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A INTEGRAL EQUATION RELATED TO THE MACROECONOMIC MODELS

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Abstract. The paper is devoted to the study of a Volterra integral equation of the second kind related to the macroeconomic models. Using contraction mapping principle and some techniques of nonlinear analysis, the uniqueness existence, approximation and asymptotic expansions of solutions with respect to small parameters appeared in the model are established.

1. Introduction

In [1], [2], from the analysis of macroeconomic models as the Harrod-Domar model, Phillips, or Leontief model and other models, S. I. Chernyshov and the authors have pointed out the need to establish a model in which the integral equation is used to model the process of economic. Specifically, in [1], a methodology based on balance relations for modelling of economic dynamics, including forecast estimates, is developed. The problems considered are reduced to Volterra and Fredholm integral equations of the second kind, such

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as the following integral

$$x(t) = q(t) + \int_1^t k(t,s)x(s)ds,\ t \geq 1, \tag{1.1}$$

where

$$\begin{cases} k(t,s) = -\alpha - \beta - \frac{\beta - s}{t}, \\ q(t) = -\alpha - \frac{\beta-(1-t)}{t}Y_1' + \frac{\gamma}{t}Y_1, \end{cases} \tag{1.2}$$

$\alpha, \beta, \gamma, Y_1, Y_1'$ are given constants, the solution of this equation can be found via the procedure of successive approximations (see [1], page 15).
The above ideas leads to study the existence and some properties of the solution of Eq. (1.1). Note that, by the transformation $t = \tau + 1$, $s = r + 1$, $\tau \geq 0$, $r \geq 0$, and setting

$$\begin{cases} \bar{x}(\tau) = x(t) = x(\tau+1), \\ \bar{q}(\tau) = q(t) = q(\tau+1) = -\alpha - \frac{\beta+\tau}{\tau+1}Y_1' + \frac{\gamma}{\tau+1}Y_1, \\ \bar{k}(\tau, r) = k(t,s) = k(\tau+1, r+1) = -\alpha - \beta - \frac{\beta-1-r}{\tau+1}, \end{cases} \tag{1.3}$$

then Eq. (1.1) has the form

$$\bar{x}(\tau) = \bar{q}(\tau) + \int_0^\tau \bar{k}(\tau, r)\bar{x}(r)dr,\ \tau \geq 0, \tag{1.4}$$

with $\bar{q} \in C^\infty(\mathbb{R}_+)$, $\bar{k} \in C^\infty(\triangle)$, $\triangle = \{(\tau, r) \in \mathbb{R}_+^2 : r \leq \tau\}$.

This paper consists of five sections. In section 2, the uniqueness existence and some properties of the solution are proved. Section 3 presents an approximate of a solution of Eq. (1.1) by a convergent sequence of polynomials. Finally, in sections 4 and 5, asymptotic expansions of solutions with respect to small parameters appeared in the model are established.

2. The existence and some properties of solutions

We rewrite Eq. (1.4) as follows

$$x(t) = q(t) + \int_0^t k(t,s)x(s)ds,\ t \geq 0, \tag{2.1}$$

where

$$\begin{cases} q(t) = -\alpha - \frac{\beta+t}{t+1}Y_1' + \frac{\gamma}{t+1}Y_1,\ t \geq 0, \\ k(t,s) = -\alpha - \beta - \frac{\beta-1-s}{t+1},\ 0 \leq s \leq t. \end{cases} \tag{2.2}$$

Because $\alpha, \beta, \gamma, Y_1, Y_1'$ are given constants, it is clear that $q \in C^\infty(\mathbb{R}_+)$ and $k \in C^\infty(\triangle)$, $\triangle = \{(t,s) \in \mathbb{R}_+^2 : s \leq t\}$.

Theorem 2.1. *With the functions q, k defined by* (2.2), *Eq.*(2.1) *has a unique solution $x \in C^\infty(\mathbb{R}_+)$.*

Proof. Using contraction mapping principle, it is not difficult to show that for any $T > 0$, Eq. (2.1) has a unique solution $x \in C([0,T])$ satisfying

$$x(t) = q(t) + \int_0^t k(t,s)x(s)ds, \ 0 \le t \le T. \tag{2.3}$$

On the other hand, we have

$$|x(t)| \le \sup_{0\le t\le T} |q(t)| + \sup_{0\le s\le t\le T} |k(t,s)| \int_0^t |x(s)|\, ds, \ 0 \le t \le T, \tag{2.4}$$

by applying Gronwall's lemma, it implies that

$$|x(t)| \le \sup_{0\le t\le T} |q(t)| \exp\left(T \sup_{0\le s\le t\le T} |k(t,s)|\right) \equiv C_T, \ 0 \le t \le T. \tag{2.5}$$

Therefore, it can be extended to obtain a unique solution $x \in C^0(\mathbb{R}_+)$ of Eq. (2.1). Furthermore, the right hand side of (2.1) is the function $H_1(t)$ with its derivative as follows

$$H_1'(t) = q'(t) + k(t,t)\ x(t) + \int_0^t \frac{\partial k}{\partial t}(t,s)x(s)ds, \tag{2.6}$$

so $H_1 \in C^1(\mathbb{R}_+)$. Hence, $x \in C^1(\mathbb{R}_+)$. The same manner leads to $x \in C^2(\mathbb{R}_+)$, $x \in C^3(\mathbb{R}_+)$, $\cdots$, $x \in C^k(\mathbb{R}_+)$, for all $k = 1, 2, \cdots$, it means that $x \in C^\infty(\mathbb{R}_+)$. □

Theorem 2.2. *Assume* $x_0 \in C([0,T])$ *and let the recurrent sequence* $\{x_m\}$ *be defined by*

$$x_m(t) = q(t) + \int_0^t k(t,s)x_{m-1}(s)ds, \ 0 \le t \le T, \ m = 1, 2, \cdots. \tag{2.7}$$

Then

$$\sup_{0\le t\le T} |x_m(t) - x(t)| \to 0, \quad \text{as } m \to +\infty. \tag{2.8}$$

Proof. By applying the contraction mapping principle, Theorem 2.2 can easily be proven. □

According to the Weierstrass Theorem, for any $T > 0$, the functions $q(t)$, $k(t,s)$ are approximated by sequences of polynomials with an variable or two variables $\{q_n(t)\}$, $\{k_n(t,s)\}$, respectively. Let $x_n(t)$ be the solution of (2.1) corresponding to $q(t) = q_n(t)$, $k(t,s) = k_n(t,s)$.

Then we have the following theorem.

Theorem 2.3. *Assume that*

$$\begin{cases} \|q_n - q\|_{C([0,T])} \equiv \sup\limits_{0\le t\le T} |q_n(t) - q(t)| \to 0, \\ \|k_n - k\|_{C(\triangle_T)} \equiv \sup\limits_{0\le s\le t\le T} |k_n(t,s) - k(t,s)| \to 0, \end{cases} \tag{2.9}$$

as $n \to +\infty$, *with* $\triangle_T = \{(t,s) : 0 \le s \le t \le T\}$. *Then*

$$\lim_{n\to+\infty} \|x_n - x\|_{C([0,T])} = 0. \tag{2.10}$$

Furthermore

$$\|x_n - x\|_{C([0,T])} \le C \left(\|q_n - q\|_{C([0,T])} + \|k_n - k\|_{C(\triangle_T)} \right), \tag{2.11}$$

for all $n \in \mathbb{N}$, *where* C *is a constant independent of* n.

Proof. Note that $x_n(t)$ satisfies the following equation

$$x_n(t) = q_n(t) + \int_0^t k_n(t,s)x_n(s)ds,\ 0 \le t \le T. \tag{2.12}$$

Putting $y_n(t) = x_n(t) - x(t)$, it follows from (2.1) and (2.12) that

$$y_n(t) = q_n(t) - q(t) + \int_0^t (k_n(t,s) - k(t,s))\, x_n(s)ds + \int_0^t k(t,s)y_n(s)ds. \tag{2.13}$$

Consequently

$$\begin{aligned} |y_n(t)| &\le |q_n(t) - q(t)| + \textstyle\int_0^t |k_n(t,s) - k(t,s)|\, |x_n(s)|\, ds \\ &\quad + \textstyle\int_0^t |k(t,s)|\, |y_n(s)|\, ds \\ &\le \|q_n - q\|_{C([0,T])} + T\, \|k_n - k\|_{C(\triangle_T)}\, \|x_n\|_{C([0,T])} \\ &\quad + \|k\|_{C(\triangle_T)} \textstyle\int_0^t |y_n(s)|\, ds. \end{aligned} \tag{2.14}$$

Applying Gronwall's Lemma, (2.14) leads to

$$\begin{aligned} &|y_n(t)| \\ &\le \left[\|q_n - q\|_{C([0,T])} + T\, \|k_n - k\|_{C(\triangle_T)}\, \|x_n\|_{C([0,T])} \right] \exp(T\, \|k\|_{C(\triangle_T)}), \end{aligned} \tag{2.15}$$

for all $t \in [0,T]$. Thus

$$\begin{aligned} &\|x_n - x\|_{C([0,T])} \\ &\le \left[\|q_n - q\|_{C([0,T])} + T\, \|k_n - k\|_{C(\triangle_T)}\, \|x_n\|_{C([0,T])} \right] \exp(T\, \|k\|_{C(\triangle_T)}). \end{aligned} \tag{2.16}$$

Moreover, (2.9) and (2.12) give the following inequality

$$|x_n(t)| \le \|q_n\|_{C([0,T])} + \|k_n\|_{C(\triangle_T)} \int_0^t |x_n(s)|\, ds,\ 0 \le t \le T. \tag{2.17}$$

Continuing applying Gronwall's Lemma, we conclude from (2.9) and (2.17) that

$$\|x_n\|_{C([0,T])} \leq \|q_n\|_{C([0,T])} \exp(T \|k_n\|_{C(\triangle_T)}) \leq C, \ \forall n = 1, 2, \cdots , \tag{2.18}$$

where C is a constant independent of n. Combining (2.16), (2.18), we obtain (2.11) and (2.10) follows. $\square$

3. An approximate of a solution by a convergent sequence of polynomials

In the above section, $\{x_n\}$ is defined by (2.12) so it may be not a sequence of polynomials. However, we can establish $\{x_n\}$ such that it is a sequence of polynomials and it convergences to a solution of (2.1).

For every $T > 0$, according to the Weierstrass Theorem, the functions $q(t)$, $k(t, s)$ are approximated by sequences of polynomials with an variable or two variables $\{q_n(t)\}$, $\{k_n(t, s)\}$, respectively, given by

$$\begin{aligned} &q_n(t) = \textstyle\sum_{k=0}^{n} q_k^{(n)} t^k, \ k_n(t,s) = \sum\limits_{i+j\leq n} k_{ij}^{(n)} t^i s^j, \\ &\|q_n - q\|_{C([0,T])} \to 0, \ \|k_n - k\|_{C(\triangle_T)} \to 0. \end{aligned} \tag{3.1}$$

We shall construct the recurrent sequence $\{x_n\}$ below

$$\begin{cases} x_0(t) = 0, \\ x_n(t) = q_n(t) + \int_0^t k_n(t,s) x_{n-1}(s) ds, \ 0 \leq t \leq T, \ n = 1, 2, \cdots . \end{cases} \tag{3.2}$$

It is clear to see that $\{x_n(t)\}$ is a sequence of polynomials with an variable t, and the following property is fulfilled.

Theorem 3.1.

$$\lim_{n\to+\infty} \|x_n - x\|_{C([0,T])} = 0.$$

Proof. In order to prove Theorem 3.1, we need the following lemma.

Lemma 3.2. *Let $A, B \geq 0$ be reals given, let $\{z_n\} \subset C(\mathbb{R}_+)$ be a sequence of continuous functions and positive such that*

$$\begin{aligned} &z_n(t) \leq A + B \textstyle\int_0^t z_{n-1}(s) ds, \ t \geq 0, \ n = 1, 2, \cdots . \\ &z_0 \in C(\mathbb{R}_+), \ \ z_0(t) \geq 0, \ \ t \geq 0. \end{aligned} \tag{3.3}$$

Then

$$\begin{aligned}
&\text{(i) } z_n(t) \le A\left[1 + Bt + B^2\tfrac{t^2}{2} + B^3\tfrac{t^3}{3!} + \cdots + B^k\tfrac{t^k}{k!}\right] \\
&\qquad + B^{k+1}\tfrac{t^k}{k!}\textstyle\int_0^t z_{n-k-1}(r)dr, \\
&\qquad t \ge 0, \quad k = 0,1,2,\cdots,n-1; \quad n = 1,2,\cdots, \qquad (3.4)\\
&\text{(ii) } z_n(t) \le A\left[1 + Bt + B^2\tfrac{t^2}{2} + B^3\tfrac{t^3}{3!} + \ldots + B^{n-1}\tfrac{t^{n-1}}{(n-1)!}\right] \\
&\qquad + B^n\tfrac{t^{n-1}}{(n-1)!}\textstyle\int_0^t z_0(r)dr, \quad t \ge 0, \; n = 1,2,\cdots.
\end{aligned}$$

Proof. The proof of Lemma 3.2 can be done relatively easily by induction. □

Now, we show that $\{x_n(t)\}$ is bounded in $C([0,T])$.

Because $\|q_n - q\|_{C([0,T])} \to 0$, $\|k_n - k\|_{C(\triangle_T)} \to 0$, there exists two positive reals A, B such that

$$\|q_n\|_{C([0,T])} \le A, \;\; \|k_n\|_{C(\triangle_T)} \le B, \; \forall n = 1,2,\cdots. \tag{3.5}$$

On the other hand, by (3.2) we get

$$|x_n(t)| \le A + B\int_0^t |x_{n-1}(s)|\, ds, \; 0 \le t \le T, \; n = 1,2,\cdots. \tag{3.6}$$

Using Lemma 3.2, (3.4) (ii), in which $z_n(t) = |x_n(t)|$, $z_0(t) = |x_0(t)| = 0$, (3.6) leads to

$$\begin{aligned}
|x_n(t)| &\le A\left[1 + Bt + B^2\tfrac{t^2}{2} + B^3\tfrac{t^3}{3!} + \ldots + B^{n-1}\tfrac{t^{n-1}}{(n-1)!}\right] \\
&\le A\exp(Bt), \quad 0 \le t \le T, \quad n = 1,2,\cdots.
\end{aligned} \tag{3.7}$$

Next, we show that $\lim\limits_{n\to+\infty} \|x_n - x\|_{C([0,T])} = 0$.

Putting $y_n(t) = x_n(t) - x(t)$ and

$$\bar{q}_n(t) = q_n(t) - q(t), \;\; \bar{k}_n(t,s) = k_n(t,s) - k(t,s). \tag{3.8}$$

It implies from (2.1) and (3.2) that

$$y_n(t) = \bar{q}_n(t) + \int_0^t \bar{k}_n(t,s)x_{n-1}(s)ds + \int_0^t k(t,s)y_{n-1}(s)ds. \tag{3.9}$$

Since $|x_n(t)| \le A\exp(Bt)$, (3.5) and (3.9) yield

$$|y_n(t)| \le \|\bar{q}_n\|_{C([0,T])} + \frac{A}{B}\exp(BT)\left\|\bar{k}_n\right\|_{C(\triangle_T)} + B\int_0^t |y_{n-1}(s)|\, ds. \tag{3.10}$$

For all $\varepsilon > 0$, by

$$\|\bar{q}_n\|_{C([0,T])} = \|q_n - q\|_{C([0,T])} \to 0, \;\; \left\|\bar{k}_n\right\|_{C(\triangle_T)} = \|k_n - k\|_{C(\triangle_T)} \to 0,$$

there exists $n_0 \in \mathbb{N}$ such that

$$\|\bar{q}_n\|_{C([0,T])} + \frac{A}{B}\exp(BT)\left\|\bar{k}_n\right\|_{C(\triangle_T)} < \frac{\varepsilon}{2\exp(BT)}, \quad \forall n \geq n_0. \tag{3.11}$$

Consequently

$$|y_n(t)| \leq \frac{\varepsilon}{2\exp(BT)} + B\int_0^t |y_{n-1}(s)|\, ds,\ \forall n \geq n_0. \tag{3.12}$$

Using Lemma 3.2 again, (3.4)(i), with $z_n(t) = |y_n(t)|$, $A = \frac{\varepsilon}{2\exp(BT)}$, $k = n - n_0$, we obtain

$$\begin{aligned}
|y_n(t)| &\leq \tfrac{\varepsilon}{2\exp(BT)}\left[1 + Bt + B^2\tfrac{t^2}{2} + B^3\tfrac{t^3}{3!} + \cdots + B^{n-n_0}\tfrac{t^{n-n_0}}{(n-n_0)!}\right] \\
&\quad + B^{n-n_0+1}\tfrac{t^{n-n_0}}{(n-n_0)!}\int_0^t |y_{n_0-1}(r)|\, dr \\
&\leq \tfrac{\varepsilon}{2\exp(BT)}\exp(BT) + B\tfrac{(BT)^{n-n_0}}{(n-n_0)!}\int_0^T |y_{n_0-1}(r)|\, dr \\
&\leq \tfrac{\varepsilon}{2} + B\tfrac{(BT)^{n-n_0}}{(n-n_0)!}\int_0^T |y_{n_0-1}(r)|\, dr,\ \forall n \geq n_0.
\end{aligned} \tag{3.13}$$

Since $B\frac{(BT)^{n-n_0}}{(n-n_0)!}\int_0^T |y_{n_0-1}(r)|\, dr \to 0$, there exists $n_1 > n_0$ such that

$$B\frac{(BT)^{n-n_0}}{(n-n_0)!}\int_0^T |y_{n_0-1}(r)|\, dr < \frac{\varepsilon}{2}, \quad \forall n \geq n_1. \tag{3.14}$$

Combining (3.13) and (3.14), the result is

$$|y_n(t)| < \varepsilon, \quad \forall n \geq n_1, \quad \forall t \in [0,T]. \tag{3.15}$$

This ends the proof of Theorem 3.1. □

4. An asymptotic expansion of the solution with respect to β

In this section, we consider an asymptotic expansion of the solution of (2.1) with respect to β. When $\beta = 0$, Eq. (2.1) has the form

$$x(t) = q_0(t) + \int_0^t k_0(t,s)x(s)ds,\ t \geq 0, \tag{4.1}$$

where

$$\begin{cases} q_0(t) = -\alpha - \frac{Y_1' t - \gamma Y_1}{t+1},\ t \geq 0, \\ k_0(t,s) = -\alpha + \frac{1+s}{t+1},\ \ t \geq 0,\ s \geq 0. \end{cases} \tag{4.2}$$

Assume that $x_0(t)$ is a solution of (4.1). Let us denote by $x_1(t), \cdots, x_N(t)$, the solutions of the following integral equations respectively

$$\begin{cases} x_1(t) = q_1(t) + \int_0^t k_0(t,s)x_1(s)ds, \quad t \geq 0, \\ x_2(t) = q_2(t) + \int_0^t k_0(t,s)x_2(s)ds, \quad t \geq 0, \\ \qquad \vdots \\ x_N(t) = q_N(t) + \int_0^t k_0(t,s)x_N(s)ds, \quad t \geq 0, \end{cases} \tag{4.3}$$

in which

$$\begin{cases} q_1(t) = -\frac{Y_1'}{t+1} + \frac{t+2}{t+1}\int_0^t x_0(s)ds, \quad t \geq 0, \\ q_i(t) = \frac{t+2}{t+1}\int_0^t x_{i-1}(s)ds, \quad i = 2, 3, \cdots, t \geq 0. \end{cases} \tag{4.4}$$

Suppose that $x(t)$ is a solution of (2.1). Then $y(t) = x(t) - \sum_{i=0}^N \beta^i x_i(t)$ is a solution of the following integral equation

$$y(t) = \beta^{N+1}\bar{q}(t) + \int_0^t k(t,s)y(s)ds, \ \ t \geq 0, \tag{4.5}$$

with

$$\bar{q}(t) = \frac{t+2}{t+1}\int_0^t x_N(s)ds, \ t \ \geq 0. \tag{4.6}$$

It follows from (4.5) that

$$z(t) \leq 2a^2(t) + 2b^2(t)\int_0^t z(s)ds, \tag{4.7}$$

where

$$z(t) = y^2(t), \ \ a(t) = |\beta|^{N+1}|\bar{q}(t)|, \ \ b(t) = \sqrt{\int_0^t k^2(t,s)ds}, \ \ t \geq 0. \tag{4.8}$$

Using Gronwall's Lemma again, from (4.7), we obtain

$$z(t) \leq 2a^2(t) + 4b^2(t)\int_0^t a^2(s)\exp\left[2\int_s^t b^2(r)dr\right]ds, \ t \geq 0. \tag{4.9}$$

or

$$y^2(t) \leq 2\Phi(t)\beta^{2N+2}, \ t \geq 0, \tag{4.10}$$

with

$$\Phi(t) = \bar{q}^2(t) + 2b^2(t)\int_0^t \bar{q}^2(s)\exp\left[2\int_s^t b^2(r)dr\right]ds. \tag{4.11}$$

Chosing β small enough, such as $|\beta| \leq 1$, the function $k(t,s)$ is bounded by a constant independent of β, because of the fact that

$$\begin{aligned} &|k(t,s)| \leq \left|-\alpha - \beta - \tfrac{\beta-1-s}{t+1}\right| \leq |\alpha| + |\beta| + \tfrac{|\beta|+1+s}{t+1} \leq |\alpha| + 3, \\ &\forall \alpha \in \mathbb{R}, \quad \forall \beta \in [-1,1], \quad t, s \in \mathbb{R}_+, \quad s \leq t. \end{aligned} \tag{4.12}$$

The result is

$$\begin{aligned} & b^2(t) = \int_0^t k^2(t,s)ds \le (|\alpha|+3)^2\, t, \\ & \int_s^t b^2(r)dr \le (|\alpha|+3)^2 \left(\tfrac{t^2}{2} - \tfrac{s^2}{2}\right). \end{aligned} \tag{4.13}$$

We get

$$\begin{aligned} \Phi(t) &\le \bar{q}^2(t) + 2\,(|\alpha|+3)^2\, t \int_0^t \bar{q}^2(s) \exp\left[(|\alpha|+3)^2 \left(t^2 - s^2\right)\right] ds \\ &\equiv \Psi^2(t), \end{aligned} \tag{4.14}$$

for all $t \ge 0$, where $\Psi(t)$ is continuous in $\mathbb{R}_+$, independent of β. Obviously, $\Psi(t) \ge 0,\ \forall\, t \ge 0$. From (4.10) and (4.14), we obtain the following theorem.

Theorem 4.1. *There exists the function $\Psi \in C(\mathbb{R}_+)$ independent of β such that*

$$\left| x(t) - \sum_{i=0}^{N} \beta^i x_i(t) \right| \le \sqrt{2}\Psi(t)\, |\beta|^{N+1}, \quad t \ge 0, \tag{4.15}$$

where $x_0(t),\ x_1(t),\ \cdots, x_N(t)$ are defined by (4.1)-(4.4).

5. An asymptotic expansion of the solution with respect to α, β

This section gives an asymptotic expansion of the solution of (2.1) with respect to α, β. When $\alpha = \beta = 0$, (2.1) has the form

$$x(t) = \bar{q}_0(t) + \int_0^t \bar{k}_0(t,s)x(s)ds, \quad t \ge 0, \tag{5.1}$$

where

$$\begin{cases} \bar{q}_0(t) \equiv \bar{q}_{0,0}(t) = -\frac{Y_1' t - \gamma Y_1}{t+1}, & t \ge 0, \\ \bar{k}_0(t,s) \equiv \bar{k}_{0,0}(t,s) = \frac{1+s}{t+1}, & t \ge 0, \quad s \ge 0. \end{cases} \tag{5.2}$$

Let $\bar{x}_0(t) \equiv \bar{x}_{0,0}(t)$ be a solution of (5.1). It is not difficult to see that

$$\bar{x}_0(t) = \frac{Y_1'}{t+1} + (\gamma Y_1 - Y_1')\frac{e^t}{t+1}, \ t \ge 0,$$

is a unique solution of (5.1). Assume that $\bar{x}_{ij}(t) \equiv \bar{x}_{i,j}(t),\ i,\, j = 0, 1, \cdots, N$, $1 \le i+j \le N$, respectively, are the solutions of the following integral equations

$$\begin{aligned} & x_{ij}(t) = q_{ij}(t) + \int_0^t \bar{k}_0(t,s)x_{ij}(s)ds, \quad t \ge 0; \\ & i, j = 0, 1, \cdots, N, \quad 1 \le i+j \le N, \end{aligned} \tag{5.3}$$

in which

$$\begin{aligned} &q_{10}(t) \equiv q_{1,0}(t) = -1 - \textstyle\int_0^t \bar{x}_0(s)ds, \quad t \geq 0, \\ &q_{01}(t) \equiv q_{0,1}(t) = -\tfrac{Y_1'}{t+1} - \tfrac{t+2}{t+1} \textstyle\int_0^t \bar{x}_0(s)ds, \quad t \geq 0, \\ &q_{ij}(t) \equiv q_{i,j}(t) = -\textstyle\int_0^t \bar{x}_{i-1,j}(s)ds - \tfrac{t+2}{t+1} \textstyle\int_0^t \bar{x}_{i,j-1}(s)ds, \quad t \geq 0, \end{aligned} \tag{5.4}$$

$$i, j = 0, 1, 2, \cdots, N; \;\; 2 \leq i + j \leq N, \;\; (i,j) \neq (0,1), \;\; (i,j) \neq (1,0).$$

Let $x(t)$ be a solution of (2.1). Then $\bar{y}(t) = x(t) - \sum_{0 \leq i,j \leq N, i+j \leq N} \bar{x}_{ij}(t)\alpha^i\beta^j$ is a solution of the following integral equation

$$\bar{y}(t) = E_N(\alpha, \beta, t) + \int_0^t k(t,s)\bar{y}(s)ds, \quad t \geq 0, \tag{5.5}$$

where

$$E_N(\alpha, \beta, t) = -\sum_{0 \leq i,j \leq N, \; i+j=N} \left[\alpha + \beta\frac{t+2}{t+1}\right] \alpha^i\beta^j \int_0^t \bar{x}_{ij}(s)ds, \; t \; \geq 0. \tag{5.6}$$

Then we have the following lemma.

Lemma 5.1. *There exists the function $\bar{\Psi} \in C(\mathbb{R}_+)$ independent of α, β such that*

$$|E_N(\alpha, \beta, t)| \leq \bar{\Psi}(t) \left(\sqrt{\alpha^2 + \beta^2}\right)^{N+1}, \quad t \geq 0. \tag{5.7}$$

Proof. The proof of Lemma 5.1 is as follows. We have, for all $t \geq 0$,

$$|E_N(\alpha, \beta, t)| \leq \sum_{0 \leq i,j \leq N, \; i+j=N} \left(|\alpha| + |\beta| \frac{t+2}{t+1}\right) |\alpha|^i |\beta|^j \int_0^t |\bar{x}_{ij}(s)| \, ds. \tag{5.8}$$

Note that

$$\begin{aligned} &\left(|\alpha| + |\beta| \tfrac{t+2}{t+1}\right) |\alpha|^i |\beta|^j \leq \left(1 + \tfrac{t+2}{t+1}\right) \left(\sqrt{\alpha^2 + \beta^2}\right)^{i+j+1} \\ &\leq 3 \left(\sqrt{\alpha^2 + \beta^2}\right)^{N+1}, \;\; 0 \leq i,j \leq N, \;\; i+j = N, \;\; s \leq t. \end{aligned} \tag{5.9}$$

So

$$\begin{aligned} |E_N(\alpha, \beta, t)| &\leq 3 \sum_{0 \leq i,j \leq N, \; i+j=N} \textstyle\int_0^t |\bar{x}_{ij}(s)| \, ds \left(\sqrt{\alpha^2 + \beta^2}\right)^{N+1} \\ &\equiv \bar{\Psi}_*(t) \left(\sqrt{\alpha^2 + \beta^2}\right)^{N+1}, \quad t \geq 0. \end{aligned} \tag{5.10}$$

$\square$

The main result of this section is the following.

Theorem 5.2. *There exists the function* $\bar{\Psi}_* \in C(\mathbb{R}_+)$ *independent of* α, β *such that*

$$\left| x(t) - \sum_{0 \le i,j \le N, i+j \le N} \bar{x}_{ij}(t) \alpha^i \beta^j \right| \le \bar{\Psi}(t) \left(\sqrt{\alpha^2 + \beta^2} \right)^{N+1}, \quad t \ge 0, \tag{5.11}$$

where $\sqrt{\alpha^2+\beta^2} \le 1$ *and* $\bar{x}_{ij}(t),\ i,j \ge 0,\ i+j \le N$ *are defined by* (5.3), (5.4).

Proof. We have $\bar{y}$ satisfies the equation

$$\bar{y}(t) = E_N(\alpha, \beta, t) + \int_0^t k(t,s)\bar{y}(s)ds,\ t \ge 0. \tag{5.12}$$

It implies from (5.12) that

$$z(t) \le 2a^2(t) + 2b^2(t) \int_0^t z(s)ds, \tag{5.13}$$

where $z(t) = \bar{y}^2(t)$, and

$$a(t) = \bar{\Psi}(t) \left(\sqrt{\alpha^2+\beta^2} \right)^{N+1}, \quad b(t) = \sqrt{\int_0^t k^2(t,s)ds}, \ \ t \ge 0. \tag{5.14}$$

Using Gronwall's Lemma, (5.13) yields

$$z(t) \le 2a^2(t) + 4b^2(t) \int_0^t a^2(s) \exp\left[2 \int_s^t b^2(r)dr \right] ds, \ \ t \ge 0. \tag{5.15}$$

or

$$y^2(t) \le \bar{\Phi}(t) \left(\sqrt{\alpha^2+\beta^2} \right)^{2N+2}, \ \ t \ge 0, \tag{5.16}$$

with

$$\bar{\Phi}(t) = 2\bar{\Psi}^2(t) + 4b^2(t) \int_0^t \bar{\Psi}^2(s) \exp\left[2 \int_s^t b^2(r)dr \right] ds. \tag{5.17}$$

Note that, with α, β small enough such that $\sqrt{\alpha^2+\beta^2} \le 1$, the function $k(t,s)$ is bounded by a constant independent of α, β, t, s, because of

$$\begin{aligned} |k(t,s)| &\le \left| -\alpha - \beta - \tfrac{\beta-1-s}{t+1} \right| \le |\alpha| + |\beta| + \tfrac{|\beta|+1+s}{t+1} \\ &\le 3\left(\sqrt{\alpha^2+\beta^2} \right) + \tfrac{1+s}{t+1} \le 4, \ \ \forall\, t,s \in \mathbb{R}_+, \ \ s \le t. \end{aligned} \tag{5.18}$$

Thus

$$b^2(t) = \int_0^t k^2(t,s)ds \le 16t, \quad \int_s^t b^2(r)dr \le 8\left(t^2 - s^2\right). \tag{5.19}$$

Consequently

$$\bar{\Phi}(t) \le 2\bar{\Psi}^2(t) + 64t \int_0^t \bar{\Psi}^2(s) \exp\left[16\left(t^2-s^2\right)\right] ds \equiv \bar{\Psi}_*^2(t), \ \ \forall t \ge 0. \tag{5.20}$$

It is obviously that $\bar{\Psi}_*(t)$ is continuous in $\mathbb{R}_+$, independent of α, β. Theorem 5.2 is proved completely. $\square$

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