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APPROXIMATION RESULTS FOR GENERALIZED NONEXPANSIVE MAPPINGS AND THEIR APPLICATIONS IN VARIOUS NONLINEAR PROBLEMS

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Abstract. In this research, we establish convergence findings concerning generalized α-Reich-Suzuki nonexpansive($GARSN$) mappings using a rapid iterative approach called the F-iterative scheme. To support our findings, we present a numerical example. Furthermore, we compare our results with other widely recognized iterative schemes. Lastly, we demonstrate the practical applications of these results in solving high-degree polynomial equations, systems of linear equations, and delay composite functional differential equations($DCFDE$) of the Volterra-Stieljes Type(VST).

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1. Introduction

The theory of nonexpansive(NE) mappings holds significant importance in nonlinear analysis due to its wide range of applications. In 1965, Browder [6], Gohde [11], and Kirk [20] independently established fixed point results for nonexpansive mappings. Since then, numerous researchers have contributed to this field by introducing different concepts, extending the existing results, and proposing new notions related to nonexpansive mappings.

In 1980, Gregus [12] generalized the work of Kannan [19] by combining the ideas of nonexpansive and Kannan mappings, leading to the development of a distinct class known as Reich nonexpansive(RNE) mappings. In 2008, Suzuki [28] introduced another class of mappings called Suzuki's generalized nonexpansive($SGNE$) mappings. More recently, Ali et al. [4] utilized a three-step iterative scheme to establish both weak and strong convergence results for $SGNE$ mappings in uniformly convex Banach spaces($UCBS$).

In the year 2011, Aoyama and Kohsaka presented a concept called α-nonexpansive (ANE) mapping, which is a type of generalized nonexpansive (GNE) mapping. They derived certain outcomes pertaining to this category of mappings.

More recently, a distinct expansion of NE mappings was proposed by Pandey et al. [25] Their extension encompasses ANE mappings as well as $SGNE$ mappings. They termed this extension as $GARSN$ mappings and derived intriguing results associated with this class of mappings.

The numerical computation of nonlinear operators poses an intriguing research challenge in nonlinear analysis. However, finding the fixed points of certain operators is not a straightforward task. To address this difficulty, numerous iterative procedures have been developed over time. Among these procedures, the three fundamental algorithms employed to approximate the fixed points of NE mappings are Mann's algorithm [22], Ishikawa's algorithm [18], and Halpern's algorithm [17].

Motivated by the aforementioned iterative techniques, numerous algorithms have been developed by researchers to approximate the fixed points of various nonlinear mappings. Some of these algorithms include the Noor iteration [23], the Agarwal et al. algorithm [2], the Abbas and Nazir iteration [1], the Thakur New iteration [29], the Picard S-iteration [14], the normal S-iteration [15, 16], the Ullah and Arshad (M)-iteration [30], the Garodia and Uddin algorithm [9], and many others. However, all these algorithms converges slow to the fixed points.

In 2020, Ali and Ali [3] employed a novel and a faster iterative procedure called the F-iteration:

$$\begin{cases} \rho_1 = \rho \in D, \\ \omega_n = \Gamma((1-\delta_n)\rho_n + \delta_n \Gamma \rho_n), \\ \sigma_n = \Gamma \omega_n, \\ \rho_{n+1} = \Gamma \sigma_n, \end{cases} \tag{1.1}$$

where $\delta_n \in (0,1)$.

In their work [3], Ali and Ali asserted that the F-iteration (1.1) exhibits faster convergence compared to any generalized contraction mappings. They also provided proofs for several convergence results, including both weak and strong convergence.

Inspired by the aforementioned research, we establish convergence outcomes for $GARSN$ mappings utilizing algorithm (1.1). The findings of this study expand upon and supplement existing literature by presenting novel and significant contributions. We introduce a new example and demonstrate that the F-iteration process yields superior approximation results compared to various other iterative procedures, particularly in the case of the F-iteration process. Consequently, our study contributes valuable results derived from the existing body of literature.

2. Preliminaries

We will commence by introducing several essential definitions and results. In this context, the notation $F(\Gamma)$ denotes the fixed point set associated with the operator Γ.

Definition 2.1. ([24]) A Banach space G is said to have the Opial property if, for any sequence $\{\rho_n\}$ that weakly converges to a point x in the space, the inequality

$$\limsup_{n\to\infty} ||\rho_n - x|| < \limsup_{n\to\infty} ||\rho_n - y||$$

holds true for all $y \in G$ with $y \neq x$.

This property has significant implications for the existence and uniqueness of fixed points in various mathematical settings.

Definition 2.2. ([10]) Let D be a nonempty closed subset of a Banach space G, and consider a bounded sequence $\{\rho_n\}$ in G. For any $\rho \in D$ set:

$$r(\rho, \{\rho_n\}) = \limsup_{n\to\infty} ||\rho - \rho_n||.$$

The asymptotic radius of D with respect to $\{\rho_n\}$ is then defined as:

$$r(D, \{\rho_n\}) = \inf\{r(\rho, \{\rho_n\}) : \rho \in D\}.$$

Furthermore, the asymptotic center is defined as the set:

$$A(D, \{\rho_n\}) = \{\rho \in D : r(\rho, \{\rho_n\}) = r(D, \{\rho_n\})\}.$$

It is important to note that for a $UCBS$, the set $A(D, \{\rho_n\})$ consists of a single element.

Definition 2.3. ([10]) If D represents a nonempty closed convex subset of a Banach Space G, then a mapping $\Gamma : D \to G$ is considered to be demiclosed with respect to $\sigma \in G$ if, for any sequence $\{\rho_n\} \subset D$ and for any $\rho \in D$, the following condition holds: if $\{\rho_n\}$ converges weakly to ρ and $\{\Gamma\rho_n\}$ converges strongly to σ, then it follows that $\Gamma\rho = \sigma$.

Let us revisit the definition of $GARSN$ mappings.

Definition 2.4. ([25]) A mapping Γ defined on a nonempty subset D of a Banach Space G is referred to as a generalized α-Reich-Suzuki nonexpansive ($GARSN$) mapping if, for all $\rho, \sigma \in D$ and $\delta \in [0, 1]$, the following conditions hold:

$$\frac{1}{2}||\rho - \Gamma\rho|| \leq ||\rho - \sigma|| \Rightarrow ||\Gamma\rho - \Gamma\sigma|| \leq \max\{P(\rho, \sigma), Q(\rho, \sigma)\},$$

where

$$P(\rho, \sigma) = \delta||\Gamma\rho - \rho|| + \delta||\Gamma\sigma - \sigma|| + (1 - 2\delta)||\rho - \sigma||$$

and

$$Q(\rho, \sigma) = \delta||\Gamma\rho - \sigma|| + \delta||\Gamma\sigma - \rho|| + (1 - 2\delta)||\rho - \sigma||.$$

Lemma 2.5. ([25]) *Every nonexpansive mapping is also considered a $GARSN$ mapping.*

Lemma 2.6. ([25]) *If Γ is a $GARSN$ mapping on a nonempty subset D of a Banach space G, then for all $\rho, \sigma \in D$, the following condition hold:*

$$||\rho - \Gamma\sigma|| \leq \left(\frac{3 + \delta}{1 - \delta}\right) ||\rho - \Gamma\rho|| + ||\rho - \sigma||.$$

Lemma 2.7. ([26]) *Suppose $\{q_n\}$ is a sequence in $[y, z]$ for some $y, z \in (0, 1)$ and $\{x_n\}, \{y_n\}$ are sequences in a uniformly convex Banach space G satisfying $\limsup_{n\to\infty} ||x_n|| \leq d$, $\limsup_{n\to\infty} ||y_n|| \leq d$ and $\lim_{n\to\infty} ||(1 - q_n)x_n + q_n y_n|| = d$ for any $d \geq 0$. Then,*

$$\lim_{n\to\infty} ||x_n - y_n|| = 0.$$

3. Approximation results

In this section, we demonstrate several convergence results utilizing the F-iterative scheme (1.1) for $GARSN$ mappings.

Theorem 3.1. *Let $\Gamma : D \to D$ be a mapping that satisfies Definition 2.4 with $F(\Gamma) \neq \emptyset$, where $F(\Gamma)$ is defined on a nonempty, convex, and closed subset D of a uniformly convex Banach space G. If the sequence $\{\rho_n\}$ is defined by Algorithm (1.1), then the $\lim_{n\to\infty} ||\rho_n - g||$ exists for all $g \in F(\Gamma)$.*

Proof. Let $g \in F(\Gamma)$ and $\omega \in D$. As Γ satisfies the Definition 2.4, we have

$$||\Gamma\omega - \Gamma g|| \leq ||\omega - g||.$$

Using the Algorithm(1.1), we get

$$\begin{aligned} ||\omega_n - g|| &= ||\Gamma((1-\delta_n)\rho_n + \delta_n\Gamma\rho_n) - g|| \\ &\leq ||(1-\delta_n)\rho_n + \rho_n\Gamma\rho_n - g|| \\ &\leq (1-\delta_n)||\rho_n - g|| + \delta_n||\Gamma\rho_n - g|| \\ &\leq (1-\delta_n)||\rho_n - g|| + \delta_n||\rho_n - g|| \\ &= ||\rho_n - g|| \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} ||\sigma_n - g|| &= ||\Gamma\omega_n - g|| \\ &\leq ||\omega_n - g|| \\ &\leq ||\rho_n - g||. \end{aligned}$$

So

$$\begin{aligned} ||\rho_{n+1} - g|| &= ||\Gamma\sigma_n - g|| \leq ||\sigma_n - g|| \\ &\leq ||\omega_n - g|| \\ &\leq ||\rho_n - g||. \end{aligned} \tag{3.2}$$

Consequently, for any $n \in \mathbb{N}$ and $g \in F(\Gamma)$, we can observe that $||\rho_{n+1} - g|| \leq ||\rho_n - g||$. This inequality demonstrates that the sequence $\{||\rho_n - g||\}$ is both bounded and non-increasing. As a result, we can deduce that the limit of the sequence $\lim_{n\to\infty} ||\rho_n - g||$ exists for every $g \in F(\Gamma)$. □

The aforementioned theorem is essential for the subsequent findings.

Theorem 3.2. *Suppose G, D, Γ, and the sequence $\{\rho_n\}$ is defined as in Theorem 3.1, the condition $F(\Gamma) \neq \emptyset$ holds if and only if the sequence $\{\rho_n\}$ is bounded, and the limit as n approaches infinity of $||\Gamma\rho_n - \rho_n||$ is equal to zero.*

Proof. First we consider $\lim_{n\to\infty} ||\rho_n - \Gamma\rho_n|| = 0$ and the sequence $\{\rho_n\}$ is bounded then we will prove $F(\Gamma) \neq \emptyset$.

For any $g \in A(D, \{\rho_n\})$. Therefore,

$$\begin{aligned} r(\Gamma g, \{\rho_n\}) &= \limsup_{n\to\infty} ||\rho_n - \Gamma g|| \\ &\leq \limsup_{n\to\infty} \{\left(\frac{3+\delta}{1-\delta}\right) ||\Gamma\rho_n - \rho_n|| + ||\rho_n - g||\} \\ &= \left(\frac{3+\delta}{1-\delta}\right) \limsup_{n\to\infty} ||\Gamma\rho_n - \rho_n|| + \limsup_{n\to\infty} ||\rho_n - g|| \\ &= \limsup_{n\to\infty} ||\rho_n - g|| \\ &= r(g, \{\rho_n\}). \end{aligned}$$

This implies that $\Gamma g \in A(D, \{\rho_n\})$. Since $A(D, \{\rho_n\})$ consists a single element in UCBS, $g \in F(\Gamma)$. Hence proved that $F(\Gamma) \neq \emptyset$.

Conversely, we assume that $F(\Gamma) \neq \emptyset$, we can establish that $\{\rho_n\}$ is a bounded sequence and that $\lim_{n\to\infty} ||\rho_n - \Gamma\rho_n|| = 0$. The boundedness of $\{\rho_n\}$ can be inferred from the proof of Theorem 3.1. Since $F(\Gamma)$ is nonempty, we can choose an arbitrary element g from $F(\Gamma)$. According to Theorem 3.1, it follows that $\lim_{n\to\infty} ||\rho_n - g||$ exists. Let us denote this limit as:

$$\xi = \lim_{n\to\infty} ||\rho_n - g||. \tag{3.3}$$

From (3.1) , we get

$$||\omega_n - g|| \leq ||\rho_n - g||,$$

this implies that

$$\limsup_{n\to\infty} ||\omega_n - g|| \leq \limsup_{n\to\infty} ||\rho_n - g|| = \xi. \tag{3.4}$$

Since

$$||\Gamma\rho_n - g|| \leq ||\rho_n - g||,$$

also, we have

$$\limsup_{n\to\infty} ||\Gamma\rho_n - g|| \leq \limsup_{n\to\infty} ||\rho_n - g|| = \xi. \tag{3.5}$$

Similarly, from (3.2), we have

$$||\rho_{n+1} - g|| \leq ||\omega_n - g||,$$

this implies

$$\xi = \liminf_{n\to\infty} ||\rho_{n+1} - g|| \leq \liminf_{n\to\infty} ||\omega_n - g||. \tag{3.6}$$

By combining (3.4) and (3.6), we obtain

$$\xi = \lim_{n\to\infty} ||\omega_n - g||. \tag{3.7}$$

From (3.7), we have

$$\begin{aligned} \xi &= \lim_{n\to\infty} ||\omega_n - g|| \\ &= \lim_{n\to\infty} ||\Gamma((1-\delta_n)\rho_n + \delta_n\Gamma\rho_n) - g|| \\ &\leq \lim_{n\to\infty} ||(1-\delta_n)\rho_n + \delta_n\Gamma\rho_n - g|| \\ &= \lim_{n\to\infty} ||(1-\delta_n)(\rho_n - g) + \delta_n(\Gamma\rho_n - g)|| \\ &\leq (1-\delta_n)\lim_{n\to\infty} ||\rho_n - g|| + \delta_n \lim_{n\to\infty} ||\Gamma\rho_n - g|| \\ &\leq (1-\delta_n)\lim_{n\to\infty} ||\rho_n - g|| + \delta_n \lim_{n\to\infty} ||\rho_n - g|| \\ &\leq \lim_{n\to\infty} ||\rho_n - g|| \\ &= \xi. \end{aligned}$$

Hence, we have

$$\xi = \lim_{n\to\infty} ||(1-\delta_n)(\rho_n - g) + \delta_n(\Gamma\rho_n - g)||. \tag{3.8}$$

If we apply Theorem 3.1, we must have

$$\lim_{n\to\infty} ||\Gamma\rho_n - \rho_n|| = 0.$$

This completes the proof. □

We establish weak convergence results based on the conditions that satisfy Opial's property.

Theorem 3.3. *Suppose G, D, Γ, and the sequence $\{\rho_n\}$ is defined as in Theorem 3.1, the condition $F(\Gamma) \neq \emptyset$ holds If the set G satisfies the Opial's condition, then the sequence $\{\rho_n\}$ weakly converges to an element in $F(\Gamma)$.*

Proof. As $F(\Gamma)$ is nonempty, where Γ is the mapping defined in Definition 2.4, we can conclude from Theorems 3.1 and 3.2 that the limit $\lim_{n\to\infty} ||\rho_n - g||$ exists and that

$$\lim_{n\to\infty} ||\Gamma\rho_n - \rho_n|| = 0.$$

Subsequently, we aim to demonstrate that the sequence $\{\rho_n\}$ cannot possess two distinct weakly subsequential limits in $F(\Gamma)$. Let x and y be the two weak subsequential limits of $\{\rho_{n_j}\}$ and $\{\rho_{n_k}\}$, respectively. According to Theorem 3.2, we know that $(I-\Gamma)$ is demiclosed at 0, indicating that $(I-\Gamma)x = 0$. Consequently, we have $\Gamma x = x$, and similarly, $\Gamma y = y$.

Furthermore, we proceed to establish the uniqueness of the limit. If $x \neq y$, employing the Opial's condition, we can deduce that

$$\begin{aligned}\lim_{n\to\infty} ||\rho_n - x|| &= \lim_{n\to\infty} ||\rho_{n_j} - x|| < \lim_{n\to\infty} ||\rho_{n_j} - y|| \\ &= \lim_{n\to\infty} ||\rho_n - y|| = \lim_{n\to\infty} ||\rho_{n_k} - y|| \\ &< \lim_{n\to\infty} ||\rho_{n_k} - x|| \\ &= \lim_{n\to\infty} ||\rho_n - x||,\end{aligned}$$

which is a contradiction to our hypothesis hence $x = y$. Therefore, $\rho_n \rightharpoonup x \in F(\Gamma)$. □

Afterward, we demonstrate several powerful convergence theorems.

Theorem 3.4. *Suppose G, D, Γ, and the sequence $\{\rho_n\}$ is defined as in Theorem 3.1, the condition $F(\Gamma) \neq \emptyset$ holds. If D is compact, then the sequence $\rho_n \to z \in F(\Gamma)$.*

Proof. Given that $F(\Gamma) \neq \emptyset$, according to Theorem 3.2, we can conclude that $\lim_{n\to\infty} ||\rho_n - \Gamma\rho_n|| = 0$. Additionally, exploiting the compactness of D, we can find a subsequence $\{\rho_{n_j}\}$ of $\{\rho_n\}$ that converges to $z \in D$. Applying Lemma 2.6, we obtain the following result.

$$||\rho_{n_j} - \Gamma z|| \leq \left(\frac{3+\delta}{1-\delta}\right) ||\Gamma\rho_{n_j} - \rho_{n_j}|| + ||\rho_{n_j} - z||.$$

As we let $j \to \infty$, we can observe that $\Gamma z = z$, implying that $z \in F(\Gamma)$. This result can be obtained by utilizing Theorem 3.1. Consequently, we can conclude that $\lim_{n\to\infty} ||\rho_n - z||$ exists for any $z \in F(\Gamma)$, indicating that the sequence $\{\rho_n\}$ converges strongly to z. □

Theorem 3.5. *Suppose G, D, Γ, and the sequence $\{\rho_n\}$ is defined as in Theorem 3.1, the condition $F(\Gamma) \neq \emptyset$ holds. Then, the sequence $\{\rho_n\}$ converges to a fixed point of Γ if and only if $\lim_{n\to\infty} \varrho(\rho_n, F(\Gamma)) = 0$, where*

$$\varrho(\rho, F(\Gamma)) = \inf\{||\rho - g|| : g \in F(\Gamma)\}.$$

Proof. The proof of the first part can be easily demonstrated. Now we will establish the converse part. Assume that for any $g \in F(\Gamma)$, we have

$$\lim_{n\to\infty} \inf \varrho(\rho_n, F(\Gamma)) = 0.$$

By utilizing Theorem 3.1, we know that $\lim_{n\to\infty} ||\rho_n - g||$ exists for all $g \in F(\Gamma)$. Consequently, $\lim_{n\to\infty} \inf \varrho(\rho_n, F(\Gamma)) = 0$.

Moving forward, we will demonstrate that $\{\rho_n\}$ is a Cauchy sequence in D. Given that $\lim_{n\to\infty} \inf \varrho(\rho_n, F(\Gamma)) = 0$, we can choose any $\gamma > 0$, which implies that there exists an $m_0 \in \mathbb{N}$ such that for all $n \geq m_0$,

$$\varrho(\rho_n, F(\Gamma)) < \frac{\gamma}{2},$$

this implies that

$$\inf\{||\rho_n - g|| : g \in F(\Gamma)\} < \frac{\gamma}{2}.$$

Considering $\inf\{||\rho_{m_0} - g|| : g \in F(\Gamma)\} < \frac{\gamma}{2}$, we can conclude that there exists $g \in F(\Gamma)$ satisfying the inequality:

$$||\rho_{m_0} - g|| < \frac{\gamma}{2}.$$

For t and n greater than or equal to m_0, we can observe the following:

$$\begin{aligned} ||\rho_{n+t} - \rho_n|| &\leq ||\rho_{n+t} - g|| + ||\rho_n - g|| \\ &\leq ||\rho_{m_0} - g|| + ||\rho_{m_0} - g|| \\ &= 2||\rho_{m_0} - g|| \\ &< \gamma. \end{aligned}$$

Thus, $\{\rho_n\}$ forms a Cauchy sequence in D. As D is complete, we can conclude that $\lim_{n\to\infty} \rho_n = g$ for some $g \in D$. Moreover, since $\lim_{n\to\infty} \varrho(\rho_n, F(\Gamma)) = 0$, it implies that g belongs to $F(\Gamma)$. □

In the paper by Senter et al. [27], Condition (I) was introduced with the following formulation:

The mapping $\Gamma : D \to D$ is said to fulfill Condition (I) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$. Additionally, the condition requires that

$$||\rho - \Gamma\rho|| \geq f(\varrho(\rho, F(\Gamma)))$$

holds for all $\rho \in D$, where

$$\varrho(\rho, F(\Gamma)) = \inf\{||\rho - g|| : g \in F(\Gamma)\}.$$

Theorem 3.6. *Suppose G, D, Γ, and the sequence $\{\rho_n\}$ are defined as in Theorem* 3.1 *such that the condition $F(\Gamma) \neq \emptyset$ holds. If Γ satisfies Condition (I), then the sequence $\rho_n \to z \in F(\Gamma)$.*

Proof. For any $z \in F(\Gamma)$, let's consider the limit $\lim_{n\to\infty} ||\rho_n - z||$ which we denote as r. Similarly, the limit $\lim_{n\to\infty} \varrho(\rho_n, F(\Gamma))$ also exists. If $r = 0$, then there is no further action required.

However, if $r > 0$, based on Condition (I), we have the inequality:

$$f(\varrho(\rho_n, F(\Gamma))) \leq ||\rho_n - \Gamma\rho_n||.$$

Referring to Theorem 3.2, we know that

$$\lim_{n\to\infty} ||\rho_n - \Gamma\rho_n|| = 0.$$

Consequently, we can deduce:

$$\lim_{n\to\infty} f(\varrho(\rho_n, F(\Gamma))) = 0.$$

Given that f is a non-decreasing function with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, we can conclude that

$$\lim_{n\to\infty} \varrho(\rho_n, F(\Gamma)) = 0.$$

By applying Theorem 3.5, we can establish the desired result. □

4. Example

In this section, we present a numerical experiment involving a $GARSN$ mapping.

Example 4.1. Let $G = \mathbb{R}$ and $D = [-1, 1]$. Define a mapping $\Gamma : D \to D$ by

$$\Gamma\rho = \begin{cases} \frac{-\rho}{3}, \text{ if } \rho \in [-1, 0), \\ -\rho, \text{ if } [0, 1] \setminus \{\frac{1}{3}\}, \\ 0 \text{ if } \rho = \frac{1}{3}. \end{cases}$$

For $\rho = \frac{1}{3}$ and $\sigma = 1$, then $\Gamma\rho = 0;\ \Gamma\sigma = 1$. Then, we have

$$\frac{1}{2}||\rho - \Gamma\rho|| = \frac{1}{2}|\frac{1}{3} - 0| = \frac{1}{6} \leq \frac{2}{3} = ||\rho - \sigma||$$

but

$$||\Gamma\rho - \Gamma\sigma|| = |0 + 1| = 1 > \frac{2}{3} = ||\rho - \sigma||.$$

Thus Γ does not satisfy condition (C).

Next, we show that Γ is $GARSN$ mapping with $\delta = \frac{1}{2}$.

(i) Let $\rho, \sigma \in [-1, 0)$. Then $\Gamma\rho = \frac{-\rho}{3};\ \Gamma\sigma = \frac{-\sigma}{3}$. So,

$$\begin{aligned} P(\rho,\sigma) &= \frac{1}{2}||\rho - \Gamma\rho|| + \frac{1}{2}||\sigma - \Gamma\sigma|| \\ &= \frac{1}{2}|\rho + \frac{\rho}{3}| + \frac{1}{2}|\sigma + \frac{\sigma}{3}| \\ &= \frac{1}{2}(|\frac{4\rho}{3}| + |\frac{4\sigma}{3}|) \\ &\geq \frac{1}{3}|\sigma - \rho| \\ &= ||\Gamma\rho - \Gamma\sigma||, \end{aligned}$$

and

$$\begin{aligned} Q(\rho,\sigma) &= \frac{1}{2}||\rho-\Gamma\sigma||+\frac{1}{2}||\sigma-\Gamma\rho|| \\ &= \frac{1}{2}|\rho+\frac{\sigma}{3}|+\frac{1}{2}|\sigma+\frac{\rho}{3}| \\ &\geq \frac{1}{3}|\sigma-\rho| \\ &= ||\Gamma\rho-\Gamma\sigma||. \end{aligned}$$

(ii) Let $\rho,\sigma\in[0,1]/\{\frac{1}{3}\}$. Then $\Gamma\rho=-\rho$ and $\Gamma\sigma=-\sigma$. So,

$$\begin{aligned} P(\rho,\sigma) &= \frac{1}{2}||\rho-\Gamma\rho||+\frac{1}{2}||\sigma-\Gamma\sigma|| \\ &= \frac{1}{2}|\rho+\rho|+\frac{1}{2}|\sigma+\sigma| \\ &= \frac{1}{2}(|2\rho|+|2\sigma|) \\ &= |\rho|+|\sigma| \\ &\geq |\sigma-\rho| \\ &= ||\Gamma\rho-\Gamma\sigma||, \end{aligned}$$

and

$$\begin{aligned} Q(\rho,\sigma) &= \frac{1}{2}||\rho-\Gamma\sigma||+\frac{1}{2}||\sigma-\Gamma\rho|| \\ &= \frac{1}{2}|\rho+\sigma|+\frac{1}{2}|\sigma+\rho| \\ &= |\rho+\sigma| \\ &\geq |\sigma-\rho| \\ &= ||\Gamma\rho-\Gamma\sigma||. \end{aligned}$$

(iii) Let $\rho\in[-1,0)$ and $\sigma\in[0,1]/\{\frac{1}{3}\}$. Then $\Gamma\rho=\frac{-\rho}{3}$ and $\Gamma\sigma=-\sigma$. So,

$$\begin{aligned} P(\rho,\sigma) &= \frac{1}{2}||\rho-\Gamma\rho||+\frac{1}{2}||\sigma-\Gamma\sigma|| \\ &= \frac{1}{2}|\rho+\frac{\rho}{3}|+\frac{1}{2}|\sigma+\sigma| \\ &= \frac{1}{2}(|\frac{4\rho}{3}|+|2\sigma|) \\ &\geq |\sigma-\frac{\rho}{3}| \\ &= ||\Gamma\rho-\Gamma\sigma||, \end{aligned}$$

and

$$
\begin{aligned}
Q(\rho,\sigma) &= \frac{1}{2}||\rho-\Gamma\sigma||+\frac{1}{2}||\sigma-\Gamma\rho|| \\
&= \frac{1}{2}|\rho+\sigma|+\frac{1}{2}|\sigma+\frac{\rho}{3}| \\
&= \frac{1}{2}(|\rho+\sigma|+|\sigma+\frac{\rho}{3}|) \\
&\geq |\sigma-\frac{\rho}{3}| \\
&= ||\Gamma\rho-\Gamma\sigma||.
\end{aligned}
$$

(iv) Let $\rho\in[-1,0)$ and $\sigma=\frac{1}{3}$. Then $\Gamma\rho=\frac{-\rho}{3}$ and $\Gamma\sigma=0$. So,

$$
\begin{aligned}
P(\rho,\sigma) &= \frac{1}{2}||\rho-\Gamma\rho||+\frac{1}{2}||\sigma-\Gamma\sigma|| \\
&= \frac{1}{2}|\rho+\frac{\rho}{3}|+\frac{1}{2}|\frac{1}{3}+0| \\
&= \frac{1}{2}|\frac{4\rho}{3}|+\frac{1}{6} \\
&\geq \frac{1}{3}|\rho| \\
&= ||\Gamma\rho-\Gamma\sigma||,
\end{aligned}
$$

and

$$
\begin{aligned}
Q(\rho,\sigma) &= \frac{1}{2}||\rho-\Gamma\sigma||+\frac{1}{2}||\sigma-\Gamma\rho|| \\
&= \frac{1}{2}|\rho-0|+\frac{1}{2}|\frac{1}{3}+\frac{\rho}{3}| \\
&\geq \frac{1}{3}|\rho| \\
&= ||\Gamma\rho-\Gamma\sigma||.
\end{aligned}
$$

(v) Let $\rho\in[0,1]/\{\frac{1}{3}\}$ and $\sigma=\frac{1}{3}$. Then $\Gamma\rho=-\rho$ and $\Gamma\sigma=0$. So,

$$
\begin{aligned}
P(\rho,\sigma) &= \frac{1}{2}||\rho-\Gamma\rho||+\frac{1}{2}||\sigma-\Gamma\sigma|| \\
&= \frac{1}{2}|\rho+\rho|+\frac{1}{2}|\frac{1}{3}+0| \\
&= |\rho|+\frac{1}{6} \\
&\geq |\rho| \\
&= ||\Gamma\rho-\Gamma\sigma||,
\end{aligned}
$$

and

$$\begin{aligned} Q(\rho,\sigma) &= \frac{1}{2}||\rho-\Gamma\sigma||+\frac{1}{2}||\sigma-\Gamma\rho|| \\ &= \frac{1}{2}|\rho-0|+\frac{1}{2}|\frac{1}{3}+\rho| \\ &\geq |\rho| \\ &= ||\Gamma\rho-\Gamma\sigma||. \end{aligned}$$

In all of the aforementioned cases, we can observe that

$$||\Gamma\rho-\Gamma\sigma|| \leq \max\{P(\rho,\sigma),Q(\rho,\sigma)\}$$

holds true when $\delta=\frac{1}{2}$.

Assuming that $\delta_k=0.85$, $\eta_k=0.65$, and $\gamma_k=0.65$, we observe the strong convergence of the F, M, Thakur, Abbas, Agarwal, Noor, Ishikawa, and Mann iteration processes to a fixed point $g=0$ of the mapping Γ. The convergence is demonstrated in Tables 1, 2, 3, 4, 5, and 6.

TABLE 1. Numerical comparison of iterations.

n	**F**	**M**	**Thakur**	**Abbas**
1	-0.9	-0.9	-0.9	-0.9
2	-0.04000000	0.04000000	-0.06925000	0.01435000
3	-0.00177777	-0.00933333	-0.00532840	0.00142423
4	-0.00007901	0.00041481	-0.00040999	0.00014135
5	-0.00000351	-0.00009679	-0.00003154	0.00001402
6	-0.00000001	-0.00000530	-0.00000242	0.00000139
7	0	-0.00000100	- 0.00000018	0.00000013
8	0	0.00000004	- 0.00000001	0.00000001
9	0	- 0.00000001	0	0
10	0	0	0	0
11	0	0	0	0
12	0	0	0	0
13	0	0	0	0

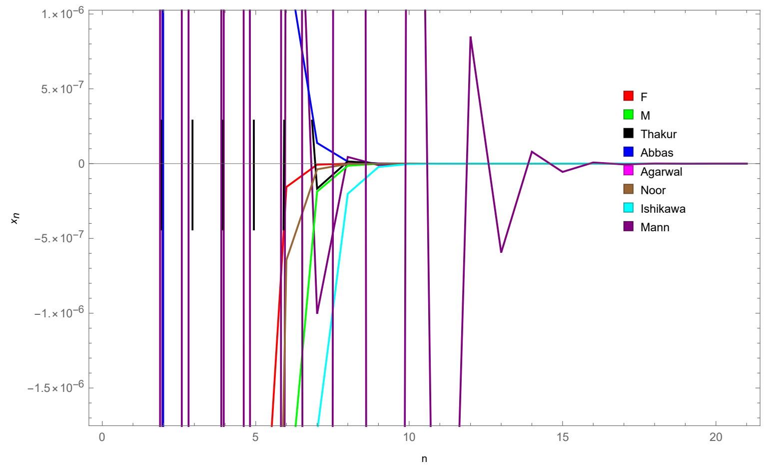

FIGURE 1. The convergence behaviors of the iterative schemes.

TABLE 2. Numerical comparison of of iterations.

n	**Agarwal**	**Noor**	**Ishikawa**	**Mann**
1	-0.9	-0.9	-0.9	-0.9
2	0.07900000	-0.05311666	-0.10100000	-0.12000000
3	-0.00513500	-0.00313486	-0.01133444	-0.08400000
4	0.00045073	-0.00018501	-0.00127197	0.01120000
5	-0.00002929	-0.00001091	-0.00014274	-0.00784000
6	0.00000257	- 0.00000064	-0.00001601	0.00104533
7	-0.00000016	-0.00000003	-0.00000179	-0.00073173
8	0.00000001	0	-0.00000020	0.000097564
9	0	0	-0.00000002	-0.00006829
10	0	0	0	0.00000910
11	0	0	0	-0.00000637
12	0	0	0	0.000000084
13	0	0	0	-0.00000059
14	0	0	0	0.00000007
15	0	0	0	-0.00000005
16	0	0	0	0
17	0	0	0	0
18	0	0	0	0
19	0	0	0	0
20	0	0	0	0
21	0	0	0	0

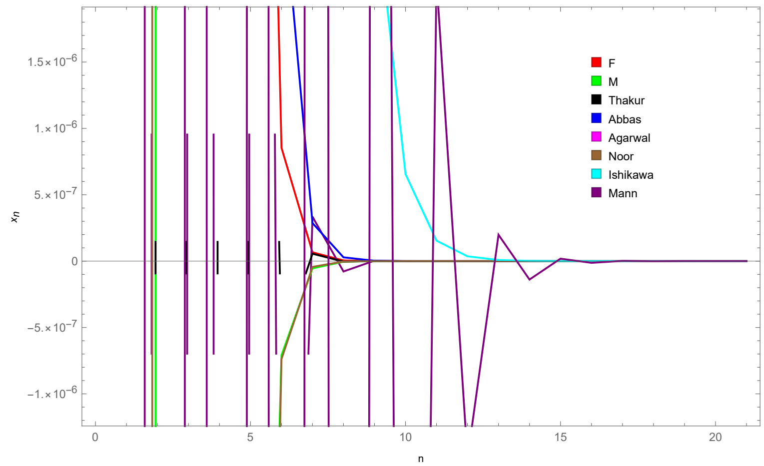

FIGURE 2. The convergence behaviors of the iterative schemes.

TABLE 3. Numerical comparison of of iterations.

n	**F**	**M**	**Thakur**	**Abbas**
1	0.3	0.3	0.3	0.3
2	0.02333333	-0.07000000	-0.02025000	0.02977500
3	0.00181481	0.00311111	-0.00155812	0.00295516
4	0.00014115	-0.00072592	-0.00011988	0.00029330
5	0.00001097	0.00003226	-0.00000929	0.00002911
6	0.00000082	-0.00000752	-0.00000070	0.00000288
7	0.00000006	0.00000033	- 0.00000005	0.00000028
8	0	- 0.00000752	0	0
9	0	0.00000033	0	0
10	0	-0.00000007	0	0
11	0	0	0	0
12	0	0	0	0
13	0	0	0	0
14	0	0	0	0
15	0	0	0	0
16	0	0	0	0
17	0	0	0	0
18	0	0	0	0
19	0	0	0	0
20	0	0	0	0

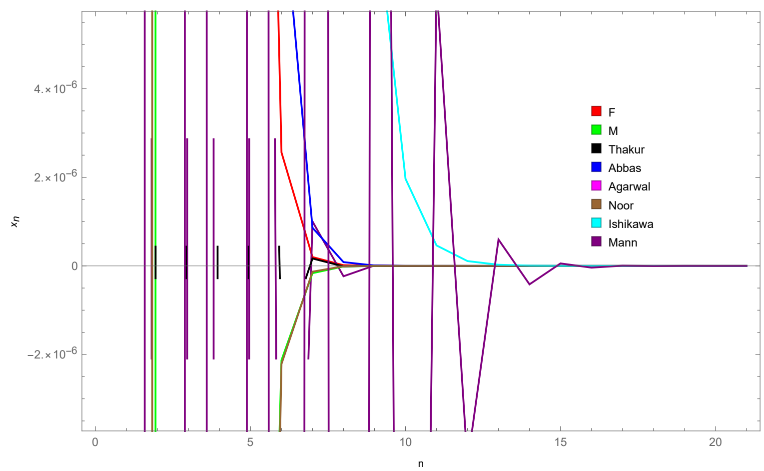

FIGURE 3. The convergence behaviors of the iterative schemes.

TABLE 4. The convergence behaviors of the iterative schemes.

n	**Agarwal**	**Noor**	**Ishikawa**	**Mann**
1	0.3	0.3	0.3	0.3
2	-0.01950000	-0.06082500	0.07050000	-0.21000000
3	0.00171166	-0.00358980	0.01656750	0.02800000
4	-0.00011125	-0.00021186	0.00389336	-0.01960000
5	0.00000976	-0.00001250	0.00091494	0.00261333
6	- 0.00000063	- 0.00000073	0.00021501	-0.00182933
7	0.00000005	-0.00000004	0.00005052	0.00024391
8	0	0	0.00001187	-0.00017073
9	0	0	0.00000379	0.00002276
10	0	0	0.00000065	-0.00001593
11	0	0	0.00000015	0.00000212
12	0	0	0.00000003	-0.00000148
13	0	0	0	0.00000019
14	0	0	0	- 0.00000013
15	0	0	0	0.00000001
16	0	0	0	-0.00000001
17	0	0	0	0
18	0	0	0	0
19	0	0	0	0
20	0	0	0	0

Table 5. The convergence behaviors of the iterative schemes.

n	**F**	**M**	**Thakur**	**Abbas**
1	0.9	0.9	0.9	0.9
2	0.07000000	-0.21000000	-0.06075000	0.08932500
3	0.00544444	0.00933333	-0.00467437	0.00886550
4	0.00042345	-0.00217777	-0.00035966	0.00087990
5	0.00003293	0.00009679	-0.00002767	0.00008733
6	0.00000256	-0.00002258	-0.00000212	0.00000866
7	0.00000019	0.00000100	- 0.00000016	0.00000086
8	0.00000001	- 0.00000023	-0.00000001	0.00000008
9	0	0.00000001	0	0
10	0	0	0	0
11	0	0	0	0
12	0	0	0	0
13	0	0	0	0
14	0	0	0	0
15	0	0	0	0
16	0	0	0	0

Table 6. The convergence behaviors of the iterative schemes.

n	**Agarwal**	**Noor**	**Ishikawa**	**Mann**
1	0.9	0.9	0.9	0.9
2	-0.05850000	-0.18547500	0.21150000	-0.63000000
3	0.00513500	-0.01076940	0.04970250	0.08400000
4	-0.00033377	-0.00063559	0.01168008	-0.05880000
5	0.00002929	-0.00003751	0.00274482	0.00784000
6	- 0.00000190	- 0.00000221	0.00064503	-0.00548800
7	0.00000016	-0.00000013	0.00015158	0.00073173
8	-0.00000001	0	0.00003562	-0.00051221
9	0	0	0.00000837	0.00006829
10	0	0	0.00000196	-0.00004780
11	0	0	0.00000046	0.00000637
12	0	0	0.00000010	-0.00000446
13	0	0	0.00000002	0.00000059
14	0	0	0	- 0.00000041
15	0	0	0	0.00000005
16	0	0	0	-0.00000003
17	0	0	0	0

Based on the information provided in Tables 1, 2, 3, 4, 5, 6, and Figures 1, 2, 3, it can be observed that the F-iterative process exhibits a faster convergence towards the fixed point $g = 0$ of the mapping Γ compared to the other iterations.

5. Application

5.1. Polynomial equations. To further support our theorem, we present the following application as evidence of its validity.

Theorem 5.1. *Let us consider the equation:*

$$\rho^q + 1 = (q^4 - 1)\rho^{q+1} + q^4\rho \tag{5.1}$$

where q represents a natural number greater than or equal to 3*, that is, $q \geq 3$. We aim to show that the sequence $\{\rho_n\}$ defined by Algorithm* (1.1) *converges to a solution of equation* (5.1) *if* $\lim_{n\to\infty} \varrho(\rho_n, u) = 0$*, where u represents the solution.*

Proof. Firstly, it is worth noting that if $|\rho| > 1$, the solution set of equation (5.1) would be u. Therefore, we consider $D = [-1, 1]$ as the solution set. For any $\rho, \sigma \in D$, we define the distance between them as $||\rho - \sigma|| = |\rho - \sigma|$.

Next, let us define the mapping $\Gamma : D \to D$ as follows:

$$\Gamma\rho = \frac{\rho^q + 1}{(q^4 - 1)\rho^q + q^4}.$$

It is important to note that since $q \geq 2$, we can conclude that $q^4 \geq 6$. Thus, our goal is to prove that the mapping Γ defined earlier is an $GARSN$ mapping.

$$\begin{aligned}
||\Gamma\rho - \Gamma\sigma|| &= |\frac{\rho^q + 1}{(q^4 - 1)\rho^q + q^4} - \frac{\sigma^q + 1}{(q^4 - 1)\sigma^q + q^4}| \\
&= |\frac{\rho^q - \sigma^q}{((q^4 - 1)\rho^q + q^4)((q^4 - 1)\sigma^q + q^4)}| \\
&\leq \frac{|\rho - \sigma|}{q^4} \\
&\leq \frac{|\rho - \sigma|}{6} \\
&\leq |\rho - \sigma| \\
&\leq \max\{P(\rho, \sigma), Q(\rho, \sigma)\},
\end{aligned}$$

where

$$P(\rho, \sigma) = \delta||\Gamma\rho - \rho|| + \delta||\Gamma\sigma - \sigma|| + (1 - 2\delta)||\rho - \sigma||,$$

and

$$Q(\rho, \sigma) = \delta||\Gamma\rho - \sigma|| + \delta||\Gamma\sigma - \rho|| + (1 - 2\delta)||\rho - \sigma||.$$

Therefore,

$$||\Gamma\rho - \Gamma\sigma|| \leq \delta \max\{P(\rho,\sigma), Q(\rho,\sigma)\}.$$

Consequently, based on Theorem 3.5, we can conclude that the sequence $\{\rho_n\}$ converges to the fixed point of Γ. However, since $F(\Gamma) = u$, we can further assert that $\{\rho_n\}$ converges to a solution of equation (5.1). $\square$

5.2. **Linear system of equations.** Let D be a set consisting of n-dimensional real numbers, denoted as $\mathbb{R}^n$, where $\mathbb{R}$ represents the set of real numbers and n is a positive integer.

We define the distance between two points $\rho = (\rho_1, ..., \rho_n)$ and $\sigma = (\sigma_1, ..., \sigma_n)$ in D as the maximum absolute difference between their corresponding components, given by

$$||\rho - \sigma|| = \max_{1 \leq i \leq n} |\rho_i - \sigma_i|.$$

Theorem 5.2. *Consider the following system:*

$$\begin{cases} u_{11}\rho_1 + u_{12}\rho_2 + u_{13}\rho_3 + ... + u_{1n}\rho_n = \omega_1, \\ u_{21}\rho_1 + u_{22}\rho_2 + u_{23}\rho_3 + ... + u_{2n}\rho_n = \omega_2, \\ \vdots \\ u_{n1}\rho_1 + u_{n2}\rho_2 + u_{n3}\rho_3 + ... + u_{nn}\rho_n = \omega_n. \end{cases}$$

If $\theta = \max_{1\leq i \leq n}(\sum_{j=1, j\neq i}^{n} |u_{ij}| + |1 + u_{ii}|) < 1$*, then the sequence of F-iterates converges to the sought solution of the aforementioned linear system.*

Proof. Consider the map $\Gamma : D \to D$ defined by $\Gamma\rho = (E + I_n)\rho - \omega$, where

$$E = \begin{pmatrix} u_{11} \ u_{12} \ . \ . \ .u_{1n} \\ u_{21} \ u_{22} \ . \ . \ .u_{2n} \\ \vdots \\ u_{n1} \ u_{n2} \ . \ . \ .u_{nn} \end{pmatrix}.$$

$\rho = (\rho_1, ..., \rho_n), \sigma = (\sigma_1, ..., \sigma_n) \in \mathbb{R}^n$, I_n is the identity matrix for $n \times n$ matrices and $\omega = (\omega_1, ..., \omega_n) \in \mathbb{D}^n$.

Let us prove that $||\Gamma\rho - \Gamma\sigma|| \leq \theta||\rho - \sigma||$ for all $\rho, \sigma \in \mathbb{R}^n$.

We denote by

$$\tilde{E} = E + I_n = (\tilde{e}_{ij}), i, j = 1, ..., n$$

with

$$\tilde{e}_{ij} = \begin{cases} u_{ij}, & \text{if } j \neq i, \\ 1 + u_{ij}, & \text{if } j = i. \end{cases}$$

Hence, $\max_{1\leq i\leq n} \sum_{j=1}^{n} |\tilde{e}_{ij}| = \max_{1\leq i\leq n}(\sum_{j=1,j\neq i}^{n} |u_{ij}| + |1 + u_{ii}|) = \theta < 1$.

On the other hand, for all $i = 1, ..., n$, we have

$$(\Gamma\rho)_i - (\Gamma\sigma)_i = \sum_{j=1}^{n} \tilde{e}_{ij}(\rho_j - \sigma_j). \tag{5.2}$$

Therefore, using (5.2), we obtain

$$\begin{aligned} ||\Gamma\rho - \Gamma\sigma|| &= \max_{1\leq i\leq n}(|(\Gamma\rho)_i - (\Gamma\sigma)_i|) \\ &\leq \max_{1\leq i\leq n}(\sum_{j=1}^{n} |\tilde{e}_{ij}||\rho_j - \sigma_j|) \\ &\leq \max_{1\leq i\leq n}\sum_{j=1}^{n} |\tilde{e}_{ij}| \max_{1\leq k\leq n}(|\rho_k - \sigma_k|) \\ &= \theta||\rho - \sigma|| \\ &= \max\{P(\rho,\sigma), Q(\rho,\sigma)\}. \end{aligned}$$

Hence, Γ is Reich-Suzuki type nonexpansive. Hence the sequence of F-iterates converges to the sought solution. □

5.3. **Differential and integral equations.** In a publication by El-Sayed and Omar in 2022 [7], they proved the existence and uniqueness of the weak solution for a delay composite functional differential equations ($DCFDE$) of the Volterra-Stieljes Type (VST). Several authors have also addressed the problem of solving this type of $DCFDE$, as documented in [13, 21]. In this section, our focus is on estimating the weak solution of a $DCFDE$.

Consider a reflexive Banach space denoted by G, equipped with the norm $||.||_G$. Let G^* represent the dual space of G. We define $C^{'}[J, G]$ as the class of continuous functions on the interval $J = [0, M]$, where M is a positive constant. The norm of ρ in $C^{'}[J, G]$ is given by:

$$||\rho||_{C'} = \sup_{t\in J} ||\rho(t)||_G, \rho \in C^{'}[J, G],$$

where ρ belongs to $C^{'}[J, G]$.

Let's examine the subsequent $DCFDE$ of the VST:

$$\frac{\varrho}{\varrho t}\rho(t) = f_1(t, \int_0^{h(t)} f_2(t, u, \rho(u))\varrho_u g(t, u)), \quad t \in J \tag{5.3}$$

along with the initial condition:

$$\rho(0) = \rho_0. \tag{5.4}$$

Let's assume the following conditions hold:

(i) The function $h : J \to J$ is continuous and increasing, satisfying $h(t) \leq t$.

(ii) The function $f_1 : J \times G \to G$ is weakly continuous and satisfies the weak Lipschitz condition with Lipschitz constant L_1. This condition can be expressed as:

$$|\Gamma(f_1(t,\rho)) - f_1(t,\sigma)| \leq L_1|\Gamma(\rho - \sigma)|,$$

for $L_1 > 0, (t,\rho), (t,\sigma) \in J \times G, \Gamma \in G^*$.

(iii) The function $f_2 : J \times J \times G \to G$ is weakly continuous and weakly satisfies the Lipschitz condition with Lipschitz constant L_2, This condition can be expressed as:

$$|\Gamma(f_2(t,u,\rho)) - f_2(t,u,\sigma)| \leq L_2|\Gamma(\rho - \sigma)|,$$

for all $(t,u,\rho), (t,u,\sigma) \in J \times J \times G$, where $\Gamma \in G^*$ and $L_2 > 0$.

(iv) The function $g : J \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exists a constant $\mathfrak{w}$ such that:

$$\mathfrak{w} = \max\{\sup|g(t,h(t))| + \sup|g(t,0)|\} \in J.$$

(v) The condition $L_1 L_2 \mathfrak{w} t < 1$ holds.

The task of finding the solution to equations (5.3) and (5.4) can be equivalently stated as finding the solution to the following integral equation, as mentioned in [7].

$$\rho(t) = \rho_0 + \int_0^t f_1(u, \int_0^{h(t)} f_2(u,\theta,\rho(\theta)))\varrho_\theta g^{'}(u,\theta))\varrho u.$$

In the subsequent theorem, we establish an approximation of the solution to the equations (5.3) and (5.4) by employing Algorithm (1.1).

Theorem 5.3. *Assuming that Assumptions* $(i)-(v)$ *are met, it can be inferred that Problems* (5.3) *and* (5.4) *possess a sole solution* g *in the space* $C^{'}[J,G]$, *and the sequence* $\{\rho_n\}$ *defined in* (1.1) *exhibits convergence towards* g.

Proof. Consider the sequence $\{\rho_n\}$ defined in (1.1). Let's define an operator Γ on $C^{'}[J,G]$ as follows:

$$\Gamma(\rho(t)) = \rho_0 + \int_0^t f_1(u, \int_0^{h(t)} f_2(u,\theta,\rho(\theta)))\varrho_\theta g^{'}(u,\theta))\varrho u.$$

Now

$$\begin{aligned} ||\omega_n - g||_{C'} &= ||\Gamma[(1-\delta_n)\rho_n + \delta_n\Gamma\rho_n] - g||_{C'} \\ &\leq ||(1-\delta_n)\rho_n + \delta_n\Gamma\rho_n - g||_{C'} \\ &\leq (1-\delta_n)||\rho_n - g||_{C'} + ||\Gamma\rho_n - g||_{C'}, \end{aligned}$$

and

$$\begin{aligned}
||\Gamma(\rho_n) - g||_{C'} &= ||\Gamma(\rho_n) - \Gamma g||_{C'} \\
&\leq ||\rho_0 + \int_0^t f_1(u, \int_0^{h(u)} f_2(u,\theta,\rho_n(\theta)))\varrho_\theta g^{'}(u,\theta))\varrho u - \rho_0 \\
&\quad - \int_0^t f_1(u, \int_0^{h(u)} f_2(u,\theta,p(\theta)))\varrho_\theta g^{'}(u,\theta))\varrho u||_{C'} \\
&= |\Gamma[\int_0^t f_1(u, \int_0^{h(u)} f_2(u,\theta,\rho_n(\theta)))\varrho_\theta g^{'}(u,\theta))\varrho u \\
&\quad - \int_0^t f_1(u, \int_0^{h(u)} f_2(u,\theta,p(\theta)))\varrho_\theta g^{'}(u,\theta))\varrho u]| \\
&\leq \int_0^t L_1|\Gamma[\int_0^{h(u)} f_2(u,\theta,\rho_n(\theta)))\varrho_\theta g^{'}(u,\theta)) \\
&\quad - \int_0^{h(u)} f_2(u,\theta,p(\theta))]\varrho_\theta g^{'}(u,\theta)|\varrho u \\
&\leq L_1 \int_0^t \int_0^{h(u)} |\Gamma[f_2(u,\theta,\rho_n(\theta)) \\
&\quad -f_2(u,\theta,p(\theta)]\varrho_\theta g^{'}(u,\theta)|\varrho u \\
&\leq L_1 \int_0^t \int_0^{h(u)} L_2|\Gamma[\rho_n(\theta) - p(\theta)]\varrho_\theta g^{'}(u,\theta)|\varrho u \\
&= L_1 L_2 ||\rho_n - g||_{C'} \int_0^t \int_0^{h(u)} \varrho_\theta g^{'}(u,\theta)\varrho u \\
&= L_1 L_2 ||\rho_n - g||_{C'} \int_0^t (g^{'}(u,h(u)) - g^{'}(u,0))\varrho u \\
&\leq L_1 L_2 ||\rho_n - g||_{C'} \int_0^t \varrho u \\
&= L_1 L_2 \mathfrak{w} t ||\rho_n - g||_{C'} \\
&\leq ||\rho_n - g||_{C'}.
\end{aligned}$$

So,

$$\begin{aligned}
||\omega_n - g||_{C'} &\leq (1-\delta_n)||\rho_n - g||_{C'} + \delta_n||\rho_n - g||_{C'} \\
&\leq ||\rho_n - g||_{C'}.
\end{aligned}$$

Therefore,

$$\begin{aligned} ||\sigma_n - g||_{C'} &= ||\Gamma\omega_n - g||_{C'} \\ &= ||\Gamma\omega_n - \Gamma g||_{C'} \\ &\leq ||\omega_n - g||_{C'} \\ &\leq ||\rho_n - g||_{C'}. \end{aligned}$$

If we define $||\rho_n - g||_{C'} = \nu_n$, it follows that:

$$\nu_{n+1} \leq \nu_n \quad \text{for all } n \in \mathbb{N}.$$

This inequality implies that:

$$\lim_{n\to\infty} \nu_n = 0.$$

Therefore, we can conclude that $\{\rho_n\}$ converges to g, that is, $\rho_n \to g$. Hence the sequence of F-iterates converges to the sought solution. □

6. Conclusions

The class of Reich-Suzuki type nonexpansive maps is analyzed with a faster iterative scheme and some new convergence results are obtained. We shown by a new example that the class of α-Reich-Suzuki type nonexpansive maps properly includes many classical classes of nonlinear maps. Moreover, it has been found by carrying out some numerical computations that the effectiveness of F-iterative approach is more considerable and faster than the other classical iterative schemes when dealing with the class of α-Reich-Suzuki type nonexpansive maps. Eventually, three new applications are established of our main outcome.

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