\max\{n_1,n_2\}$, we get

$$\begin{aligned}d(x_n,x_m,z)&\leq[d(x_n,x_m,x)]^s[d(x_m,z,x)]^s[d(z,x_n,x)]^s\\&<\sqrt[3]{\varepsilon}\sqrt[3]{\varepsilon}\sqrt[3]{\varepsilon}=\varepsilon,\end{aligned}$$

which implies that $\{x_n\}_{n\in\mathbb{N}}$ is a multiplicative Cauchy sequence. □

Definition 2.11. Let (X,d) be a multiplicative b_2-metric space and $A\subset X$ then A is bounded if for $x,y\in A$ there exists $M>1$ such that $B_M(x,y)\subset A$.

Theorem 2.12. *Let (X,d) is a multiplicative b_2-metric space. Every multiplicative Cauchy sequence is bounded.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a multiplicative Cauchy sequence in X. Then, for $\varepsilon=2>1$, there exist $n_1\in\mathbb{N}$ such that $d(x_n,x_m,z)<2$ for all $n,m\geq n_1$. Hence, set $M^{\frac{1}{3s}}=\max\{2,d(x_1,x_{n_1},z),\cdots,d(x_n,x_{n_1},z)\}$, then it follows that $d(x_n,x_{n_1},z)<M^{\frac{1}{3s}}$ for all $n\in\mathbb{N}$. Thus, we get

$$\begin{aligned}d(x_n,x_m,z)&\leq[d(x_n,x_m,x_{n_1})]^s[d(x_m,z,x_{n_1})]^s[d(x_n,z,x_{n_1})]^s\\&<\sqrt[3]{M}\sqrt[3]{M}\sqrt[3]{M}=M\end{aligned}$$

for $n,m\in\mathbb{N}$, which implies the sequence is bounded. □

3. Main results

In the following theorem, we provide fixed point results for a contraction type mapping in a multiplicative b_2-metric space.

Theorem 3.1. *Let (X, d) be a complete multiplicative b_2-metric space. If a mapping $T : X \to X$ satisfies the condition:*

$$d(Tx, Ty, z) \le [d(x, Tx, z)]^\alpha [d(y, Ty, z)]^\beta [d(x, y, z)]^\gamma \tag{3.1}$$

with α, β, γ are non-negative real numbers such that $\alpha+\beta+\gamma < 1$, then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X. For each $n \in \mathbb{N}$, define the sequence $x_{n+1} = Tx_n$. For the sequence $\{x_n\}_{n\in\mathbb{N}}$, we get

$$\begin{aligned} d(x_{n+1}, x_n, z) &= d(Tx_n, Tx_{n-1}, z) \\ &\le [d(x_n, Tx_n, z)]^\alpha [d(x_{n-1}, Tx_{n-1}, z)]^\beta [d(x_{n-1}, x_n, z)]^\gamma \\ &= [d(x_n, x_{n+1}, z)]^\alpha [d(x_{n-1}, x_n, z)]^\beta [d(x_{n-1}, x_n, z)]^\gamma. \end{aligned} \tag{3.2}$$

It follows from (3.2) that

$$d(x_{n+1}, x_n, z) \le [d(x_{n-1}, x_n, z)]^{\frac{\beta+\gamma}{1-\alpha}} = [d(x_{n-1}, x_n, z)]^\xi. \tag{3.3}$$

Since $\beta + \gamma < 1 - \alpha$, it follows that $\xi = \frac{\beta+\gamma}{1-\alpha} < 1$. Thus T is a multiplicative contraction mapping. Repeated use of (3.3), we get

$$\begin{aligned} d(x_{n+1}, x_n, z) &\le [d(x_{n-1}, x_n, z)]^\xi \\ &\le [d(x_{n-2}, x_{n-1}, z)]^{\xi^2} \\ &\vdots \\ &\le [d(x_1, x_0, z)]^{\xi^n}. \end{aligned} \tag{3.4}$$

We claim that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. For $n, m \in \mathbb{N}$ and $t \in X$, we get

$$d(x_n, x_{n+m}, z) \le [d(x_n, x_{n+m}, t)]^s [d(x_{n+m}, z, t)]^s [d(z, x_n, t)]^s. \tag{3.5}$$

Taking $t = x_{n+1}$ in (3.5), we obtain

$$\begin{aligned} d(x_n, x_{n+m}, z) &\le [d(x_n, x_{n+m}, x_{n+1})]^s [d(x_{n+m}, z, x_{n+1})]^s [d(z, x_n, x_{n+1})]^s \\ &\le [d(x_1, x_0, z)]^{s\xi^n} [d(x_{n+m}, z, x_{n+1})]^s [d(x_1, x_0, z)]^{s\xi^n}. \end{aligned} \tag{3.6}$$

It follows that

$$\begin{aligned} &d(x_{n+m}, z, x_{n+1}) \\ &\quad \le [d(x_{n+m}, z, x_{n+2})]^s [d(z, x_{n+1}, x_{n+2})]^s [d(x_{n+1}, x_{n+m}, x_{n+2})]^s \\ &\quad \le [d(x_1, x_0, z)]^{s\xi^{n+1}} [d(x_{n+m}, z, x_{n+2})]^s [d(x_1, x_0, z)]^{s\xi^{n+1}}. \end{aligned} \tag{3.7}$$

Using (3.7) in (3.6), we obtain

$$\begin{aligned} d(x_n, x_{n+m}, z) \le & [d(x_1, x_0, z)]^{s\xi^n}[d(x_1, x_0, z)]^{s^2\xi^{n+1}}[d(x_{n+m}, z, x_{n+2})]^{s^2} \\ & \times [d(x_1, x_0, z)]^{s^2\xi^{n+1}}[d(x_1, x_0, z)]^{s\xi^n}. \end{aligned} \tag{3.8}$$

Proceeding in a similar manner, we get

$$\begin{aligned} d(x_n, x_{n+m}, z) &\le \left([d(x_1, x_0, z)]^{s\xi^n}[d(x_1, x_0, z)]^{s^2\xi^{n+1}} \cdots [d(x_1, x_0, z)]^{s^m\xi^{m-1}}\right)^2 \\ &\le \left([d(x_1, x_0, z)]^{s\xi^n\left(1+s\xi+s^2\xi^2+\cdots+s^{m-1}\xi^{m-1}\right)}\right)^2 \\ &= \left([d(x_1, x_0, z)]^{s\xi^n\frac{1-(s\xi)^m}{1-s\xi}}\right)^2. \end{aligned}$$

Taking the limit as $n \to \infty$, we get $d(x_n, x_{n+m}, z) \to 1$ as $n \to \infty$, thus $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is a complete multiplicative metric space, there exists $x^* \in X$ such that $d(x_n, x^*, z) \to 1$ as $n \to \infty$.

We claim that x^* is a fixed point of T. From

$$d(x, Tx, z) \le [d(x, Tx, t)]^s[d(Tx, z, t)]^s[d(z, x, t)]^s, \tag{3.9}$$

taking $t = x_n$, we get

$$\begin{aligned} d(x, Tx, z) &\le [d(x, Tx, x_n)]^s[d(Tx, z, x_n)]^s[d(z, x, x_n)]^s \\ &= [d(x, Tx, x_n)]^s[d(Tx, z, Tx_{n-1})]^s[d(z, x, x_n)]^s. \end{aligned} \tag{3.10}$$

Using the contraction condition, we get

$$\begin{aligned} d(x, Tx, z) &\le [d(x, Tx, x_n)]^s[d(Tx, z, Tx_{n-1})]^s[d(z, x, x_n)]^s \\ &\le [d(x, Tx, x_n)]^s[d(x_{n-1}, Tx_{n-1}, z)]^{s\alpha} \\ &\quad \times [d(x, Tx, z)]^{s\beta}[d(x_{n-1}, x, z)]^{s\gamma}[d(z, x, x_n)]^s \\ &= [d(x, Tx, x_n)]^s[d(x_{n-1}, x_n, z)]^{s\alpha} \\ &\quad \times [d(x, Tx, z)]^{s\beta}[d(x_{n-1}, x, z)]^{s\gamma}[d(z, x, x_n)]^s \\ &\le [d(x, Tx, x_n)]^s[d(x_1, x_0, z)]^{\xi^{n-1}s\alpha} \\ &\quad \times [d(x, Tx, z)]^{s\beta}[d(x_{n-1}, x, z)]^{s\gamma}[d(z, x, x_n)]^s. \end{aligned} \tag{3.11}$$

It follows that

$$\begin{aligned} 1 &\le d(x, Tx, z) \\ &\le [d(x, Tx, x_n)]^{\frac{s}{1-s\beta}}[d(x_1, x_0, z)]^{\frac{\xi^{n-1}s\alpha}{1-s\beta}}[d(x_{n-1}, x, z)]^{\frac{s\gamma}{1-s\beta}}[d(z, x, x_n)]^{\frac{s}{1-s\beta}}. \end{aligned} \tag{3.12}$$

Taking the limit as $n \to \infty$, we get $d(x, Tx, z) \to 1$ as $n \to \infty$.

Next, we show the uniqueness of the fixed point. Assume that there exists $y^* \neq x^* \in X$ such that $Ty^* = y^*$. It follows that

$$\begin{aligned} d(x^*, y^*, z) &= d(Tx^*, Ty^*, z) \\ &\leq [d(x^*, Tx^*, z)]^\alpha [d(y^*, Ty^*, z)]^\beta [d(x^*, y^*, z)]^\gamma \\ &\leq [d(x^*, y^*, z)]^\gamma. \end{aligned} \tag{3.13}$$

This is a contradiction, unless we take $d(x^*, y^*, z) = 1$ which implies that $x^* = y^*$. □

In the results to follow, we investigate fixed point results extended to a multiplicative 2-metric for weakly multiplicative contraction mappings using control functions as defined in [1].

Definition 3.2. Let (X, d) be a multiplicative 2-metric space. A mapping T is a weakly multiplicative contraction mapping if there exists a continuous non-decreasing function $\phi : [1, \infty) \to [1, \infty)$ satisfying:

$$d(Tx, Ty, z) \leq \frac{d(x, y, z)}{\phi(d(x, y, z))} \tag{3.14}$$

for all $x, y, z \in X$ with $\phi(t) = 1$ if and only if $t = 1$.

One can easily verify that, if we choose $\phi(t) = t^{1-\alpha}$ for $0 < \alpha < 1$, we get $d(Tx, Ty, z) \leq d(x, y, z)^\alpha$. Every multiplicative contraction mapping is a weakly multiplicative contraction mapping.

Theorem 3.3. *Let (X, d) be a multiplicative 2-metric space and $T : X \to X$ be a weakly multiplicative contraction mapping. If (X, d) is complete, then T has a unique fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. Then define the sequence $\{x_n\}_{n\in\mathbb{N}}$ as follows $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. We claim that the sequence $\{t_n\}_{n\in\mathbb{N}}$, where $t_n = d(x_{n+1}, x_n, z)$, converges in the interval $[1, \infty)$. Using (3.14) for the sequence $\{t_n\}_{n\in\mathbb{N}}$, we get

$$\begin{aligned} t_{n+1} &= d(x_{n+2}, x_{n+1}, z) \\ &= d(Tx_{n+1}, Tx_n, z) \\ &\leq \frac{d(x_{n+1}, x_n, z)}{\phi(d(x_{n+1}, x_n, z))} \\ &= \frac{t_n}{\phi(t_n)}. \end{aligned} \tag{3.15}$$

Since $\phi(t) \geq 1$, we conclude from (3.15) that $t_{n+1} \leq t_n$. Thus the sequence is decreasing and bounded from below by 1, hence the sequence converges to some $t \in \mathbb{R}$. Taking the limit as $n \to \infty$ in (3.15) and using the continuity of the function ϕ, we get

$$t \leq \frac{t}{\phi(t)}, \tag{3.16}$$

which holds true, unless we take $\phi(t) = 1$ or $t = 1$. In either case, we conclude that $t = 1$. Thus, we get $\lim_{n\to\infty} d(x_{n+1}, x_n, z) = \lim_{n\to\infty} t_n = 1$.

Next, we prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, that is, $d(x_n, x_m, z) \to 1$ as $n, m \to \infty$. We assume that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence in X, then exists $\varepsilon > 1$ and sequences $\{m_k\}_{k\in\mathbb{N}}$ and $\{n_k\}_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $d(x_{n_k}, x_{m_k}, z) \geq \varepsilon$ for $n_k > m_k > k$. It follows that $d(x_{n_k}, x_{m_k-1}, z) < \varepsilon$. Since $d(x_{n+1}, x_n, z) \to 1$ as $n \to \infty$ and $\{d(x_{m_k-1}, x_{m_k}, z)\}_{k\in\mathbb{N}}$ is a subsequence, it follows that $d(x_{m_k-1}, x_{m_k}, z) \to 1$ as $k \to \infty$. From the multiplicative inequality, we obtain

$$\begin{aligned} \varepsilon &\leq d(x_{n_k}, x_{m_k}, z) \\ &\leq [d(x_{n_k}, x_{m_k}, x_{m_k-1})][d(x_{m_k}, z, x_{m_k-1})][d(z, x_{n_k}, x_{m_k-1})], \end{aligned} \tag{3.17}$$

which implies that

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}, z) = \varepsilon. \tag{3.18}$$

Next, we shall show that $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}, z) = \varepsilon$. From the inequality that follows

$$d(x_{m_k-1}, x_{n_k}, z) \leq d(x_{m_k-1}, x_{n_k}, x_{m_k}, z) d(x_{n_k}, z, x_{m_k}) d(z, x_{m_k-1}, x_{m_k}), \tag{3.19}$$

we get

$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}, z) \leq \varepsilon. \tag{3.20}$$

From the inequality that follows

$$d(x_{m_k}, x_{n_k}, z) \leq d(x_{m_k}, x_{n_k}, x_{m_k-1}) d(x_{n_k}, z, x_{m_k-1}) d(z, x_{m_k}, x_{m_k-1}), \tag{3.21}$$

we get

$$\varepsilon \leq \lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}, z). \tag{3.22}$$

Combining inequalities (3.20) and (3.22), we get obtain

$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}, z) = \varepsilon. \tag{3.23}$$

In a similar manner, we shall show that

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}, z) = \varepsilon.$$

Consider the inequality that follows

$$d(x_{m_k}, x_{n_k+1}, z) \leq d(x_{m_k}, x_{n_k+1}, x_{n_k}) d(x_{n_k+1}, z, x_{n_k}) d(z, x_{m_k}, x_{n_k}), \tag{3.24}$$

from which we obtain

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}, z) \leq \varepsilon. \tag{3.25}$$

Now, consider the inequality

$$d(x_{m_k}, x_{n_k}, z) \leq d(x_{m_k}, x_{n_k}, x_{n_k+1}) d(x_{n_k+1}, z, x_{n_k+1}) d(z, x_{m_k}, x_{n_k+1}) \tag{3.26}$$

from which, we get

$$\varepsilon \leq \lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}, z). \tag{3.27}$$

Combining inequalities (3.25) and (3.27), we get

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}, z) = \varepsilon. \tag{3.28}$$

Finally, using the inequality (3.14), we obtain

$$\begin{aligned} d(x_{n_k+1}, x_{m_k}, z) &= d(Tx_{n_k}, Tx_{m_k-1}, z) \\ &\leq \frac{d(x_{n_k}, x_{m_k-1}, z)}{\phi(d(x_{n_k}, x_{m_k-1}, z))}. \end{aligned} \tag{3.29}$$

Taking the limit $k \to \infty$ on both sides of inequality (3.29), we get $\varepsilon \leq \frac{\varepsilon}{\phi(\varepsilon)}$. This leads to a contradiction, as we require that $\phi(\varepsilon) = 1$ which implies that $\varepsilon = 1$. Hence, we conclude that $\lim_{m,n\to\infty} d(x_n, x_m, z) = 1$. Thus the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X.

Since (X, d) is a complete 2-metric space, we get that there exists $x^* \in X$ such that $d(x_n, x, z) \to 1$ as $n \to \infty$. We claim that x^* is a fixed point for T. Using the contraction (3.14), we get

$$\begin{aligned} d(Tx^*, x^*, z) &\leq [d(Tx^*, x^*, x_{n+1})][d(x^*, z, x_{n+1})][d(z, Tx^*, x_{n+1})] \\ &\leq [d(Tx^*, x^*, x_{n+1})][d(x^*, z, x_{n+1})][d(z, Tx^*, Tx_n)] \\ &\leq [d(Tx^*, x^*, x_{n+1})][d(x^*, z, x_{n+1})] \left(\frac{d(x^*, x_n, z)}{\phi(d(x^*, x_n, z))} \right). \end{aligned} \tag{3.30}$$

Taking the limit as $n \to \infty$ in (3.30), we get $1 \leq d(Tx^*, x^*, z) \leq 1$. Thus, we get $Tx^* = x^*$.

Finally, we prove the uniqueness of the fixed point. Assume that there exists $y^* \in X$ such that $Ty^* = y^*$. Using the contraction (3.14), we obtain

$$\begin{aligned} d(x^*, y^*, z) &= d(Tx^*, Ty^*, z) \\ &\leq \frac{d(x^*, y^*, z)}{\phi(d(x^*, y^*, z))}. \end{aligned} \tag{3.31}$$

From inequality (3.31), we get that $\phi(d(x^*, y^*, z)) = 1$, which implies that $d(x^*, y^*, z) = 1$, thus $x^* = y^*$. □

Corollary 3.4. *Let (X, d) be a multiplicative 2-metric space and $T : X \to X$ satisfying the contraction*

$$d(T^n x, T^n y, z) \leq \frac{d(x, y, z)}{\phi(d(x, y, z))} \tag{3.32}$$

for $x, y, z \in X$, $n \in \mathbb{N}$ and where $\phi : [1, \infty) \to [1, \infty)$ is a continuous function such that $\phi(t) = 1$ if and only if $t = 1$. If (X, d) is complete, then T has a unique fixed point.

Proof. Define $T = T^n$, then the proof follows in a similar manner as Theorem 3.3. □

Example 3.5. Let $X = [1, \infty)$, define $d : X \times X \times X \to [1, \infty)$ by

$$d(x, y, z) = e^{|x-y|^\xi + |y-z|^\xi + |z-x|^\xi}$$

for all $x, y, z \in X$ and $T : X \to X$ by

$$Tx = \sqrt{x}.$$

Using the Mean value theorem, we obtain

$$\begin{aligned} d(Tx, Ty, z) &= e^{\left|\sqrt{x}-\sqrt{y}\right|^\xi + \left|\sqrt{y}-z\right|^\xi + \left|z-\sqrt{x}\right|^\xi} \\ &\leq e^{\frac{1}{2}|x-y|^\xi + \frac{1}{2}|y-z|^\xi + \frac{1}{2}|z-x|^\xi} \\ &= \frac{e^{|x-y|^\xi + |y-z|^\xi + |z-x|^\xi}}{e^{\frac{1}{2}|x-y|^\xi + \frac{1}{2}|y-z|^\xi + \frac{1}{2}|z-x|^\xi}} \\ &= \frac{d(x, y, z)}{\phi(d(x, y, z))}, \end{aligned} \tag{3.33}$$

where $\phi(t) = t^{\frac{1}{2}}$. It follows from Theorem 3.1 that T has a unique fixed point in X.

4. Conclusion

In this paper, some fixed point results are proved in relation to the multiplicative b_2-metric space. We provided examples for the generalization. The results can be applied for other contraction type mappings which is for future research.

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