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ITERATIVE ALGORITHM FOR MULTIVALUED GENERALIZED (α, β)-NONEXPANSIVE MAPPINGS IN AN ORDERED CAT(0) SPACE WITH APPLICATION

Jong Kyu Kim[1], Samir Dashputre[2], Rashmi Verma[3] and Padmavati[4]

[1]Department of Mathematics Education, Kyungnam University,
Changwon, Gyeongnam 51767, Korea
e-mail: jongkyuk@kyungnam.ac.kr

[2]Department of Mathematics, Shahid Durwasha Nishad, Govt.College Arjunda,
Balod (C.G), India
e-mail: samir231973@gmail.com

[3]Department of Applied Mathematics, Shri Shankaracharya Institute of
Professional Management and Technology, Raipur(C.G), India
e-mail: nayakrashmi88@gmail.com

[4]Department of Mathematics, Govt. V.Y.T. Autonomous P.G. College,
Durg(C.G), India
e-mail: padmavati.sudha62@gmail.com

Abstract. This paper examines the SR iterative algorithm, which approximates fixed points of monotone multivalued generalized (α, β)-nonexpansive mappings in the setting of an ordered CAT(0) space. We establish convergence results for the SR iterative algorithm applied to monotone generalized (α, β)-nonexpansive mappings. Additionally, we demonstrate the application of these results to fractional differential equations and equivalent nonlinear integral equations.

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[0]Corresponding author: Rashmi Verma(nayakrashmi88@gmail.com).

1. Introduction

A metric space (X, d) is a $CAT(0)$ space if it is geodesically connected and every geodesic triangle within X is no thicker than its comparison triangle in the Euclidean plane. It is a well-established fact that any complete, simply connected Riemannian manifold with nonpositive sectional curvature qualifies as a CAT(0) space. Additional examples include pre-Hilbert spaces, and R-trees (refer to [13]), among others. For further information on these spaces, one can refer to [6, 8, 9].

The investigation into sufficient conditions for the existence of fixed points of multivalued contraction and nonexpansive mappings using the Hausdorff metric was initially conducted by Markin [21] and later expanded by Nadler [22]. The study of fixed point theory in $CAT(0)$ spaces began with Kirk [17, 18], who demonstrated that every nonexpansive mapping on a bounded, closed, convex subset of a complete $CAT(0)$ space possesses a fixed point. In fact, $CAT(0)$ spaces offer an appropriate setting for deriving fixed points of nonexpansive mappings and their various generalizations [10, 11, 27, 30].

The conditions for a mapping to be nonexpansive traditionally apply to all points within the mapping's domain. However, relaxed conditions have been introduced to ensure that these mappings do not affect the outcomes of fixed point results. Addressing this issue, Suzuki [28] introduced a new class of mappings, formally known as mappings satisfying Condition (C), particularly in the context of uniform convex Banach spaces. It's worth noting that the class of nonexpansive mappings is a specific subclass within this broader class of mappings satisfying Condition (C), which may not necessarily be continuous.

Building on this framework, Akbar and Eslamian [2] extended this concept from single-valued mappings to multivalued mappings, successfully obtaining fixed points within the framework of Banach spaces.

2. Preliminaries

Let M be a nonempty subset of a metric space X. The subset M is termed proximal if for each $v \in X$, there exists an element $k \in M$ such that

$$d(v, k) = d(v, M) = \inf\{d(v, y) : y \in M\} \tag{2.1}$$

where $d(v, M)$ represents the distance from the point v to the set M. Let $P(X)$ denote the family of nonempty closed bounded subsets of X, $D(X)$ denote the family of nonempty bounded proximal subsets of X, and $\kappa(X)$ denote the family of nonempty compact subsets of X. The Hausdorff distance H on $P(X)$ is defined by

$$H(E,F) = \max\left\{\sup_{e\in E} d(e,F), \sup_{f\in F} d(f,E)\right\}. \tag{2.2}$$

This mapping H is known as the Pompeiu-Hausdorff metric induced by d.

A multivalued mapping $\Psi : X \to P(X)$ is said to have a fixed point if there exists an element $p \in X$ such that $p \in \Psi(p)$. The set $F(\Psi)$ denotes the set of all fixed points of Ψ. An element $p \in X$ is said to be a strict fixed point (or end point of Ψ) if

$$\Psi(p) = \{p\}.$$

The set of strict fixed points (end points) of Ψ is denoted by $SF(\Psi)$, and clearly, $SF(\Psi) \subseteq F(\Psi)$.

A multivalued mapping $\Psi : X \to P(X)$ is said to be:

(1) nonexpansive if

$$H(\Psi(v), \Psi(y)) \le d(v,y), \quad \forall v, y \in X.$$

(2) quasi-nonexpansive if $F(\Psi)$ is nonempty and for any $p \in F(\Psi)$,

$$H(\Psi(v), \Psi(p)) \le d(v,p), \quad \forall v \in X.$$

(3) satisfying condition (C) if for any $v, y \in X$ with $\frac{1}{2}d(v, \Psi(v)) \le d(v,y)$,

$$H(\Psi(v), \Psi(y)) \le d(v,y).$$

(4) a generalized α-nonexpansive mapping if there exists an $\alpha \in [0,1)$ such that for each $v, y \in X$ with $\frac{1}{2}d(v, \Psi(v)) \le d(v,y)$,

$$H(\Psi(v), \Psi(y)) \le \alpha d(v, \Psi(y)) + \alpha d(y, \Psi(v)) + (1-2\alpha) d(v,y).$$

A geodesic path in a metric space (X,d) is a map ξ joining two points v and y in X from a closed interval $[0,l] \subset \mathbb{R}$ such that $\xi(0) = v$, $\xi(l) = y$, and $d(\xi(q), \xi(q')) = |q - q'|$ for all $q, q' \in [0,l]$. In particular, $l = d(v,y)$. The image of ξ is called the geodesic or metric segment joining v and y. If the image is unique, then it is denoted by $[v,y]$. The space (X,d) is called a geodesic space if any two points of X are connected by a geodesic, whereas X is known to be uniquely geodesic if for each $v, y \in X$, there is exactly one metric segment which joins v and y.

A subset M of X containing every geodesic segment joining any two of its points is said to be convex. In a geodesic metric space (X,d), a geodesic triangle $\triangle(a,b,c)$ consists of three points in X where a, b, and c are the vertices of $\triangle$, and geodesic segments between them are the sides of $\triangle$. A comparison triangle for $\triangle(a,b,c)$ in (X,d) is a triangle $\bar{\triangle}(a,b,c) = \triangle(\bar{a}, \bar{b}, \bar{c})$ in the Euclidean plane $\mathbb{R}^2$ such that $d(a,b) = d_{\mathbb{R}^2}(\bar{a}, \bar{b})$, $d(a,c) = d_{\mathbb{R}^2}(\bar{a}, \bar{c})$, and $d(b,c) = d_{\mathbb{R}^2}(\bar{b}, \bar{c})$.

Suppose that $\triangle$ is a geodesic triangle in E and $\bar{\triangle}$ is a comparison triangle for $\triangle$. In a geodesic space, if all geodesic triangles of appropriate size satisfy the following comparison axiom called $CAT(0)$ inequality:

$$d(u,v) \leq d_{\mathbb{R}^2}(\bar{u},\bar{v}) \ \forall \ u,v \in \triangle, \bar{u},\bar{v} \in \bar{\triangle}$$

then such a geodesic space is said to be $CAT(0)$ space.

Aoyama and Kohshaka [4] suggested the class of α-nonexpansive mappings in Banach spaces who also explored fixed points of such mappings. Recently, Iqbal et al. [14] proposed the concept of multivalued generalized α-nonexpansive mappings and obtained existence and approximation results in the setting of a Banach space.

In 2018, Harandi et al. [3] presented the class of (α,β)-nonexpansive mappings which are properly larger than the class of α-nonexpansive mapping for a fixed point sequence. Many researchers have presented and studied iterative techniques for approximating the fixed points and established convergence results in CAT(0) spaces for the general class of multivalued mappings including Mann, Ishikawa, and S-iterative schemes [19, 26, 30, 29].

Motivated by [19, 28], we present the class of monotone multivalued generalized (α,β)-nonexpansive multivalued mappings and establish fixed points for such mappings in an ordered $CAT(0)$ space (see [5]). We will approximate the fixed points of the proposed mapping using the S-iterative scheme. Under suitable conditions, Δ-convergence and strong convergence results will be established. An application of the convergence results is also presented. Now, we recall some important definitions and results needed in the sequel. We assume that (X,d) is a $CAT(0)$ space.

Lemma 2.1. ([12]) *For $v,y \in X$ and $q \in [0,1]$, there exists a unique $h \in [v,y]$ such that*

$$d(v,h) = (1-q)d(v,y) \ \text{and} \ d(y,h) = qd(v,y).$$

We denote the unique point $h \in [v,y]$ in the above Lemma by $(1-q)v \oplus qy$.

Lemma 2.2. ([12]) *For $v,y,z \in X$ and $q \in [0,1]$, we have the following inequalities:*

(i) $d((1-q)v \oplus qy, z) \leq (1-q)d(v,z) + qd(y,z)$.
(ii) $d((1-q)v \oplus qy, z)^2 \leq (1-q)d(v,z)^2 + qd(y,z)^2 - q(1-q)d(v,y)^2$.

Let M be a bounded subset X and $\{v_n\}$ a bounded sequence in X. Then,

(1) a mapping $r(., \{v_n\}) : X \to \mathbb{R}^+$ by

$$r(v,\{v_n\}) = \limsup_{n\to\infty} d(v_n, v).$$

For each $v \in X$, the value $r(v,\{v_n\})$ is called asymptotic radius of $\{v_n\}$ at v [1].

(2) The asymptotic radius of $\{v_n\}$ [1] relative to M is the number r given by

$$r = \inf\{r(v, \{v_n\}); v \in M\}.$$

Denote asymptotic radius of $\{v_n\}$ relative to M by $r(M, \{v_n\})$.

(3) The asymptotic center of $\{v_n\}$ relative to M is the set $A(\{v_n\})$ of points in X for which $r(M, \{v_n\}) = r(v, \{v_n\})$, that is,

$$A(\{v_n\}) = \{v \in Y : r(v, \{v_n\}) = r\}.$$

Definition 2.3. ([12]) A sequence $\{v_n\}$ in a $CAT(0)$ space X is Δ-convergent to $v \in X$ if v is the unique asymptotic center of every subsequence of $\{v_n\}$. In such situation, we write $\Delta - \lim_n v_n = v$ and v is the Δ-limit of $\{v_n\}$.

Given $\{v_n\} \subset X$ such that $\{v_n\}$ is Δ-convergent to v if we take $y \in X$ such that $v \neq y$, then by the uniqueness of the asymptotic center, we have

$$\limsup_{n\to\infty} d(v_n, v) < \limsup_{n\to\infty} d(v_n, y).$$

Lemma 2.4. ([12]) *In a complete CAT(0) space, every bounded sequence admits a Δ-convergent subsequence.*

Lemma 2.5. ([12]) *If $\{v_n\}$ is a bounded sequence in a closed convex subset M of a complete CAT(0) space, then the asymptotic center of $\{v_n\}$ is in M.*

Lemma 2.6. ([19]) *Let p be an element of a complete CAT(0) space X. Assuming $\{t_n\}$ is a sequence in $[\theta, \eta]$ for some $\theta, \eta \in (0, 1)$ and that $\{v_n\}$, $\{y_n\}$ are two sequences in X satisfying the following for some $r \geq 0$:*

$$\limsup_{n\to\infty} d(v_n, p) \leq r, \quad \limsup_{n\to\infty} d(y_n, p) \leq r$$

and

$$\limsup_{n\to\infty} d(t_n v_n + (1 - t_n) y_n, p) = r.$$

Then $\lim_{n\to\infty} d(v_n, y_n) = 0$.

Let M be a nonempty convex subset of X and $\Psi : M \to P(M)$ with $p \in F(\Psi)$. Then, the modification of S-iterative scheme [1] in the framework of $CAT(0)$ spaces is given as follows:

Let $v_1 \in M$. Define, for $\alpha_n, \beta_n \in (0, 1)$

$$\begin{cases} y_n = (1 - \beta_n) x_n \oplus \beta_n s_n, \\ x_{n+1} = (1 - \alpha_n) x_n \oplus \alpha_n s_n^{'}, \end{cases} \tag{2.3}$$

with $s_n \in \Psi x_n,\ s_n^{'} \in \Psi y_n$ *and* $d(s_n, s_n^{'}) \leq H(\Psi x_n, \Psi y_n)$ *satisfying*

$$d(s_{n+1}, s_n^{'}) \leq H(\Psi x_{n+1}, \Psi y_n).$$

Consider a complete $CAT(0)$ space, X, endowed with partial order $\preccurlyeq$. Two elements v, y are comparable if $v \preccurlyeq y$ *or* $y \preccurlyeq v$. For any $a \in X$, define

$$[a, \to) = \{v \in X : a \preccurlyeq v\} \quad \text{and} \quad (\leftarrow, a] = \{v \in X : v \preccurlyeq a\}$$

for every $v, y \in X$. An order interval $[v, y]$ is the set given by

$$[v, y] = \{w \in X : v \preccurlyeq w \preccurlyeq y\}.$$

Throughout in this paper, we consider the order intervals to be closed convex subsets of an ordered $CAT(0)$ space $(X, \preccurlyeq)$. Let M be a nonempty closed convex subset of $(X, \preccurlyeq)$. A mapping $\Psi : M \to P(M)$ is called monotone if for any $u_v \in \Psi v$ there exists $u_y \in \Psi y$ *such that* $u_v \preccurlyeq u_y$ *whenever* $v \preccurlyeq y$ *for all* $v, y \in M$. Moreover, the mapping Ψ is said to be :

(1) monotone nonexpansive, if Ψ is monotone and such that for any comparable $v, y \in M$,

$$H(\Psi v, \Psi y) \leq d(v, y), \tag{2.4}$$

(2) monotone quasi-nonexpansive, if Ψ is monotone with $p \in F(\Psi) \neq \phi$ and $v \in M$ such that whenever v, p are comparable,

$$H(\Psi v, \Psi p) \leq d(v, p), \tag{2.5}$$

holds.

(3) monotone Suzuki generalized nonexpansive, if

$$\frac{1}{2}d(v, \Psi v) \leq d(v, y) \implies H(\Psi v, \Psi y) \leq d(v, y). \tag{2.6}$$

(4) monotone α-generalized nonexpansive, if for some $\alpha < 1$,

$$H(\Psi v, \Psi p)^2 \leq \alpha d(\Psi v, p)^2 + \alpha d(v, \Psi p)^2 + (1 - 2\alpha)d(v, p)^2. \tag{2.7}$$

3. Multivalued generalized (α, β)-nonexpansive mapping

Recently, Abbas et. al. [1] initiated the notion of monotone generalized (α, β)-nonexpansive mapping, which is a wider class of nonexpansive type mapping that properly contains nonexpansive mapping satisfying condition (C) and generalized α-nonexpansive mappings in setting of $CAT(0)$ space as follows.

Definition 3.1. Let M be a nonempty subset of a $CAT(0)$ space (X, d). A multivalued mapping $\Psi : M \to P(M)$ satisfies the Condition $(C_{(\alpha,\beta)})$ if there exists $(\alpha, \beta) \in (0,\, 1)$ such that for any $v, y \in M$,

$$\frac{1}{2} d(v, \Psi v) \leq d(v, y) \tag{3.1}$$

implies that

$$H(\Psi v, \Psi y) \leq \alpha d(y, \Psi v) + \beta d(v, \Psi y) + (1 - \alpha - \beta) d(v, y). \tag{3.2}$$

If a multivalued mapping satisfies the Condition $(C_{(\alpha,\beta)})$ in a $CAT(0)$ space then we say Ψ is the multivalued generalized (α, β)-nonexpansive mapping.

Let M be a nonempty closed subset of an ordered $CAT(0)$ space $(X, \preccurlyeq)$. A mapping $\Psi : M \to P(M)$ is said to be a monotone multivalued generalized (α, β)-nonexpansive mapping if

(a) Ψ is monotone,
(b) Ψ satisfies (3.1) for all $v, y \in M$ and either $v \preccurlyeq y$ *or* $y \preccurlyeq v$.

Remark 3.2. (1) Multivalued generalized (α, β)-nonexpansive mappings extend and generalize the class of mappings introduced by [14]. Indeed, if $\alpha = \beta$ then the mapping is reduced to multivalued generalized α-nonexpansive mapping.
(2) Multivalued generalized (α, β)-nonexpansive mappings contain the class of mappings satisfying the Condition (C). Substituting $\alpha = \beta = 0$ we get our desired mapping.
(3) Every nonexpansive mapping is generalized (0,0)-nonexpansive mapping.

The following example demonstrates that a multivalued generalized (α, β)-nonexpansive mappings in the settings of an ordered $CAT(0)$ space which is neither nonexpansive nor satisfies the Condition (C).

Example 3.3. ([1]) Consider an Example 18 of [24] where

$$d(v, y) = |v_1 - y_1| + |v_1 v_2 - y_1 y_2|.$$

Define an order on X as follows: for $v = (v_1, v_2)$ and $y = (y_1, y_2), v < y$ if and only if $v_1 \leq y_1$ and $v_2 \leq y_2$. Thus $(X, d, \preccurlyeq)$ is an ordered Hyperbolic space which is an example of an ordered $CAT(0)$ space.

Let $M = [0, 2] \times [0, 2] \subset Y$ and $\Psi : M \to P(M)$ be defined by

$$\Psi(v_1, v_2) = \begin{cases} \{(0, \frac{1}{4}), (\frac{1}{2}, 1)\}, & \text{if } (v_1, v_2) \neq (2, 2), \\ \{(\frac{3}{2}, \frac{3}{2}), (\frac{19}{10}, \frac{19}{10})\}, & \text{if } (v_1, v_2) = (2, 2). \end{cases}$$

Then, the mapping Ψ does not satisfy the Condition (C) on M and therefore is not nonexpansive. Indeed, for $(v_1, v_2) = (1, 1)$ and $(y_1, y_2) = (2, 2)$, we have

$$\frac{1}{2}d(v, \Psi v) = \frac{1}{2}\min\{\frac{1}{2}, \frac{1}{4}\} = \frac{1}{8}$$

and

$$d(v, y) = |1 - 2| + |1(1) - 2(2)| = 4.$$

Thus

$$\frac{1}{2}d(v, \Psi v) < d(v, y).$$

And also, the mapping Ψ is multivalued generalized (α, β)-nonexpansive mapping, for $\alpha = \frac{7}{8}$ and $\beta = \frac{1}{8}$.

In 2021, Abbas et al. [1] introduced multivalued generalized (α, β)-nonexpansive mapping and its properties as follows:

(i) If Ψ satisfies the Condition (C) as defined in (3.2), then Ψ satisfies the Condition $(C_{(\alpha,\beta)})$ but the converse is not true in general (see [1]).

(ii) If Ψ satisfies the Condition $(C_{(\alpha,\beta)})$ with $F(\Psi) \neq \phi$, then Ψ is quasi-nonexpansive.

Indeed,

$$\tfrac{1}{2}dist(p, \Psi p) \leq d(v, p), \ \forall v \in X, p \in F(\Psi).$$

As Ψ satisfying the Condition $(C_{(\alpha,\beta)})$ for some $\alpha, \beta \in [0, 1)$ such that

$$H(\Psi v, \Psi p) \leq \alpha \ dist(p, \Psi v) + \beta \ dist(v, \Psi p) + (1 - \alpha - \beta)d(v, p)$$

holds. Then

$$H(\Psi v, \Psi y) \leq \alpha H(\Psi p, \Psi v) + \beta \ dist(p, \Psi p) + \beta d(v, p)(1 - \alpha - \beta)d(v, p),$$

implies that

$$(1 - \alpha)H(\Psi v, \Psi y) \leq d(v, p),$$

since, $1 - \alpha > 0$, it follows that

$$H(\Psi v, \Psi y) \leq d(v, p).$$

Lemma 3.4. ([1]) *Let us assume that M is a nonempty subset of $CAT(0)$ space X and $\Psi : M \to P(M)$ is a multivalued mapping satisfying the condition $(C_{(\alpha,\beta)})$ for some $\alpha, \beta \in [0, 1)$. Then*

(1) *If M is closed then $F(\Psi)$ is closed. Moreover, if M is convex and $F(\Psi) \neq \phi$ with $SF(\Psi) = F(\Psi)$, then $F(\Psi)$ is convex.*

(2) *For each $v, y \in M$ and $p \in \Psi v$, we have the followings:*

  (i) *$H(\Psi v, \Psi p) \leq d(v, p)$.*

  (ii) *Either $\frac{1}{2}dist(v, \Psi v) \leq d(v, y)$ or $\frac{1}{2}dist(p, \Psi p) \leq d(y, p)$.*

(iii) *Either* $H(\Psi v,\Psi y)\le \alpha dist(y,\Psi v)+\beta dist(v,\Psi y)+(1-\alpha-\beta)d(v,y)$ *or* $H(\Psi v,\Psi p)\le \alpha dist(p,\Psi v)+\beta dist(v,\Psi p)+(1-\alpha-\beta)d(v,p)$.

(3) *Let* M *be closed and convex. Then*

$$H(\Psi v,\Psi y)\le \frac{(1+\alpha+\beta)}{(1-\beta)}dist(v,\Psi v)+d(v,y) \tag{3.3}$$

holds for all $v,y\in M$.

4. Convergence results for monotone generalized (α,β)-nonexpansive mapping type 1

In this section, we establish strong and Δ-convergence results for monotone generalized (α,β)-nonexpansive mapping type I in a $CAT(0)$ space.

Assume that (X,d) is an ordered $CAT(0)$ space and M is a nonempty, convex, and closed subset of X. Let $\Psi: M\to P(M)$ be multivalued mapping. The sequence $\{\tau_m\}$ is defined by

$$\begin{cases}\tau_1\in M,\\ z_m=(1-\beta_m)\tau_m\oplus\beta_m s_m,\\ y_m=s_m^{'},\\ \tau_{m+1}=(1-\alpha_m)s_m^{'}\oplus\alpha_m s_m^{''},\quad \forall m\in\mathbb{N},\end{cases} \tag{4.1}$$

where $\{\alpha_m\}$ and $\{\beta_m\}$ are real sequences in $(0,1)$, with $s_m\in\Psi(\tau_m)$, $s_m^{'}\in\Psi(z_m)$ and $s_m^{''}\in\Psi(y_m)$ and $d(s_m,s_m^{'})\le H(\Psi(\tau_m),\Psi(z_m))$, $d(s_m^{'},s_m^{''})\le H(\Psi(z_m),\Psi(y_m))$ and $d(s_m,s_m^{''})\le H(\Psi(\tau_m),\Psi(y_m))$. The sequence defined in (4.1) is referred as SR iterative algorithm.

Lemma 4.1. ([1]) *Assume that* M *is a nonempty, closed, and convex subset of a complete ordered* $CAT(0)$ *space* (X,d) *and* $\Psi: M\to P(M)$ *is a monotone multivalued generalized* (α,β)*-nonexpansive mapping. Then*

$$\tfrac{1}{2}d(p,\Psi(x))\le d(p,x)$$

for all $x\in M$ *and* $p\in F(\Psi)$ *such that either* $x\preceq p$ *or* $p\preceq x$.

Lemma 4.2. *Let* M *and* $\Psi: M\to P(M)$ *be as in Lemma* 4.1. *Let* $\tau_1\in M$ *be such that* $\tau_1\preceq\Psi(\tau_1)$(*or* $\Psi(\tau_1)\preceq\tau_1$). *Then, for sequence* $\{\tau_m\}$ *defined by SR iterative algorithm* (4.1), *we have*

(1) $\tau_m\preceq s_m\preceq\tau_{m+1}$(*or* $\tau_{m+1}\preceq s_m\preceq\tau_m$); *for any* $m\ge 1$ *and* $s_m\in\Psi(\tau_m)$.

(2) $\tau_m\le\tau^*$(*or* $\tau^*\preceq\tau_m$), *provided* $\{\tau_m\}$ *is* Δ*-convergent to a point* $\tau^*\in M$ *for all* $m\in\mathbb{N}$.

Proof. If $\tau_1 \preceq s_1$, then by convexity of order interval $[\tau_1, s_1]$ and (4.1) we have

$$\tau_1 \preceq (1-\beta_1)\tau_1 \oplus \beta_1 s_1 \preceq s_1.$$

Thus, there exists z_1 such that

$$\tau_1 \preceq z_1 \preceq s_1. \tag{4.2}$$

Since Ψ is monotone there exists $s_1^{'} \in \phi(z_1)$ such that $s_1 \preceq s_1^{'}$. Again by convexity of order interval $[s_1^{'}, s_1^{''}]$, $s_1^{'} \preceq s_1^{''}$. As Ψ is monotone there exists $s_1^{''} \in \phi(y_1)$ such that $s_1^{'} \preceq s_1^{''}$. Again by convexity of order interval $[s_1^{'}, s_1^{''}]$, and by (4.1), we have

$$s_1^{'} \preceq (1-\alpha_1)s_1^{'} \oplus \alpha_1 s_1^{''},$$

thus

$$s_1^{'} \preceq x_2 \preceq s_1^{''}. \tag{4.3}$$

From (4.2) and (4.3) the above argument, we conclude that

$$x_1 \preceq s_1 \preceq x_2.$$

Hence, the statement is true for $m = 1$. Assuming that the statement is true for all m, that is $s_m \in \Psi(\tau_m)$, we have

$$x_m \preceq s_m \preceq x_{m+1}. \tag{4.4}$$

Now, we show that (4.4) is true for $(m+1)$. By convexity of order interval $[x_m, s_m]$ and (4.1)

$$x_m \preceq (1-\beta_m)x_m \oplus \beta_m s_m \preceq s_m,$$

thus there exists z_m such that

$$x_m \preceq z_m \preceq s_m, \tag{4.5}$$

by monotoncity of Ψ there exists $s_m^{'} \in \Psi(z_m)$ such that $s_m \preceq s_m^{'}$. Again by the convexity of order interval $[s_m, s_m^{'}]$ and (4.1), we have

$$s_m \preceq y_m \preceq s_m^{'}, \tag{4.6}$$

by monotoncity of Ψ there exists $s_m^{''} \in \Psi(y_m)$ such that $s_m^{'} \preceq s_m^{''}$. Again by the convexity of order interval $[s_m^{'}, s_m^{''}]$ and (4.1), we have

$$s_m^{'} \preceq (1-\alpha_m)s_m^{'} \oplus \alpha_m s_m^{''} \preceq s_m^{''},$$

thus

$$s_m^{'} \preceq \tau_{m+1} \preceq s_m^{''}. \tag{4.7}$$

From (4.5), (4.6) and (4.7), we have

$$x_m \preceq z_m \preceq s_m \preceq \ s_m^{'} \preceq x_{m+1} \preceq s_m^{''},$$

and therefore

$$s_m^{''} \preceq s_{m+1}. \tag{4.8}$$

Hence, we have

$$\tau_{m+1} \preceq s_{m+1}.$$

By convexity of order interval $[\tau_{m+1}, s_{m+1}]$ and (4.1), we obtain that

$$\tau_{m+1} \preceq (1-\beta_{m+1})\tau_{m+1} \oplus \beta_{m+1} s_{m+1} z_m \preceq s_{m+1},$$

and hence

$$s_{m+1} \preceq z_{m+1} \preceq s_{m+1}^{'}. \tag{4.9}$$

The monotonicity of Ψ yields that there exists $s_{m+1}^{'} \in \Psi(z_{m+1})$ such that

$$s_{m+1} \preceq s_{m+1}^{'},$$

again by the convexity of order interval $[s_{m+1}, s_{m+1}^{'}]$, we have

$$s_{m+1} \preceq y_{m+1} \preceq s_{m+1}^{'},$$

again by convexity of order interval $[s_{m+1}^{'}, s_{m+1}^{'}]$ and (4.1)

$$s_{m+1}^{'} \preceq (1-\alpha_{m+1})s_{m+1}^{'} \oplus \alpha_{m+1} s_{m+1}^{''} \preceq s_{m+1}^{''},$$

which implies that

$$s_{m+1}^{'} \preceq \tau_{m+2} \preceq s_{m+1} \tag{4.10}$$

so combine all the above inequalities, we obtain

$$\tau_{m+1} \preceq s_{m+1} \preceq \tau_{m+2}.$$

Hence it is true for all m.

Suppose that x is $\Delta-\lim$ of $\{\tau_m\}$. From (4.1), we have $\tau_m \preceq \tau_{m+1}$ for all $m \geq 1$. Since the order interval $[\tau_m, \rightarrow)$ is closed and convex and the sequence $\{\tau_m\}$ is increasing, we deduced that $\tau \in [\tau_m, \rightarrow)$ and fixed $m \in \mathbb{N}$, if not, that is, if $\tau \notin [\tau_m, \rightarrow)$, then a subsequence $\{\tau_k\}$ of $\{\tau_m\}$ may be constructed by leaving the first $m-1$ terms of the sequence $\{\tau_m\}$ and then asymptotic center of $\{\tau_r\}$ would not be τ which contradicts that τ is the $\Delta-\lim$ of the sequence $\{\tau_m\}$. This completes the proof. □

Lemma 4.3. *Let M and $\Psi : M \to P(M)$ be as in Lemma* 4.1 *and $\{\tau_m\}$ be a SR iteration process defined by* (4.1) *where $F(\Psi) \neq \phi$ such that $SF(\Psi) = F(\Psi)$. Suppose that there exists $\tau_1 \in M$ such that $\tau_1 \preceq s_1$, where $s_1 \in \Psi(\tau_1)$. Also, assume that either τ_1 and p are comparable. Then*

(i) $\lim\limits_{n\to\infty} d(\tau_m, p)$ *exists for all $p \in F(\Psi)$.*

(ii) $\lim_{n\to\infty} d(\tau_m, s_m) = 0$ *where* $s_m \in \Psi(\tau_m)$.

Proof. Let $p \in F(\Psi)$. If $p \preceq \tau_1$. Then Lemma 4.2 and transitivity of order implies that $p \preceq \tau_2$. Applying the mathematical induction, we obtain $p \preceq \tau_m$ for all $m \geq 1$. On the other hand assume that $\tau_1 \preceq p$. Since there exists $s_1 \in \Psi(\tau_1)$, we have $s_1 \preceq p$ as $F(\Psi) = SF(\Psi)$. Further (4.1) yields

$$z_1 = (1-\beta_1)\tau_1 \oplus \beta_1 s_1 \preceq p.$$

Again, there exists $s_1^{'} \in \Psi(z_1)$ which implies that $s_1^{'} \preceq p$ as $F(\Psi) = SF(\Psi)$ and $y_1 = s_1^{'}$. Again there exists $s_1^{''} \in \Psi(y_1)$ which implies that $s_1^{''} \preceq p$, finally

$$\tau_2 \preceq (1-\alpha_2)s_1^{'} \oplus \alpha_2 s_1^{''} \preceq p,$$

continue in this manner, we obtain $z_n \preceq p$, $y_n \preceq p$ and $s_m^{'} \preceq p$ $s_m^{''} \preceq p$ and $\tau_m \preceq p$. Therefore in both case τ_m and p are comparable. Now from (4.1), we have

$$\begin{aligned} d(\tau_{m+1}, p) &= d((1-\alpha_m)s_m^{'} \oplus \alpha_m s_m^{''}, p) \\ &\leq (1-\alpha_m)d(s_m^{'}, p) + \alpha_m d(s_m^{''}, p) \\ &\leq (1-\alpha_m)\,dist(s_m^{'}, \Psi(p)) + \alpha_m\,dist(s_m^{''}, \Psi(p)) \\ &\leq (1-\alpha_m)H(\Psi(z_m), \Psi(p)) + \alpha_m H(\Psi(y_m), \Psi(p)). \end{aligned} \tag{4.11}$$

As $d(p, \Psi(p)) = 0 \leq \frac{1}{2}d(\tau_m, p)$ implies that

$$\begin{aligned} H(\Psi(z_m), \Psi(p)) &\leq \alpha\ dist(p, \Psi(z_m)) + \beta\ dist(z_m, \Psi(p)) + (1-\alpha-\beta)d(z_m, p) \\ &\leq \alpha\{\ dist(p, \Psi(p)) + dist\ (\Psi(p), \Psi(z_m)\ \} \\ &\quad + (1-\alpha-\beta)\ d(z_m, p) + \beta\{d(z_m, p) + dist\ (p, \Psi(p))\} \\ &\leq \alpha\ H(\Psi(z_m), \Psi(p)) + (1-\alpha)d(z_m, p) \\ &\leq d(z_m, p). \end{aligned} \tag{4.12}$$

Next, we compute

$$\begin{aligned} H(\Psi(y_m), \Psi(p)) &\leq \alpha\ dist\ (p, \Psi(p)) + \beta\ dist\ (y_m.\Psi(p)) + (1-\alpha-\beta)\ d(y_m, p) \\ &\leq \alpha\ \{dist\ (p, \Psi(p)) + dist\ (\Psi(p), \Psi(y_m)\} \\ &\quad + (1-\alpha-\beta)\ d(y_m, p) + \beta\ \{d(y_m, p) + dist\ (p, \Psi(p))\} \\ &\leq \alpha\ H(\Psi(y_m), \Psi(p)) + (1-\alpha)d(y_m, p) \\ &\leq d(y_m, p). \end{aligned} \tag{4.13}$$

From (4.11), (4.12) and (4.13), we have

$$d(\tau_{m+1}, p) \leq (1-\alpha_m)\ d(z_m, p) + \alpha_m\ d(y_m, p). \tag{4.14}$$

From (4.1), we compute that

$$\begin{aligned} d(z_m, p) &= d((1-\beta_m)\tau_m \oplus \beta_m s_m, p) \\ &\leq (1-\beta_m)\ d(\tau_m, p) + \beta_m\ d(s_m, p) \\ &\leq (1-\beta_m)\ d(\tau_m, p) + \beta_m\ dist\ (s_m, p) \\ &\leq (1-\beta_m)\ d(\tau_m, p) + \beta_m\ H(\Psi(\tau_m), \Psi(p)). \end{aligned} \tag{4.15}$$

But

$$\begin{aligned} H(\Psi(\tau_m), \Psi(p)) &\leq \alpha\ dist\ (p, \Psi(\tau_m)) + \beta\ dist\ (\tau_m, \Psi(p)) + (1-\alpha-\beta) d(\tau_m, p) \\ &\leq \alpha\ \{dist\ (p, \Psi(p)) + dist\ (\Psi(p), \Psi(\tau_m)\} \\ &\quad + (1-\alpha-\beta)\ d(\tau_m, p) + \beta\ \{d(\tau_m, p) + dist\ (p, \Psi(p))\} \\ &\leq \alpha\ H(\Psi(\tau_m), \Psi(p)) + (1-\alpha) d(\tau_m, p) \\ &\leq d(\tau_m, p). \end{aligned} \tag{4.16}$$

From (4.15) and (4.16), we have

$$d(z_m, p) \leq d(\tau_m, p). \tag{4.17}$$

Again from (4.1) and (4.17), we have

$$\begin{aligned} d(y_m, p) &= d(s_m^{'}, p) \\ &\leq dist\ (s_m^{'}, p) \\ &\leq d(z_m, p) \\ &\leq d(\tau_m, p). \end{aligned} \tag{4.18}$$

From (4.14), (4.17), and (4.18), we get

$$d(\tau_{m+1}, p) \leq d(\tau_m, p). \tag{4.19}$$

Thus, sequence $\{d(\tau_m, p)\}$ is decreasing and consequently $\lim_{n\to\infty} d(\tau_m, p)$ exists, for $m \geq 1$, complete the proof of (i).

From part (i), we have $\lim_{n\to\infty} d(\tau_m, p)$ exists. Suppose $\lim_{n\to\infty} d(\tau_m, p) = r \geq 0$. Now from (4.16), we get

$$\begin{aligned} \limsup_{n\to\infty} d(s_m, p) &\leq \limsup_{n\to\infty} H(\Psi(\tau_m), \Psi(p)) \\ &\leq \limsup_{n\to\infty} d(\tau_m, p) \\ &= r. \end{aligned} \tag{4.20}$$

$$\begin{aligned}
\limsup_{n\to\infty} d(z_m,p) &= \limsup_{n\to\infty} \{(1-\beta_m)\ \tau_m \oplus \beta_m\ dist\ (s_m,p)\} \\
&\leq \limsup_{n\to\infty}\ (1-\beta_m)\ d(\tau_m,p) + \limsup_{n\to\infty} \beta_m\ H(\Psi(\tau_m),\Psi(p)) \\
&\leq \limsup_{n\to\infty} d(\tau_m,p) \\
&= r. \qquad (4.21)
\end{aligned}$$

From equation (4.1), (4.12), and (4.13), we get

$$\begin{aligned}
d(\tau_{m+1},p) &= d((1-\alpha_m)s_m^{'} \oplus \alpha_m s_m^{'}, p) \\
&\leq (1-\alpha_m)d(s_m^{'},p) + \alpha_m d(s_m^{''},p) \\
&\leq (1-\alpha_m)\ dist\ (s_m^{'},p) + dist\ (s_m^{''},p) \\
&\leq (1-\alpha_m)H(\Psi(z_m),\Psi(p)) + \alpha_m\ H(\Psi(y_m),\Psi(p)) \\
&\leq (1-\alpha_m)d(\tau_m,p) + \alpha_m d(z_m,p),
\end{aligned}$$

it implies that

$$\frac{d(\tau_{m+1},p) - d(\tau_m,p)}{\alpha_m} \leq d(z_m,p) - d(\tau_m,p). \qquad (4.22)$$

Because $\{\alpha_m\}$ is a sequence in $[p,q]$, for some $p,q \in (0,1)$ we obtain that

$$\begin{aligned}
\frac{d(\tau_{m+1},p) - d(\tau_m,p)}{q} &\leq \frac{d(\tau_{m+1},p) - d(\tau_m,p)}{\alpha_m} \\
&\leq d(z_m,p) - d(\tau_m,p). \qquad (4.23)
\end{aligned}$$

Taking $\liminf$ as $m \to \infty$, in above inequality, we get

$$\begin{aligned}
\liminf_{n\to\infty} \frac{d(\tau_{m+1},p) - d(\tau_m,p)}{q} &\leq \liminf_{n\to\infty} d(z_m,p) - d(\tau_m,p) \\
&\leq \liminf_{n\to\infty} d(z_m,p). \qquad (4.24)
\end{aligned}$$

From (4.21) and (4.24), we get

$$\begin{aligned}
r &= \lim_{n\to\infty} d(z_m,p) \\
&= \lim_{n\to\infty} d((1-\beta_m)\tau_m \oplus \alpha_m s_m, p).
\end{aligned}$$

Using (4.19), (4.20) and application of Lemma 2.6, we get

$$\lim_{n\to\infty} d(\tau_m, s_m) = 0.$$

Hence, the proof is complete. □

Now, we present the existence result associated with multivalued generalized (α, β)-nonexpansive mapping.

Theorem 4.4. *Let M and $\Psi : M \to P(M)$ be as in Lemma* 4.1*. Fix $\tau_1 \in M$ such that $\tau_1 \preceq s_1$. If $\{\tau_m\}$ is a sequence given by* (4.1) *then the condition $\Delta - \lim_m \tau_m = \tau$ and $\lim\limits_{n\to\infty} d(\tau_m, s_m) = 0$ are satisfied then $\tau \in F(\Psi)$.*

Proof. Since $\Delta - \lim_m \tau_m = \tau$, Lemma 4.2 implies that $\tau_m \preceq \tau$ for all $n \geq 1$.

Utilizing the (α, β)-nonexpansiveness of Ψ *and* $\lim\limits_{n\to\infty} d(\tau_m, s_m) = 0$, we have $z \in \Psi\tau$. Further,

$$\begin{aligned}
\limsup_{n\to\infty} d(z, \tau_m) &\leq \limsup_{n\to\infty} [d(z, s_m) + d(s_m, \tau_m)] \\
&\leq \limsup_{n\to\infty} d(z, s_m) \\
&\leq \limsup_{n\to\infty} dist\ d(z, s_m) \\
&\leq \limsup_{n\to\infty} H(\Psi\tau, \Psi\tau_m) \\
&\leq \limsup_{n\to\infty} d(\tau, \tau_m).
\end{aligned}$$

Thus, from the uniqueness of the asymptotic center, we have $z = \tau$ *where* $z \in \Psi x$. □

Theorem 4.5. *Let M and $\Psi : M \to P(M)$ be as in Lemma* 4.1 *with $F(\Psi) \neq \phi$. Fix $\tau_1 \preceq s_1 \in \Psi\tau_1$. If $\{\tau_m\}$ is a sequence given by* (4.1) *then $\{\tau_m\}$ is Δ-convergent to an element of $F(\Psi)$.*

Proof. It follows from Lemma 4.1 that $\lim\limits_{n\to\infty} d(\tau_m, p)$ exists for each $p \in F(\Psi)$. So, $\{\tau_m\}$ is bounded and $\lim\limits_{n\to\infty} d(\tau_m, s_m) = 0$, where $s_m \in \Psi\ \tau_m$.

Denote $\rho_l(\tau_m) = \bigcup A_k(\{u_m\})$ where the union is taken over all subsequence $\{u_m\}$ of $\{\tau_m\}$. We now prove that $\{\tau_m\}$ is Δ-convergent to a fixed point of Ψ. First we show $\rho_l(\tau_m) \subset F(\Psi)$ and therefore assert that $\rho_l(\tau_m)$ is singleton. To show $\rho_l(\tau_m) \subset F(\Psi)$. Let $y \in \rho_l(\tau_m)$. So there exists a subsequence $\{y_m\}$ of $\{\tau_m\}$ such that $A(\{y_m\}) = \{y\}$. As a consequence of Lemma 2.4 and Lemma 2.5, there exists a subsequence $\{t_m\}$ of $\{y_m\}$ so that Δ-$\lim_m t_m = t$ *and* $t \in M$.

As $\lim\limits_{n\to\infty} d(\tau_m, s_m) = 0$ *and* $\{t_n\}$ is a subsequence of $\{\tau_m\}$, we have that $\limsup\limits_{n\to\infty} d(t_m, \Psi t_m) = 0$. By Theorem 4.5, we have $t \in \Psi t$ and hence $t \in F(\Psi)$.

Now we assert that $t = y$. Indeed, $t \neq y$ leads to a contradiction as

$$\begin{aligned}\limsup_{n\to\infty} d(t_m, t) &< \limsup_{n\to\infty} d(t_n, y)\\ &\leq \limsup_{n\to\infty} d(y_m, y)\\ &\leq \limsup_{n\to\infty} d(y_m, t)\\ &= \limsup_{n\to\infty} d(\tau_m, t)\\ &= \limsup_{n\to\infty} d(t_m, t),\end{aligned}$$

and hence, $t = y \in F(\Psi)$. To show that $\rho_l(\tau_m)$ is a singleton set, let $\{y_m\}$ be a subsequence of $\{\tau_m\}$. From Lemma 2.4 and Lemma 2.5, there exists a subsequence $\{t_m\}$ of $\{y_m\}$ such that $\Delta\text{-}\lim_m t_m = t$. Let $A(\{y_m\}) = \{y\}$ *and* $A(\{\tau_m\}) = \{\tau\}$. As it is already proved that $t = y$ thus it is sufficient to demonstrate that $t = \tau$.

If $t \neq x$, then by Lemma 4.3, $\{d(\tau_m, p)\}$ converges.

By uniqueness of asymptotic centers, we have

$$\begin{aligned}\limsup_{n\to\infty} d(t_m, t) &< \limsup_{n\to\infty} d(t_n, y)\\ &\leq \limsup_{n\to\infty} d(y_m, \tau)\\ &\leq \limsup_{n\to\infty} d(\tau_m, \tau)\\ &= \limsup_{n\to\infty} d(\tau_m, t)\\ &= \limsup_{n\to\infty} d(t_m, t),\end{aligned}$$

which is a contradiction that $t = \tau$, consequently $t = y \in F(\Psi)$. Hence the conclusion follows. □

In the following, we ascertain the strong convergence result which extends Theorem 4 in [1] for multivalued generalized (α, β)-nonexpansive mapping in the setup of ordered CAT(0) space via SR iterative algorithm.

Theorem 4.6. *Let M and $\Psi : M \to P(M)$ be as in Lemma* 4.1 *with $F(\Psi) \neq \phi$. Fix $\tau_1 \preceq s_1 \in \Psi\tau_1$. If $\{\tau_m\}$ is a sequence given by* (4.1) *with $\sum_{m=1}^{\infty} \alpha_m\beta_m = \infty$, then $\{\tau_m\}$ converges to a fixed point of Ψ if and only if $\liminf_{n\to\infty} d(\tau_m, F(\Psi)) =$* 0.

Proof. If the sequence $\{\tau_m\}$ converges to a fixed point $p \in F(\Psi)$, then it is obvious that $\liminf_{n\to\infty} d(\tau_m, F(\Psi)) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(\tau_m, F(\Psi)) = 0$. From Lemma 4.3, we have $dist(\tau_{m+1}, p) \leq d(\tau_m, p)$ *for any* $p \in F(\Psi)$. So, $dist(\tau_{m+1}, F(\Psi)) \leq d(\tau_m, F(\Psi))$ and hence $\{d(\tau_m, F(\Psi)\}$ forms a decreasing sequence that is bounded below by zero which implies that $\liminf_{n\to\infty} d(\tau_m, F(\Psi))$ exists.

To show $\{\tau_m\}$ is a Cauchy sequence in M, choose an arbitrary number, say, $e > 0$. As $\liminf_{n\to\infty} d(\tau_m, F(\Psi)) = 0$, we have $\lim_{n\to\infty} d(\tau_m, F(\Psi)) = 0$. Thus, there exists m_0 such that for all $m \geq m_0$, we have

$$d(\tau_m, F(\Psi)) < \tfrac{\epsilon}{4}.$$

Specifically,

$$\inf\{d(\tau_{m_0}, p) : p \in F(\Psi)\} < \tfrac{\epsilon}{4}.$$

Thus, there must exists $p \in F(\Psi)$ such that $d(\tau_{m_0}, p) < \frac{\epsilon}{2}$. Now for $m, n \geq m_0$, we have

$$\begin{aligned} d(\tau_{m+n}, \tau_n) &\leq d(\tau_{m+n}, p) + d(p, \tau_n) \\ &< 2d(\tau_{m_0}, p) \\ &< 2\frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Since $M \subset X$ is closed, $\{\tau_m\}$ is a Cauchy sequence and consequently, converges in M. Let $\liminf_{n\to\infty} \tau_m = g$. Note that

$$\begin{aligned} dist\ (g, \Psi g) &\leq dist\ (g, \tau_m) + dist\ (\tau_m, \Psi\tau_m) + dist\ (\Psi\tau_m, g) \\ &\leq d(\tau_m, g) + d(\tau_m, s_m) + H(\Psi\tau_m, \Psi g). \end{aligned}$$

On taking the limit as $m \to \infty$, we have $g \in \Psi g$. This completes the proof. □

Remark 4.7. (1) For $\alpha = \beta = 0$, our theorems extend the results in [2] to CAT(0) spaces.
(2) For $\alpha = \beta$, these results extend the results in [14, 25] to CAT(0) spaces.
(3) Our results extend and improve results in [30] for monotone nonexpansive mapping in a CAT(0) spaces.

5. Numerical experiments

Example 5.1. Let $M = [0, 2]$ and Y be a CAT(0) space equipped with the order $\geq$ and the standard metric given by $d(v, y) = |v - y|$. Define $\Psi : M \to$

$P(M)$ by

$$\Psi(v) = \begin{cases} [0, \frac{v}{2}], & \text{if } 0 \le v < \frac{1}{2}, \\ \{0\}, & \text{if } \frac{1}{2} \le v \le 1. \end{cases}$$

Clearly, Ψ is monotone. Indeed, if $v \ge y$ for $v \in [\frac{1}{2}, 1]$ and $y \in [0, \frac{1}{2})$, then for any $u_v \in \Psi v = \{0\}$, there exists $u_y = 0 \in \Psi y$ such that $u_v \ge u_y$.

FIGURE 1. Comparison of iteration process for $\alpha_n = \frac{n}{n+1}$ *and* $\beta_n = \frac{1}{n+7}$

no. of iteration	S Iteration	SR Iteration
1	0.4	0.4
2	0.29375	0.140625
3	0.1903935185	0.04427083333
4	0.1154260706	0.01314290365
5	0.06715698653	0.00376364968
6	0.03800898891	0.001051992359
7	0.02109290044	0.0002890088897
8	0.01153517993	0.00007838075915
9	0.006237541738	0.00002104668533
10	0.003342932525	0.000005606968513
11	0.00177872613	0.000001484197548
12	0.0009408308348	0.0000003908043311
13	0.0004951741236	0.0000001024476941
14	0.0002595242951	0.00000002675531297
15	0.0001355293541	0.000000006964875121
16	0.00007055611972	0.000000001807998478
17	0.00003663143812	0.0000000004681837491
18	0.00001897288027	0.0000000001209745625
19	0.000009805983384	0
20	0.00000505856739	0

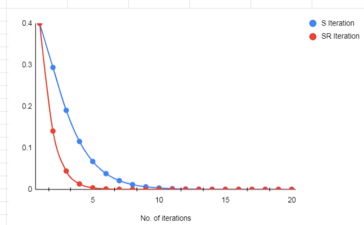


FIGURE 2. Comparison of iteration process for $\alpha_n = \frac{1}{\sqrt{n+1}}$ *and* $\beta_n = \frac{n}{n+3}$

no. of iteration	S Iteration	SR Iteration
1	0.4	0.4
2	0.2409009742	0.1131281566
3	0.1574504288	0.03621189186
4	0.1082471698	0.01244783783
5	0.07712670881	0.004487050597
6	0.05646346546	0.001673969639
7	0.04223600948	0.0006410979143
8	0.03215644754	0.0002506892032
9	0.02484816401	0.00009970593311
10	0.01944600942	0.00004022174875
11	0.01538687619	0.0000164221596
12	0.0122934738	0.000006774987071
13	0.009906756488	0.000002820474947
14	0.008045098021	0.000001183605734
15	0.006578814225	0.0000005002425793
16	0.005413815873	0.0000002127767915
17	0.004480864114	0.00000009102672596
18	0.003728357381	0.00000003914582267
19	0.003117397492	0.00000001691517908
20	0.002618357832	0.00000000734127496

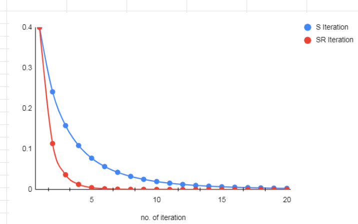


Figure 3. Comparison of iteration process for $\alpha_n = \frac{1}{n+1}$ *and* $\beta_n = \frac{1}{n+1}$

no. of iteration	S Iteration	SR Iteration
1	0.4	0.4
2	0.275	0.1125
3	0.2215277778	0.0390625
4	0.190375434	0.01495361328
5	0.1694341363	0.006056213379
6	0.154137999	0.00254445076
7	0.1423417235	0.001096969843
8	0.1328893435	0.0004820668255
9	0.1250964499	0.0002149958527
10	0.1185288863	0.00009701687854
11	0.1128963152	0.0000441988052
12	0.1079963015	0.00002029615274
13	0.1036828398	0.000009382467057
14	0.0998476327	0.000004362129135
15	0.09640843646	0.000002038083668
16	0.09330152396	0.0000009563468775
17	0.09047665083	0.0000004504592343
18	0.0878935983	0.0000002128906489
19	0.08551974073	0.0000001009166546
20	0.08332829737	0.00000004796694737

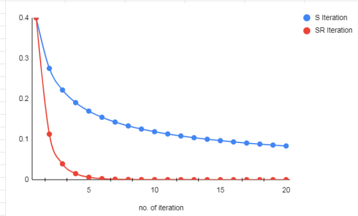
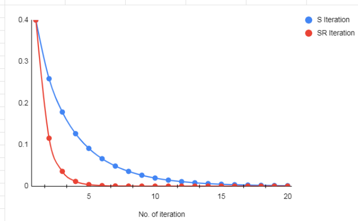


Figure 4. Comparison of iteration process for $\alpha_n = \sqrt{\frac{n+1}{5n+1}}$ *and* $\beta_n = \frac{1}{\sqrt{2n+5}}$

no. of iteration	S Iteration	SR Iteration
1	0.4	0.4
2	0.2587100572	0.1153794203
3	0.1783901926	0.03552164656
4	0.1263597198	0.01131245882
5	0.09088351628	0.003683241456
6	0.06602679278	0.001218631758
7	0.04831385596	0.0004082108016
8	0.03554521125	0.0001381031793
9	0.02626321269	0.00004710632769
10	0.01947253454	0.00001617917882
11	0.01447939177	0.000005589993943
12	0.01079293099	0.000001941381699
13	0.008061967568	0.0000006773110197
14	0.006033057808	0.0000002372601158
15	0.004522032264	0.00000008341414309
16	0.003394310627	0.00000002942258765
17	0.002551093784	0.00000001040923804
18	0.001919568442	0.000000003692690485
19	0.001445894786	0.000000001313277896
20	0.001090148939	0.0000000004681405084



6. Application to fractional differential equations

Consider the following fractional differential equation [23]:

$$^{C}D^{\alpha}x(t) = x(t)\cdot[1 - x(t)], \quad 0 < \alpha \leq 1, \tag{6.1}$$

where $^{C}D^{\alpha}$ denotes the Caputo fractional derivative of order α.

The equivalent integral equation is:

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}x(\tau)\cdot[1 - x(\tau)]\,d\tau, \tag{6.2}$$

where $\Gamma(\alpha)$ is the Gamma function.

Consider the following integral equation is given by:

$$Sx(t) = x(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}x(\tau)\,[1 - x(\tau)]\,d\tau,$$

where $\Gamma(\alpha)$ is the Gamma function defined as:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt.$$

The operator S is monotone if for any $x_1(t)$ and $x_2(t)$ in its domain:

$$(x_1(t) - x_2(t))(Sx_1(t) - Sx_2(t)) \geq 0.$$

For the given integral equation, this means showing that:

$$\begin{aligned}&(x_1(t) - x_2(t))\Big(x_1(0) - x_2(0)\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[x_1(\tau)\left[1 - x_1(\tau)\right] - x_2(\tau)\left[1 - x_2(\tau)\right]\right] d\tau\Big)\\ &\geq 0.\end{aligned}$$

The operator S is nonexpansive if for any $x_1(t)$ and $x_2(t)$ in its domain:

$$\|Sx_1(t) - Sx_2(t)\| \leq \|x_1(t) - x_2(t)\|.$$

For the integral equation, this requires:

$$\begin{aligned}&\|Sx_1(t) - Sx_2(t)\|\\ &\quad = \left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[x_1(\tau)\left[1 - x_1(\tau)\right] - x_2(\tau)\left[1 - x_2(\tau)\right]\right] d\tau\right\|.\end{aligned}$$

Using the triangle inequality, we get:

$$\begin{aligned}&\left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[x_1(\tau)\left[1 - x_1(\tau)\right] - x_2(\tau)\left[1 - x_2(\tau)\right]\right] d\tau\right\|\\ &\quad \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left\|x_1(\tau)\left[1 - x_1(\tau)\right] - x_2(\tau)\left[1 - x_2(\tau)\right]\right\| d\tau.\end{aligned}$$

Given that $x(\tau)\left[1 - x(\tau)\right]$ is Lipschitz continuous. Consider the function $f(x) = x(1-x)$. The derivative of $f(x)$ is:

$$f'(x) = 1 - 2x.$$

The absolute value of the derivative is:

$$|f'(x)| = |1 - 2x|.$$

To find the maximum value of $|f'(x)|$ for $x \in [0,1]$, we consider the endpoints and the critical points:

- At $x = 0$:
$$|f'(0)| = |1 - 2 \cdot 0| = 1.$$

- At $x = 1$:
$$|f'(1)| = |1 - 2 \cdot 1| = 1.$$
- At the critical point where $f'(x) = 0$:
$$1 - 2x = 0 \implies x = \frac{1}{2}.$$
At $x = \frac{1}{2}$:
$$|f'\left(\frac{1}{2}\right)| = |1 - 2 \cdot \frac{1}{2}| = 0.$$

Therefore, the maximum value of $|f'(x)|$ on the interval $[0, 1]$ is 1. Hence, $f(x) = x(1-x)$ is Lipschitz continuous with Lipschitz constant $L = 1$. Therefore, the integral operator S is nonexpansive. Thus, the integral operator S is monotone and nonexpansive. This implies that S is monotone generalized (α, β)-nonexpansive mapping. By Theorem 4.6, the sequence $\{x_m\}$ converges to fixed point p(say). So, the given integral equation (6.2) has a solution. Therefore, the differential equation (6.1) has a solution.

7. Conclusion

In this paper, we extend the result of Abbas et al. [1] via a new iterative algorithm for multivalued generalized (α, β)-nonexpansive mapping in ordered $CAT(0)$ space. Through numerical experiments, we have shown that our iterative algorithm is faster than the algorithm discussed in Abbas et al. [1]. Furthermore, we have also presented an application of the result in approximating the solutions of fractional differential equations.

References

[1] M. Abbas, H. Iqbal, M. De la Sen and K. Ahmad, *Approximation of Fixed Points of Multivalued Generalized (α, β)-Nonexpansive Mappings in an Ordered $CAT(0)$ Space*, Mathematics, **9** (2021), 1945.

[2] A. Abkar and M. Eslamian, *Fixed point theorems for Suzuki generalized nonexpansive multivalued mappings in Banach spaces*, Fixed Point Theory Appl., **2010** (2010), 457935.

[3] A. Amini-Harandi, M. Fakhar and H.R. Hajisharifi, *Approximate fixed points of α-nonexpansive mappings*, J. Math. Anal. Appl., **467** (2018), 1168–1173.

[4] K. Aoyama and F. Kohsaka, *Fixed point theorem for α-nonexpansive mappings in Banach spaces*, Nonlinear Anal.,**74** (2011), 4387–4391.

[5] M. Asadi, Sh. Ghasemzadehdibagi, S. Haghayeghi and N. Ahmad, *Fixed point theorems for (α, p)-nonexpansive mappings in $CAT(0)$ spaces* , Nonlinear Funct. Anal. Appl., **26**(5) (2021), 1045-1057.

[6] M.R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer, 1999.

[7] M.R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer Science & Business Media: Berlin, Germany, 2013; Volume 319.

[8] K.S. Brown, *Buildings*, Springer-Verlag: New York, 1989.

[9] D. Burago, Y. Burago and S. Ivanov, *A Course in Metric Geometry*, in Graduate Studies in Math., vol. 33, Amer. Math. Soc., Providence, RI, 2001.

[10] S. Dahputre, R. Tiwari and J. Shrivas, *Approximating fixed points for generalized α-nonexpansive mapping in $CAT(0)$ space via new iterative algorithm*, Nonlinear Funct. Anal. Appl., **29**(1) (2024), 69-81.

[11] S. Dhompongsa, A. Kaewkhao and B. Panyanak, *Lim's theorems for multivalued mappings in CAT(0) spaces*, J. Math. Anal. Appl., **312** (2005), 478–487.

[12] S. Dhompongsa and B. Panyanak, *On 4-convergence theorems in CAT(0) spaces*, Comput. Math. Appl., **56** (2008), 2572–2579.

[13] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, Inc., New York, 1984.

[14] H. Iqbal, M. Abbas and S.M. Husnine, *Existence and approximation of fixed points of multivalued generalized α-nonexpansive mappings in Banach spaces*, Numer. Algor., **85** (2020), 1029–1049.

[15] S.H. Khan and M. Abbas, *Common fixed points of two multivalued nonexpansive maps in Kohlenbach hyperbolic spaces*, Fixed Point Theory Appl., **2014**, 181.

[16] J.K. Kim, R. Verma, S. Dashputre and Padmavati *Convergence and stability of Fibonacci SR-iteration process for Monotone non-Lipschitzian mapping*, Nonlinear Funct. Anal. Appl., **30**(1) (2025), 75-97.

[17] W.A. Kirk, *Geodesic geometry and fixed point theory*, in: Seminar of Mathematical Analysis, Malaga, Seville, 2002-2003, in: Colec. Abierta, vol. 64, Univ. Sevilla Secr. Publ., Seville, 2003, pp. 195–225.

[18] W.A. Kirk, *Geodesic geometry and fixed point theory, II*, in: International Conference on Fixed Point Theory and Applications, Yokohama Publ., Yokohama, 2004, pp. 113–142.

[19] W. Laowang and B. Panyanak, *Approximating fixed points of nonexpansive nonself mappings in CAT(0) spaces*, Fixed Point Theory Appl., **2010**, 367274.

[20] H. Lu, D. Lan, Q. Hu and G. Yuan, *Fixed point theorems in CAT(0) spaces with applications*, J. Inequal. Appl., **320** (2014).

[21] J.T. Markin, *A fixed point theorem for set-valued mappings*, Bull. Amer. Math. Soc., **74** (1968), 639–640.

[22] S.B. Nadler Jr., *Multi-valued contraction mappings*, Pac. J. Math., **30** (1969), 475–488.

[23] J.J. Nieto, *Solution of a fractional logistic ordinary differential equation*, Appl. Math. Lett., **123** (2022), 107568.

[24] R. Pandey, R. Pant and A. Al-Rawashdeh, *Fixed point results for a class of monotone nonexpansive type mappings in hyperbolic spaces*, J. Funct. Spaces, **2018**, 5850181.

[25] R. Pant and R. Shukla, *Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces*, Numer. Funct. Anal. Opti., **38** (2017), 248–266.

[26] B. Panyanak, *Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces*, Comput. Math. Appl., **54** (2007), 872–877.

[27] A. Razani and S. Shabani, *Approximating fixed points for nonself mappings in CAT(0) spaces*, Fixed Point Theory Appl., **2011**, 65.

[28] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl., **340** (2008), 1088–1095.

[29] I. Uddin, J. Ali and J. J. Nieto, *An iteration scheme for a family of multivalued mappings in CAT(0) spaces with an application to image recovery*, Rev. Real Acad. Cienc. Exactas Físicas Nat. Ser. A Matemáticas, **112** (2018), 373–384.

[30] I. Uddin, J.J. Nieto and J. Ali, *One-step iteration scheme for multivalued nonexpansive mappings in CAT(0) spaces*, Mediterr. J. Math., **13** (2016), 1211–1225.