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FRACTIONAL SPECTRAL GRAPH WAVELET TRANSFORM APPROXIMATION IN TERMS OF MODULUS OF CONTINUITY

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Abstract. In recent years, many wavelets formulas were constructed for vertices functions of weighted graphs. In addition to the fractional wavelets, spectral graphs play important role in function approximation. In this paper, we define a fractional wavelet transforms in terms of discrete graph Laplacian matrix. We study general properties of fractional spectral graph wavelet transform (FRSGWT) in order to achieve the existence of best approximation of functions defined on vertices. Therefore, we prove direct and inverse theorems to get degree of approximation with upper and lower bounds in terms of modulus of continuity. These theoretical results grantees that the approximation error implements well if applied in various fields.

1. Introduction

Moduli of smoothness play a big role in estimating the degree of approximation, for example [2, 3, 19]. In particular, modulus of continuity is used earlier

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for the same proposes but different spaces, see [12]. Fourier and wavelets are transforms play important roles in signal processing.

In recent years, many generalizations were presented for Fourier and wavelets transformation, see [15]. In addition, the extension of Fourier transforms and wavelet to the fractional domain led to the creation of the fractional Fourier transform (FRFT) and fractional wavelets. The fractional order of FRFT, which is a free parameter provides generalization and greater flexibility more than the classical Fourier transform, as in [4].

The study of FRFT is divided roughly into three main research areas. First is the use of FRFT for a variety of signal processing problems [5], including compression, filtering, image encryption, pattern recognition, digital watermarking, antennas, radar, and sonar, edge detection, as well as communication. Second, FRFT's discrete algorithms see [6, 18, 17], and third, expanding the fractional idea to include other transformations, as in the fractional Fourier transforms. Mendlovic et al. [16] present first the fractional wavelet transform (FRWT) by cascading FRFT and regular wavelet transform. The authors in [21] presented a new family of wavelets in terms of fractional order, Shi in [20] also introduced a new class of fractional wavelet transforms. Generally, fractional transforms give more accurate results for signal filtering [5], image recognition [13], and better image segmentation results [14], better image compression ratios [23], and better image watermarking [8] performances since the fractional order here is an additional key that can be used. In order to extend fractional graphs resulting graph Fourier transform (GFT) and graph wavelet transform (GWT), discrete signal processing was conducted which combines both domains, graph transforms, and fractional transforms, [9, 11].

2. Preliminaries

FRFT with order ϑ is the decomposition of a function g according to κ_ϑ, that is,

$$\tilde{g}_\vartheta(s) = \langle g|\kappa_\vartheta\rangle = \int_{-\infty}^{\infty} g(x)\,\kappa_\vartheta(x,s)\,dx, \tag{2.1}$$

where κ_ϑ, is the fractional Fourier kernel, and

$$\kappa_\vartheta(x,s) = \begin{cases} M_\vartheta e^{\left(\frac{j}{2}\left(x^2+s^2\right)cot\vartheta - jxscsc\vartheta\right)}, & \vartheta \neq n\pi, \\ \delta(x-s), & \vartheta = 2n\pi, \\ \delta(x+s), & \vartheta = (2n+1)\pi, \end{cases} \tag{2.2}$$

$$M_\vartheta = \sqrt{\frac{1-jcot\vartheta}{2\pi}}, \tag{2.3}$$

where ϑ represents the transform rotation angle of the transform and δ represents direct distribution is introducing by the following in [4]:

$$g(x) = \langle \tilde{g}_{\vartheta}, \kappa_{\vartheta} \rangle = \int_{-\infty}^{+\infty} \tilde{g}_{\vartheta}(s)\, \kappa_{\vartheta}^{*}(x,s)\, ds. \tag{2.4}$$

The spectral expansion of the kernel κ_{ϑ} is given by [22]:

$$\kappa_{\vartheta}(x,s) = \sum_{\ell=0}^{\infty} e^{(-j\ \vartheta\ell\ )\xi_{\ell}(s)\xi_{\ell}(x)}, \tag{2.5}$$

where $\xi_{\ell}(x)$ is the ℓth order normalized Hermite function see [4, 15], which is the eigen-function of the Fourier transform.

To define (FRWT), we first define ϑ-order as in [22].

Let

$$W_g^{\vartheta}(s,a) = \int_{-\infty}^{\infty} g(x)\, \psi_{\vartheta,s,a}^{*}(x)\, dx, \tag{2.6}$$

where

$$\psi_{\vartheta,s,a}(x) = e^{-\frac{j}{2}\left(x^2 - a^2\left(x-\frac{a}{s}\right)^2\right)} \cot\vartheta \psi_{s,a}(x) \tag{2.7}$$

with

$$\psi_{s,a}(x) = \frac{1}{s}\psi\left(x - \frac{a}{s}\right),$$

and e is the fractional order.

$$W_g^{\vartheta}(s,a) = (T_{\vartheta}^{s} g)(a) = \sqrt{\frac{2\pi}{1 - j cot\vartheta}} \int_{-\infty}^{\infty} e^{\frac{j}{2}s^2u^2 cot\vartheta}\, \tilde{g}_{\vartheta}(u)\, \tilde{\psi}_{\vartheta}^{*}(su)\, \kappa_{\vartheta}^{*}(u,a)\, du, \tag{2.8}$$

where $\tilde{g}_{\vartheta}$ stands for the ϑ - order FRFT of g.

Now we define spectral graph fractional wavelet transforms (SGFRWT), but first the graph fractional Laplacian operator $\mathcal{L}_{\vartheta}$ is define as

$$\mathcal{L}_{\vartheta} = \upsilon R \upsilon^{D}, \tag{2.9}$$

where $0 < \vartheta \leq 1$ and the power matrix is given by

$$\upsilon = [\upsilon_0, \upsilon_1, \ldots, \upsilon_{N-1}] = \chi^{\vartheta} \tag{2.10}$$

with χ the inverse graph Fourier transform matrix.

$$R = \mathrm{diag}\left([\gamma_0, \gamma_1, \ldots, \gamma_{N-1}]\right) = \Lambda^{\vartheta}, \tag{2.11}$$

that is,

$$\gamma_i = \lambda_i^{\vartheta}, \quad i = 0, 1, \ldots, N-1. \tag{2.12}$$

The forward SGFRFT of a signal g defined on the set of vertices $\mathcal{V}$ of the graph G is define by

$$\tilde{g}_\vartheta(i) = \left\langle g, {\chi_i}^\vartheta \right\rangle = \sum_{n=1}^{N} g(n)\, \upsilon_i^*(n), \quad i = 0, 1, \ldots, N-1. \tag{2.13}$$

So SGFRFT inverse is defined by

$$g(n) = \langle \tilde{g}_\vartheta, \upsilon_i^* \rangle = \sum_{i=0}^{N-1} \tilde{g}_\vartheta(i)\, \upsilon_i(n), \quad n = 1, \ldots, N. \tag{2.14}$$

SGFRWT operator $T^r_{g_\vartheta}$ is as follow:

$$W_f(\vartheta, s, n) = \left(T^s_{g_\vartheta} f\right)(n) = \sum_{m=1}^{N} f(m)\, \psi^*_{\vartheta,s,n}(m) = \langle f, \psi_{\vartheta,s,n} \rangle, \quad n = 1, \ldots, N, \tag{2.15}$$

where

$$T^s_{g_\vartheta} = g(s\mathcal{L}_\vartheta), \tag{2.16}$$

and

$$\psi_{\vartheta,s,n}(m) = \sum_{\ell}^{N-1} g\left(s\lambda_\ell^\vartheta\right) \gamma_\ell(m)\, \gamma_\ell^*(n), \; m = 1, \ldots, N. \tag{2.17}$$

Define Ω to be the set of all SGFRWT that are generated by a graph G and a kernel κ_ϑ from (2.15).

The best approximation of $f \in L_2[a, b]$ by a SGFRWT $h \in \Omega$ given by

$$E_n(g) = \inf_{h \in \Omega} \|g - h\|. \tag{2.18}$$

In our work, to study the degree of best approximation of functions from $L_2[a, b]$, we need the following definition of modulus of continuity from [7].

$$\omega_f([a, b], \delta) = \sup_{x,y \in [a,b]} \{|f(x) - f(y)| : |x - y| \le \delta\}. \tag{2.19}$$

3. FRSGW Transform properties

Now, we provide some of the main important properties of FSGWT, including inversion, scaling and location. Considering that they are the main characteristics of wavelets, so it is important to study them in the case of fractional spectral graph. For the case of continuous functions, authors in [10], studied those properties with several applications in image processing. In [1], L_p spaces were studied for the same target, and got strong results in L_p.

3.1. **Approximation of inverse FSGWT.** For any FSGWT, with kernel κ_ϑ

$$\int_0^\infty \frac{|\kappa_\vartheta(x)|^2}{x} dx = C_{\kappa_\vartheta} < \infty$$

and $\kappa_\vartheta(0) = 0,\ f \in L_p$, we have

$$\frac{1}{C_{\kappa_\vartheta}} \sum_{n=1}^{N-1} \int_0^\infty W_f(\vartheta, t, n)\, \psi_{\vartheta,t,n}(m)\ \frac{dt}{t}\ = f(m) - \hat{f}(0)\, \chi_0(m).$$

3.2. **Small scales.** First, we need to show the approximation effect of the kernel κ_ϑ on the wavelet that generates.

Theorem 3.1. *Let κ_ϑ and $\tilde{\kappa}_\vartheta$ be the kernels of the wavelets $\psi_{\vartheta,t,n} = T^t_{\kappa_\vartheta}\delta_n$ and $\tilde{\psi}_{\vartheta,t,n} = T^t_{\tilde{\kappa}_\vartheta}\delta_n$, respectively. If for all $\lambda \in [0, \lambda_{N-1}]$, with $\|\kappa_\vartheta(t\lambda) - \tilde{\kappa}_\vartheta\ (t\lambda)\| \leq C(t,\vartheta)$, then*

$$\left\|\psi_{\vartheta,t,n} - \tilde{\psi}_{\vartheta,t,n}\right\| \leq C(t,\vartheta).$$

Proof. From Equation (2.17), and Parseval inequality $\langle f, g\rangle = \left\langle \hat{f}, \hat{g}\right\rangle$, we have

$$\begin{aligned}
&\left\|\psi_{\vartheta,t,n}(m) - \tilde{\psi}_{\vartheta,t,n}(m)\right\|^2 \\
&\quad = \left\|\sum_{i=0}^{N-1} \upsilon_i\ (m)\, \kappa_\vartheta\left(t\lambda_i^\vartheta\right) \upsilon_i^*(n) - \sum_{i=0}^{N-1} \upsilon_i\ (m)\, \tilde{\kappa}_\vartheta\left(t\lambda_i^\vartheta\right) \upsilon_i^*(n)\right\|^2 \\
&\quad \leq \sum_{i=0}^{N-1} \left\|\sum_{i=0}^{N-1} \upsilon_i\ (m) \left(\kappa_\vartheta\left(t\lambda_i^\vartheta\right) - \tilde{\kappa}_\vartheta\ \left(t\lambda_i^\vartheta\right)\right) \upsilon_i^*(n)\right\|^2 \\
&\quad \leq\ C(t,\vartheta) \sum_{i=0}^{N-1} |\upsilon_i(m)\, \upsilon_i^*(n)|^2 \\
&\quad \leq\ C(t,\vartheta)\ \sum_{i=0}^{N-1} |\upsilon_i\ (m)|^2 \leq C(t,\vartheta),
\end{aligned}$$

where the last inequality comes from $\sum_i |\upsilon_i(m)|^2 = 1$, with complete orthogonal basis υ_i. □

The following theorem is proved in [1] for any L_p SGWT, here is a special case for the usual norm in $\mathbb{R}^2$. However, it is a generalization for general fraction ϑ.

Theorem 3.2. *Let* $\kappa_\vartheta \in L_p^{(r+1)}, \kappa_\vartheta(0) = 0, {\kappa_\vartheta}^{(k)}(0) = 0$ *for all* $k < r$, *and* ${\kappa_\vartheta}^{(r)}(0) = C \neq 0$. *Assume that there is some* $\tau > 0$, *with* $\lambda \in \left[0, \tau\lambda_{N-1}^{\vartheta}\right]$ *that satisfies* $\left\|{\kappa_\vartheta}^{(r+1)}\left(\lambda^\vartheta\right)\right\| \leq C$. *Then, for* $\tilde{\kappa}_\vartheta(t\lambda^\vartheta) = \frac{C(t\lambda^\vartheta)^r}{r!}$ *we have*

$$C(t) = \left\|\kappa_\vartheta\left(t\lambda^\vartheta\right) - \tilde{\kappa}_\vartheta\ \left(t\lambda^\vartheta\right)\right\| \leq C\frac{t^{r+1}\lambda_{N-1}^{\vartheta,r+1}}{(r+1)!}$$

for all $t < \tau$.

Now, we present the main result about localization, we note that if κ_ϑ is approximately.

Theorem 3.3. *If* κ_ϑ *satisfies Theorem* 3.2 *with* τ *and* C. *Let* $m, n \in V(G)$ *with* $d_G(m,n) > r, r \in \mathbb{N}$. *Then*

$$\frac{\psi_{\vartheta,t,n}(m)}{\|\psi_{\vartheta,t,n}\|} \leq Ct$$

for some constant C.

Proof. Set $\tilde{\kappa}_\vartheta(\lambda) = \frac{{\kappa_\vartheta}^{(r)}(0)}{r!}\lambda^{\vartheta,r}$ and $\tilde{\psi}_{\vartheta,t,n} = T_{\tilde{\kappa}_\vartheta}^{t}\delta_n$, then by Lemma 5.2 in [10],

$$\tilde{\psi}_{\vartheta,t,n}(m) = \frac{{\kappa_\vartheta}^{(r)}(0)}{r!}t^r \sum_{i=0}^{N-1} \tilde{\kappa}_\vartheta\left(t\lambda_i^\vartheta\right)\chi_i^*(n)\chi_i\ (m) = 0,$$

hence, by Theorems 3.2 and 3.1, we get

$$\left\|\psi_{\vartheta,t,n}(m)\ - \tilde{\psi}_{\vartheta,t,n}(m)\right\| = \|\psi_{\vartheta,t,n}(m)\| \leq t^{r+1}\frac{\lambda_{N-1}^{\vartheta,r+1}}{(r+1)!}C,$$

also, by hypothesis

$$\left\|\tilde{\psi}_{\vartheta,t,n}\right\| = t^r\frac{{\kappa_\vartheta}^{(r)}(0)}{r!}\left\|\mathcal{L}^r\delta_n\right\|.$$

But, by Theorem 3.1,

$$\left\|\psi_{\vartheta,t,n} - \tilde{\psi}_{\vartheta,t,n}\right\|_p \leq C\frac{t^{r+1}\lambda_{N-1}^{\vartheta,r+1}}{(r+1)!}.$$

Now, by triangle inequality, we obtain the result

$$\frac{\psi_{\vartheta,t,n}(m)}{\|\psi_{\vartheta,t,n}\|} \leq \frac{\psi_{\vartheta,t,n}(m)}{\left\|\tilde{\psi}_{\vartheta,t,n}\right\| - \left\|\psi_{t,n} - \tilde{\psi}_{t,n}\right\|},$$

$$\leq \frac{\frac{\lambda_{N-1}^{\vartheta,r+1}}{(r+1)!}}{\frac{{\kappa_\vartheta}^{(r)}(0)}{r!}\left\|\mathcal{L}^r\delta_n\right\| - t\frac{\lambda_{N-1}^{\vartheta,r+1}}{(r+1)!}}Ct = Ct.$$

□

3.3. **FRSGWT Polynomial approximation.** Now, we approximate L_2 functions, then we estimate the rate of best approximation by FRSGWT in terms of modulus of continuity. First, we prove the following result.

Theorem 3.4. *Let $\lambda_{\max}^{\vartheta} \geq \lambda_{N-1}^{\vartheta}$ be any upper bound on the fractional spectrum of $\mathcal{L}_{\vartheta,r}$ for fixed $s > 0$. Then for any polynomial approximation $P(x)$ of $\kappa_{\vartheta}(sx)$ the generated FSGWT by P satisfies*

$$\left\| W_f(\vartheta, s, n) - \tilde{W}_f(\vartheta, s, n) \right\| \leq C \|f\| \, \omega(\kappa_{\vartheta}, \delta).$$

Proof. Let $\kappa_{\vartheta} \in L_2(\mathbb{R})$. Then there is a polynomial P that satisfies

$$\|P(x) - \kappa_{\vartheta}(sx)\| < \omega(\kappa_{\vartheta}, \delta).$$

Define $W_f(\vartheta, s, n) = (\kappa_{\vartheta})_n$ and $\tilde{W}_f(\vartheta, s, n) = \{(P(\mathcal{L}_{\vartheta,r})f)\}_n$.

Now, by Theorem 3.1, and Cauchy Schwarz inequality, we get

$$\begin{aligned}
&\left\| W_f(\vartheta, s, n) - \tilde{W}_f(\vartheta, s, n) \right\| \\
&= \left\| \sum_{i=0}^{N-1} \hat{f}(i) \psi_{\vartheta,r,n}^{*}(m) - \sum_{m=1}^{N} \hat{f}(i) \psi_{\vartheta,r,n}^{*}(m) \right\| \\
&= \left\| \sum_{i=0}^{N-1} \kappa_{\vartheta}\left(s\lambda_i^{\vartheta}\right) \hat{f}(i) \upsilon_i(m) \upsilon_i^{*}(n) - \sum_{i=0}^{N-1} P(\lambda_i^{\vartheta}) \hat{f}(i) \upsilon_i(m) \upsilon_i^{*}(n) \right\| \\
&\leq \left\| \sum_{i=0}^{N-1} \left(\kappa_{\vartheta}\left(s\lambda_i^{\vartheta}\right) - P\left(\lambda_i^{\vartheta}\right) \right) \left(\hat{f}(i) \upsilon_i(m) \upsilon_i^{*}(n) \right) \right\| \\
&\leq \sum_{i=0}^{N-1} \|\kappa_{\vartheta}(sm) - p(m)\| \left\| \hat{f}(i) \upsilon_i(m) \upsilon_i^{*}(n) \right\| \\
&\leq C \|f\| \, \omega(\kappa_{\vartheta}, \delta).
\end{aligned}$$

□

The following result presents the direct theorem that specifies an upper bound of the degree of approximation

Corollary 3.5. *For any $f \in L_2(\mathbb{R})$, we have*

$$\|f - W_f(\vartheta, s, n)\| \leq C\omega(f, \delta),$$

where, some $W_f \in \Omega$ from (2.15).

Proof. Choose $\tilde{W}_f$ of form (2.15), that satisfies Theorem 3.4,

$$\left\| W_f(\vartheta, s, n) - \tilde{W}_f(\vartheta, s, n) \right\| \leq C\omega(f, \delta).$$

Then

$$\begin{aligned}\|f-W_f(\vartheta,s,n)\| &\leq C\left(\left\|f-\tilde{W}_f(\vartheta,s,n)\right\|+\left\|W_f(\vartheta,s,n)-\tilde{W}_f(\vartheta,s,n)\right\|\right),\\ &< C\omega(f,\delta).\end{aligned}$$

□

The following result analyzes the lower bound of the best approximation to obtain a comprehensive picture of the optimal approximation,

Theorem 3.6. *For any* $f\in L_2(\mathbb{R}), W_f\in\Omega$ *from* (2.15)*, we have*

$$\omega(f,\delta)\leq\|f-W_f(\vartheta,s,n)\|.$$

Proof. For $m<n$, let

$$\|W_f(\vartheta,s,n)-W_f(\vartheta,s,m)\|\leq CE_n(f).$$

Now, take $b=\max\left\{i:2^{-i}<n\right\}$, then by Theorem 3.4, and by properties of modulus of continuity, we get

$$\begin{aligned}\omega(f,\delta) &\leq C\left(\omega\left(f-W_f(\vartheta,s,n),\delta\right)+\omega\left(W_f(\vartheta,s,n),\delta\right)\right),\\ &\leq C\left(\|f-W_f(\vartheta,s,n)\|+\omega\left(W_f\left(\vartheta,s,2^i\right)-W_f\left(\vartheta,s,2^{i-1}\right),2^{-b}\right)\right),\\ &\leq C\left(\omega(f,t)+\sum_{m=2}^{n}(m+1)E_m(f)\right),\\ &\leq CE_n(f).\end{aligned}$$

□

4. Conclusions

In this study, we extended the problem of FRSGWT. The fundamental contribution can be summarized as follow:

(1) A new type of FRSGWT approximation has been studied in terms of discrete graph Laplacian matrix.
(2) A new versions of direct and inverse theorems are proved.

As a conclusion, the new defined FRSGWT provides an alternative operator for the graph signal processing. For further research, we could study approximation of L_p functions by FRSGWT in terms of smoothness of functions. So that, modulus of smoothness could be a good replacement of the modulus of continuity that give faster approach to best approximation.

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