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ESTIMATES FOR FRACTIONAL INTEGRALS OF RIEMANN-LIOUVILLE TYPE USING A CLASS OF FUNCTIONS

Adel Benguessoum[1], Halim Benali[2], Mohammed Said Souid[3] and Kanokwan Sitthithakerngkiet[4]

[1]Department of Mathematics, Laboratory of Informatics and Mathematics,
University of Tiaret, Algeria
e-mail: benalihalim19@yahoo.fr, bengadel@yahoo.fr

[2]Department of Mathematics, Laboratory of Informatics and Mathematics,
University of Tiaret, Algeria
e-mail: benali.halim@univ-tiaret.dz

[3]Department of Economic Sciences, University of Tiaret, Algeria
e-mail: souimed2008@yahoo.com

[4]Intelligent and Nonlinear Dynamic Innovations Research Center,
Department of Mathematics, Faculty of Applied Science,
King Mongkut's University of Technology North Bangkok (KMUTNB)
e-mail: kanokwan.s@sci.kmutnb.ac.th

Abstract. In this paper, our research focuses on fractional integral inequalities involving h-convex functions. These inequalities, which extend classical integral inequalities to fractional orders and incorporate the concept of h-convexity. In this direction, we present a weighted fractional integral operator which generalizes than of Riemann-Liouville, characterized by two parameters, and two non-negative weight functions. This study leads to establish some fractional integral inequalities via a special class of functions called h-convex. As consequence, some estimates and bounds for Laplace transform of some functions are obtained, also bounds for left hand side and right of Riemann-Liouville integrals, which lead to the well-known Hermite-Hadamard inequality.

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[0]Corresponding author: K. Sitthithakerngkiet(`kanokwan.s@sci.kmutnb.ac.th`).

1. Introduction

Due the wide application of inequalities, integral inequalities for example in the study of existence and the uniqueness of the solutions of differential equations, integral equations, in optimization problems where the objective function is convex or h-convex and the constraints are given by fractional integral inequalities. It is natural to study integral inequalities involving fractional calculus.

Fractional calculus generalizes derivative and integral operations to non-integer orders, providing a more flexible approach to modeling complex phenomena.

In recent years, fractal and fractional problems in mathematics, especially fractional integral inequalities involving h-convex functions have garnered significant attention due to their broad applications across optimization, differential equations, signal processing, and related areas. Researchers have explored various inequalities to establish connections with existing theories and uncover new insights. Notable works such as [5, 7, 14, 16, 20, 21] have utilized Riemann-Liouville and Hadamard [9, 15, 19] integrals and their generalizations.

Wu Y. [18] investigated fractional integral inequalities for h-convex functions, providing applications to differential equations and integral equations.

Pachpatte [12] contributed to understanding these inequalities by deriving explicit bounds and highlighting the importance of h-convexity.

Bashiret al. [4] explored Hermite-Hadamard type inequalities for h-convex functions, demonstrating applications in special functions and integral transforms.

These inequalities are powerful tools for analyzing the properties of functions, normed vector spaces, and measure spaces. Their understanding and application are crucial to many fundamental results and theorems in various areas of mathematics.

This work aims to provide a comprehensive understanding of fractional integral inequalities involving h-convex functions and their significance in various mathematical domains. It establishes new inequalities, explores their applications, and contributes to advancing theoretical frameworks.

2. Preliminaries

Definition 2.1. Let $\mathbb{I} \subset \mathbb{R}$ be an interval and $\phi : \mathbb{I} \to \mathbb{R}$, $h : [0, 1] \to (0, \infty)$ be nonnegative functions, $h \neq 0$. The function ϕ is said h-convex if

$$\phi(\rho c + (1 - \rho)d) \leq h(\rho)\phi(c) + h(1 - \rho)\phi(d) \tag{2.1}$$

holds for all $c, d \in \mathbb{I}$ and $\rho \in (0,1]$. If (2.1) is reversed ϕ is said h-concave.

Remark 2.2. The class of convex functions is a special case of h-convex functions, where $h(t) = t$ for all t. Similarly, the class of concave functions is a special case of h-concave functions with $h(t) = -t$. By choosing different functions for h, one can obtain various subclasses of h. The s-convex functions (in the second sence), Godunova-Levin functions and P-functions, which are obtained by taking in (2.1) $h(t) = t^s (s \in (0.1)), h(t) = 1/t$ and $h(t) = 1$, respectively [11, 17, 20, 21].

Example 2.3. Let $h : (0,\infty) \to (0,\infty)$, defined by $h(x) = x^{-1/2}$ and $f : [a,b] \to \mathbb{R}$

$$f(x) = \begin{cases} 1, & \text{if} \quad x \neq \frac{a+b}{2} \\ 2^{3/2}, & \text{if} \quad x = \frac{a+b}{2} \end{cases},$$

we verify that f is not convex, but it is h- convex.

Definition 2.4. ([13]) Let $0 < \delta < \Delta < \infty$, and $\phi \in L_1[\delta, \Delta]$. Then the RiemannLiouville fractional integrals of ϕ of order $\mu > 0$ with $\delta > 0$ are defined by

$$J^{\mu}_{\delta+}\phi(s) = \frac{1}{\Gamma(\mu)} \int_{\delta}^{s} (s-t)^{\mu-1}\phi(t)dt, \quad s > \delta \tag{2.2}$$

and

$$J^{\mu}_{\Delta-}\phi(s) = \frac{1}{\Gamma(\mu)} \int_{s}^{\Delta} (t-s)^{\mu-1}f(t)dt, \quad s < \Delta, \tag{2.3}$$

where $\Gamma(\mu) = \int_0^\infty t^{\mu-1}\, e^{-t}\, dt, \quad \mu > 0$, is the Gamma function. We set $J^0_{\delta+}\phi = J^0_{\Delta-}\phi = \phi$.

Our objective in this work is to establish some estimates for a more general fractional integral operator than the Riemann-Liouville fractional integral using the h-convexity property of functions (see, Theorem 3.3 and 3.14) which are symmetric about the midpoint, as well as of absolute values of ordinary derivative (see, Theorem 3.9).

Remark 2.5. The class of convex functions is a special case of h-convex functions, where $h(t) = t$ for all t. Similarly, the class of concave functions is a special case of h-concave functions with $h(t) = -t$. By choosing different functions for h, one can obtain various subclasses of h. The s-convex functions (in the second sence), Godunova-Levin functions and P-functions, which are obtained by taking in (2.1) $h(t) = t^s (s \in (0.1)), h(t) = 1/t$ and $h(t) = 1$, respectively [11, 17, 20, 21].

3. Main results

Definition 3.1. Let $0 < \delta < \Delta < \infty, 1 \leq p < \infty, \mu > 0, \nu \geq 1$. Let $\mathrm{F}^{\mu,\nu}_{u,\omega}$ be the integral operator defined from $L_p([\delta,\Delta])$ to $L_p([\delta,\Delta])$ as follows:

$$\mathrm{F}^{\mu,\nu}_{u,\omega;\delta+}\phi(s) = \frac{\omega(s)}{\Gamma(\mu)} \int_\delta^s (s-t)^{\mu-1} \left[\ln \frac{s}{t}\right]^{\nu-1} \phi(t)\, u(t)dt \tag{3.1}$$

and

$$\mathrm{F}^{\mu,\nu}_{u,\omega;\Delta-}\phi(s) = \frac{\omega(s)}{\Gamma(\mu)} \int_s^\Delta (t-s)^{\mu-1} \left[\ln \frac{t}{s}\right]^{\nu-1} \phi(t)\, u(t)dt, \tag{3.2}$$

where u and ω are locally integrable, nonnegative functions. We set

$$\mathrm{F}^{0,1}_{1,1;\delta+}\phi = \mathrm{F}^{0,1}_{1,1;\Delta-}\phi = \phi.$$

Remark 3.2. (1) If $\nu = 1, \omega(s) = u(s) = 1$, then the operator $\mathrm{F}^{\mu,1}_{1,1}$ coincides with the classical Riemann-Liouville fractional integral operator J^μ.

(2) For $\mu > 0, \nu \geq 1$, necessary and sufficient conditions for the boundedness and compactness of the integral operator $\mathrm{F}^{\mu;\nu}_{u,\omega}$ from L_p to $L_q, 0 < p, q < \infty$ are found (see [6], Theorem 3.1, Theorem 4.1).

Theorem 3.3. *Let $\mu_1, \mu_2 \geq 1$ and $\nu_1, \nu_2 \geq 1$. Let $\phi : [\delta; \Delta] \to \mathbb{R}$ be a nonnegative h-convex function, where h is Lebesgue integrable on $(0,1)$. Assume that u is non-decreasing on $[\delta, s]$ and non-increasing on $[s, \Delta]$, for $s \in (\delta, \Delta)$. Then the following inequality*

$$\begin{aligned}
&\frac{1}{u(s)\omega(s)} \left(\frac{\Gamma(\mu_1) F^{\mu_1,\nu_1}_{u,\omega;\delta+}\phi(s)}{\left(\ln \frac{s}{\delta}\right)^{\nu_1-1}} + \frac{\Gamma(\mu_2) F^{\mu_2,\nu_2}_{u,\omega;\Delta-}\phi(s)}{\left(\ln \frac{\Delta}{s}\right)^{\nu_2-1}} \right) \\
&\quad \leq \phi(s)\left[(s-\delta)^{\mu_1} + (\Delta-s)^{\mu_2}\right] \int_0^1 h(1-z)dz \\
&\quad\quad + \left(\phi(\delta)(s-\delta)^{\mu_1} + \phi(\Delta)(\Delta-s)^{\mu_2}\right) \int_0^1 h(z)dz
\end{aligned} \tag{3.3}$$

holds.

Proof. Let $s \in (\delta, \Delta)$. Firstly, let us examine the function ϕ on the interval $[\delta, s]$. Therefore, for all $t \in [\delta, s]$, the following inequality

$$u(t)\left[\ln \frac{s}{t}\right]^{\nu_1-1} (s-t)^{\mu_1-1} \leq u(s)\left[\ln \frac{s}{\delta}\right]^{\nu_1-1} (s-\delta)^{\mu_1-1} \tag{3.4}$$

holds. Due to the h-convexity of ϕ, we write

$$\phi(t) \le h\left(\frac{s-t}{s-\delta}\right)\phi(\delta) + h\left(\frac{t-\delta}{s-\delta}\right)\phi(s). \tag{3.5}$$

Multiplying (3.4),(3.5) side to side and integrating the result over $[\delta, s]$, we get

$$\begin{aligned}&\int_\delta^s u(t)\left[\ln\frac{s}{t}\right]^{\nu_1-1}(s-t)^{\mu_1-1}\phi(t)dt\\ &\quad\le u(s)(s-\delta)^{\mu_1}\left[\ln\frac{s}{\delta}\right]^{\nu_1-1}\left\{\phi(s)\int_0^1 h(1-z)dz + \phi(\delta)\int_0^1 h(z)dz\right\},\end{aligned} \tag{3.6}$$

that is,

$$\begin{aligned}\Gamma(\mu_1)\mathrm{F}^{\mu_1,\nu_1}_{u,\omega;\delta+}\phi(s) &\le u(s)\omega(s)\left[\ln\frac{s}{\delta}\right]^{\nu_1-1}(s-\delta)^{\mu_1}\\ &\quad\times\left\{\phi(s)\int_0^1 h(1-z)dz + \phi(\delta)\int_0^1 h(z)dz\right\},\end{aligned} \tag{3.7}$$

thus

$$\frac{\Gamma(\mu_1)\mathrm{F}^{\mu_1,\nu_1}_{u,\omega;\delta+}\phi(s)}{u(s)\omega(s)\left[\ln\frac{s}{\delta}\right]^{\nu_1-1}} \le (s-\delta)^{\mu_1}\left\{\phi(s)\int_0^1 h(1-z)dz + \phi(\delta)\int_0^1 h(z)dz\right\}. \tag{3.8}$$

Now let $\mu_2, \nu_2 \ge 1$. Then for $t \in [s, \Delta]$ the following inequalities

$$u(t)\left[\ln\frac{t}{s}\right]^{\nu_2-1}(t-s)^{\mu_2-1} \le u(s)\left[\ln\frac{\Delta}{s}\right]^{\nu_2-1}(\Delta-s)^{\mu_2-1} \tag{3.9}$$

and

$$\phi(t) \le h\left(\frac{t-s}{\Delta-s}\right)\phi(\Delta) + h\left(\frac{\Delta-t}{\Delta-s}\right)\phi(s) \tag{3.10}$$

hold. And we proceed as in the first step. Thus it results that

$$\frac{\Gamma(\mu_2)\mathrm{F}^{\mu_2,\nu_2}_{u,\omega;\Delta-}\phi(s)}{u(s)\omega(s)\left[\ln\frac{\Delta}{s}\right]^{\nu_2-1}} \le (\Delta-s)^{\mu_2}\left\{\phi(s)\int_0^1 h(1-z)dz + \phi(\Delta)\int_0^1 h(z)dz\right\}. \tag{3.11}$$

By adding (3.8) and (3.11), we get (3.3). □

Corollary 3.4. *By setting $\mu_1 = \mu_2 = \mu \geq 1$ and $\nu_1 = \nu_2 = \nu \geq 1$ in* (3.3)*, we get*

$$\frac{\Gamma(\mu)}{u(s)\omega(s)}\left(\frac{F^{\mu,\nu}_{u,\omega;\delta+}\phi(s)}{\left(\ln\frac{s}{\delta}\right)^{\nu-1}}+\frac{F^{\mu,\nu}_{u,\omega;\Delta-}\phi(s)}{\left(\ln\frac{\Delta}{s}\right)^{\nu-1}}\right) \leq \phi(s)[(s-\delta)^{\mu}+(\Delta-s)^{\mu}]\int_0^1 h(1-z)dz + (\phi(\delta)(s-\delta)^{\mu}+\phi(\Delta)(\Delta-s)^{\mu})\int_0^1 h(z)dz. \tag{3.12}$$

Corollary 3.5. *By choosing in* (3.12) *$u = 1$, $\omega = 1, h(x) = x$ and $\nu = 1$, then*

$$\Gamma(\mu)\left(J^{\mu}_{\delta+}\phi(s)+J^{\mu}_{\Delta-}\phi(s)\right) \leq \phi(s)\frac{(s-\delta)^{\mu}+(\Delta-s)^{\mu}}{2} + \frac{\phi(\delta)(s-\delta)^{\mu}+\phi(\Delta)(\Delta-s)^{\mu}}{2}. \tag{3.13}$$

Corollary 3.6. *If we choose $\mu = 1$ and taking $s = \dfrac{\delta+\Delta}{2}$ in* (3.13)*, then we have*

$$\frac{1}{\Delta-\delta}\int_{\delta}^{\Delta}\phi(t)dt \leq \frac{1}{2}\phi\left(\frac{\delta+\Delta}{2}\right)+\frac{\phi(\delta)+\phi(\Delta)}{2}. \tag{3.14}$$

Remark 3.7. For $\mu_1, \mu_2 > 0$, $\nu_1, \nu_2 \leq 1$, and if ϕ is h-concave function, u is decreasing on $[\delta, s]$, increasing on $[s, \Delta]$ for $s \in (\delta, \Delta)$, then (3.3) is reversed.

Example 3.8. The following example shows the validity of the inequality established. Let $\phi : [\delta; \Delta] \to \mathbb{R}_+$, $\phi(t) = 1$ and $h(t) = t^k, k \leq 1, t > 0$. Let $\mu > 1, \nu = 2, u = 1, \omega = 1$. Then we verify easly that ϕ is h-convex. Hence from Corollary 3.4, we have the estimates

$$\frac{\Gamma(\mu)(F^{\mu,2}_{1,1;\delta+}1)(s)}{\ln\frac{s}{\delta}} \leq (s-\delta)^{\mu}\left\{\int_0^1(1-z)^k dz+\int_0^1 z^k dz\right\} \tag{3.15}$$

and

$$\frac{\Gamma(\mu)(F^{\mu,2}_{1,1;\Delta-}1)(s)}{\ln\frac{\Delta}{s}} \leq (\Delta-s)^{\mu}\left\{\int_0^1(1-z)^k dz+\int_0^1 z^k dz\right\} \tag{3.16}$$

or

$$\int_{\delta}^{s}(s-t)^{\mu-1}\ln\frac{s}{t}\,dt=\frac{(s-\delta)^{\mu}}{\mu}\ln\frac{s}{\delta}-\frac{1}{\mu}\int_{\delta}^{s}(s-t)^{\mu}\,t^{-1}dt$$
$$\leq\frac{2}{k+1}(s-\delta)^{\mu}\ln\frac{s}{\delta},\tag{3.17}$$

$$\int_{s}^{\Delta}(t-s)^{\mu-1}\ln\frac{t}{s}\,dt=\frac{(\Delta-s)^{\mu}}{\mu}\ln\frac{\Delta}{s}-\frac{1}{\mu}\int_{s}^{\Delta}(t-s)^{\mu}\,t^{-1}dt$$
$$\leq\frac{2}{k+1}(\Delta-s)^{\mu}\ln\frac{\Delta}{s}.\tag{3.18}$$

For $s=\dfrac{\delta+\Delta}{2}$ and $k=1$, we get

$$\int_{\delta}^{\frac{\delta+\Delta}{2}}\left(\frac{\delta+\Delta}{2}-t\right)^{\mu-1}\ln\left(\frac{\delta+\Delta}{2t}\right)dt\leq\left(\frac{\Delta-\delta}{2}\right)^{\mu}\ln\frac{\delta+\Delta}{2\delta}\tag{3.19}$$

and

$$\int_{\frac{\delta+\Delta}{2}}^{\Delta}\left(t-\frac{\delta+\Delta}{2}\right)^{\mu-1}\ln\frac{2t}{\delta+\Delta}\,dt\leq\left(\frac{\Delta-\delta}{2}\right)^{\mu}\ln\frac{2\Delta}{\delta+\Delta}.\tag{3.20}$$

Theorem 3.9. *Let $\mu_1,\mu_2,\nu_1,\nu_2\geq 1$. Let $\phi:[\delta;\Delta]\to\mathbb{R}$ be a non-negative differentiable function. Let u,ω be locally integrable, non-negative functions. Also suppose that u is absolutely continuous, non-decreasing on $[\delta,s]$ and non-increasing on $[s,\Delta]$, for $s\in(\delta,\Delta)$. If $|\phi'|$ is h-convex, then*

$$\begin{aligned}
&|\frac{\Gamma(\alpha_1+1)}{\omega(s)\left(\ln\dfrac{s}{\delta}\right)^{\Delta_1}}\left(F^{\mu_1,\nu_1+1}_{u,\omega;\delta+}+\beta_1F^{\mu_1+1,\nu_1}_{u/t,\omega;\delta+}-F^{\mu_1+1,\nu_1+1}_{u',\omega;\delta+}\right)\phi(s)\\
&\quad+\frac{\Gamma(\mu_2+1)}{\omega(s)\left(\ln\dfrac{\Delta}{s}\right)^{\nu_2}}\left(F^{\mu_2,\nu_2+1}_{u,\omega;\delta-}+\beta_2F^{\mu_2+1,\nu_2}_{u/t,\omega;\Delta-}+F^{\mu_2+1,\nu_2+1}_{u',\omega;\Delta-}\right)\phi(s)\\
&\quad-\left(\phi(\delta)u(\delta)(s-\delta)^{\mu_1}+\phi(\Delta)u(\Delta)(\Delta-s)^{\mu_2}\right)| \qquad (3.21)\\
&\leq|\phi'(s)|\left((\Delta-s)^{\mu_2+1}\left(\ln\frac{\Delta}{s}\right)^{\Delta_2}+(s-\delta)^{\mu_1+1}\left(\ln\frac{s}{\delta}\right)^{\nu_1}\right)\int_0^1h(1-z)dz\\
&\quad+\left(|\phi'(\Delta)|(\Delta-s)^{\mu_2+1}\left(\ln\frac{\Delta}{s}\right)^{\nu_2}+|\phi'(\delta)|(s-\delta)^{\mu_1+1}\left(\ln\frac{s}{\delta}\right)^{\nu_1}\right)\int_0^1h(z)dz
\end{aligned}$$

holds, where u' is the usual derivative of u and $(u/t)(t)$ denote $\dfrac{u(t)}{t}$.

Proof. First step: For $s \in (\delta, \Delta)$ consider the function ϕ on the interval $[\delta, s]$. Hence for $\mu_1, \nu_1 \geq 0$ and $t \in [\delta, s]$ the following inequality

$$\left[\ln \frac{s}{t}\right]^{\nu_1} u(t)(s-t)^{\mu_1} \leq \left[\ln \frac{s}{\delta}\right]^{\nu_1} u(s)(s-\delta)^{\mu_1+1} \tag{3.22}$$

holds. Due to the h-convexity of $|\phi'|$, it results that for $t \in [\delta, s]$

$$\begin{aligned} -\left(h\left(\tfrac{s-t}{s-\delta}\right)|\phi'(\delta)| + h\left(\tfrac{t-\delta}{s-\delta}\right)|\phi'(s)|\right) &\leq \phi'(t) \\ \leq h\left(\frac{s-t}{s-\delta}\right)|\phi'(\delta)| &+ h\left(\tfrac{t-\delta}{s-\delta}\right)|\phi'(s)|. \end{aligned} \tag{3.23}$$

Multiplying (3.22) and the right side of (3.23) and integrating the result over $[\delta, s]$. Then we have

$$\begin{aligned} &\int_\delta^s u(t)(s-t)^{\mu_1}\left[\ln \frac{s}{t}\right]^{\nu_1} \phi'(t)dt \\ &\leq u(s)(s-\delta)^{\mu_1+1}\left[\ln \frac{s}{\delta}\right]^{\nu_1} \\ &\quad \times \left(|\phi'(s)| \int_0^1 h(1-z)dz + |\phi'(\delta)| \int_0^1 h(z)dz\right). \end{aligned} \tag{3.24}$$

By integrating by parts, we obtain

$$\begin{aligned} &\int_\delta^s u(t)\,(s-t)^{\mu_1}\left(\ln \frac{s}{t}\right)^{\nu_1} \phi'(t)dt \\ &= -\phi(\delta)\left(\ln \frac{s}{\delta}\right)^{\nu_1} u(\delta)(s-\delta)^{\mu_1} \\ &\quad - \int_\delta^s \left(\ln \frac{s}{t}\right)^{\nu_1} u'(t)\,(s-t)^{\mu_1}\,\phi(t)dt \\ &\quad + \mu_1 \int_\delta^s \left(\ln \frac{s}{t}\right)^{\nu_1} u(t)\,(s-t)^{\mu_1-1}\,\phi(t)dt \\ &\quad + \nu_1 \int_\delta^s \left(\ln \frac{s}{t}\right)^{\nu_1-1} \frac{u(t)}{t}(s-t)^{\mu_1}\,\phi(t)dt \\ &\leq \left[\ln \frac{s}{\delta}\right]^{\nu_1} u(s)(s-\delta)^{\mu_1+1} \\ &\quad \times \left(|\phi'(s)| \int_0^1 h(1-z)dz + |\phi'(\delta)| \int_0^1 h(z)dz\right). \end{aligned} \tag{3.25}$$

Using Definition 3.1 and inequality (3.25), it follows that

$$\frac{\Gamma(\mu_1+1)}{\omega(s)\left(\ln\frac{s}{\delta}\right)^{\nu_1}}\left(\mathrm{F}^{\mu_1,\nu_1+1}_{u,\omega;\delta+}+\nu_1\mathrm{F}^{\mu_1+1,\nu_1}_{u/t,\omega;\delta+}-\mathrm{F}^{\mu_1+1,\nu_1+1}_{u',\omega;\delta+}\right)\phi(s)-\phi(\delta)u(\delta)(s-\delta)^{\mu_1}$$
$$\leq u(s)(s-\delta)^{\mu_1+1}\left(|\phi'(s)|\int_0^1 h(1-z)dz+|\phi'(\delta)|\int_0^1 h(z)dz\right). \tag{3.26}$$

By considering the left hand side of (3.23), we deduce a similar inequality

$$-u(s)(s-\delta)^{\mu_1+1}\left[\ln\frac{s}{\delta}\right]^{\nu_1}\left(|\phi'(s)|\int_0^1 h(1-z)dz+|\phi'(\delta)|\int_0^1 h(z)dz\right)$$
$$\leq\int_\delta^s u(t)\,(s-t)^{\mu_1}\left[\ln\frac{s}{t}\right]^{\nu_1}\,\phi'(t)dt. \tag{3.27}$$

By combining the resulting inequality and (3.26), we obtain

$$\left|\frac{\Gamma(\mu_1+1)}{\omega(s)\left(\ln\frac{s}{\delta}\right)^{\nu_1}}\left(\mathrm{F}^{\mu_1,\nu_1+1}_{u,\omega;\delta+}+\nu_1\mathrm{F}^{\mu_1+1,\nu_1}_{u/t,\omega;\delta+}-\mathrm{F}^{\mu_1+1,\nu_1+1}_{u',\omega;\delta+}\right)\phi(s)-\phi(\delta)u(\delta)(s-\delta)^{\mu_1}\right|$$
$$\leq u(s)(s-\delta)^{\mu_1+1}\left(|\phi'(s)|\int_0^1 h(1-z)dz+|\phi'(\delta)|\int_0^1 h(z)dz\right). \tag{3.28}$$

Last step: Let $t\in[s,\Delta]$, $\mu_2>0,\nu_2\geq 0$, and taking in account that $|\phi'|$ is h-convex. Then it follows that

$$(t-s)^{\mu_2}\left(\ln\frac{t}{s}\right)^{\nu_2}\leq(\Delta-s)^{\mu_2}\left(\ln\frac{\Delta}{s}\right)^{\nu_2} \tag{3.29}$$

and

$$-\left(h\left(\frac{\Delta-t}{\Delta-s}\right)|\phi'(s)|+h\left(\frac{t-s}{\Delta-s}\right)|\phi'(\Delta)|\right) \tag{3.30}$$
$$\leq\phi'(t)\leq h\left(\frac{\Delta-t}{\Delta-s}\right)|\phi'(s)|+h\left(\frac{t-s}{\Delta-x}\right)|\phi'(\Delta)|.$$

The rest is similar to the first step. Consequently

$$\left|\frac{\Gamma(\mu_2+1)}{\omega(s)\left(\ln\frac{\Delta}{s}\right)^{\nu_2}}\left(\mathrm{F}^{\mu_2,\nu_2+1}_{u,\omega;\Delta-}+\nu_2\mathrm{F}^{\mu_2+1,\nu_2}_{u/t,\omega;\Delta-}+\mathrm{F}^{\mu_2+1,\nu_2+1}_{u',\omega;\Delta-}\right)\phi(s)-\phi(\Delta)u(\Delta)(\Delta-s)^{\mu_2}\right|$$
$$\leq u(s)(\Delta-s)^{\mu_2+1}\left(|\phi'(s)|\int_0^1 h(1-z)dz+|f'(\Delta)|\int_0^1 h(z)dz\right). \tag{3.31}$$

By triangular inequality, by adding inequalities (3.28) and (3.31), the required inequality holds. □

As special cases, we have the following corollaries,

Corollary 3.10. *By setting* $\mu_1 = \mu_2 = \mu, \nu_1 = \nu_2 = \nu, h(t) = t^r, r \in (0,1]$ *in* (3.21)*, then*

$$\Big|\frac{\Gamma(\mu+1)}{\omega(s)u(s)}\Big(\Big[F^{\mu,\nu+1}_{u,\omega;\delta+} + \nu F^{\mu+1,\nu}_{u/t,\omega;\delta+} - F^{\mu+1,\nu+1}_{u',\omega;\delta+} + F^{\mu,\nu+1}_{u,\omega;\Delta-} + \nu F^{\mu+1,\nu}_{u/t,\omega;\Delta-} + F^{\mu+1,\nu+1}_{u',\omega;\Delta-}\Big]\phi\Big)(s)$$
$$- \frac{1}{u(s)}\left(\left((\Delta - s)^{\mu+1}\ln\frac{\Delta}{s}\right)^{\beta} u(\Delta)\phi(\Delta) + (s-\delta)^{\mu+1}\ln\left(\frac{s}{\delta}\right)^{\nu} u(\delta)\phi(\delta)\right)\Big|$$
$$\leq |\phi'(s)|\frac{\left((\Delta - s)^{\mu+1}\left(\ln\frac{\Delta}{s}\right)^{\nu} + (s-\delta)^{\mu+1}\left(\ln\frac{s}{\delta}\right)^{\nu}\right)}{r+1}$$
$$+ |\phi'(\Delta)|\frac{(\Delta - s)^{\mu+1}\left(\ln\frac{\Delta}{s}\right)^{\nu}}{r+1} + |\phi'(\delta)|\frac{(s-\delta)^{\mu+1}\left(\ln\frac{s}{\delta}\right)^{\nu}}{r+1} \tag{3.32}$$

holds.

Corollary 3.11. *If we choose* $u = 1, v = 1, \nu = 0,$ *and* $r = 1$ *in* (3.32)*, then*

$$\left|\Gamma(\mu+1)\left(\mathbf{J}^{\mu}_{\delta+}\phi(s) + \mathbf{J}^{\mu}_{b-}\phi(s)\right) - \left((\Delta - s)^{\mu}\phi(\Delta) + (s-\delta)^{\mu}\right)\phi(\delta)\right| \tag{3.33}$$
$$\leq |\frac{(\Delta - s)^{\mu+1} + (s-\delta)^{\mu+1}}{2}\phi'(s)| + \frac{(\Delta - s)^{\mu+1}}{2}|\phi'(\Delta)| + \frac{(s-\delta)^{\mu+1}}{2}|\phi'(\delta)|$$

holds.

Corollary 3.12. *On letting* $x = \dfrac{\delta+\Delta}{2}$ *and* $\mu = 1$ *in* (3.33)*, then*

$$\left|\frac{1}{\Delta - \delta}\int_{\delta}^{\Delta} f(t)dt - \frac{f(\Delta) + f(\delta)}{2}\right| \tag{3.34}$$
$$\leq \frac{(\Delta - \delta)}{8}\left[2\Big|f'\left(\frac{\delta+\Delta}{2}\right)\Big| + |f'(\Delta)| + |f'(\delta)|\right]$$

is valid.

We need the following result.

Lemma 3.13. *Assume that* $\phi : [\delta, \Delta] \to \mathbb{R}$ *is h-convex function and* ϕ *is symmetric about* $\dfrac{\delta+\Delta}{2}$*. Then*

$$\phi\left(\frac{\delta+\Delta}{2}\right) \leq 2h\left(\frac{1}{2}\right)\phi(x), \quad x \in [\delta, \Delta] \tag{3.35}$$

is valid.

Proof. We have

$$\frac{\delta+\Delta}{2}=\frac{1}{2}\left(\delta\frac{x-\delta}{\Delta-\delta}+\Delta\frac{\Delta-x}{\Delta-\delta}\right)+\frac{1}{2}\left(\Delta\frac{x-\delta}{\Delta-\delta}+\delta\frac{\Delta-x}{\Delta-\delta}\right).$$

Hence,

$$\begin{aligned}\phi\left(\frac{\delta+\Delta}{2}\right)&\leq h\left(\frac{1}{2}\right)\left[\phi\left(\delta\frac{x-\delta}{\Delta-\delta}+\Delta\frac{\Delta-x}{\Delta-\delta}\right)\right]\\&\quad+h\left(\frac{1}{2}\right)\left[\phi\left(\Delta\frac{x-\delta}{\Delta-\delta}+\delta\frac{\Delta-x}{\Delta-\delta}\right)\right]\\&=h\left(\frac{1}{2}\right)\phi(\delta+\Delta-x)+h\left(\frac{1}{2}\right)\phi(x)\\&=2h\left(\frac{1}{2}\right)\phi(x).\end{aligned}$$

□

Theorem 3.14. *Let $\mu_1>0,\mu_2>0,\nu_1,\nu_2\geq 1$. Let $\phi:[\delta;\Delta]\to\mathbb{R}$ be a non-negative h-convex function, where h is Lebesgue integrable on $(0,1)$. Let u,ω be integrable and non-negative functions, $\omega(\delta)\neq 0,\omega(\Delta)\neq 0$. Also suppose that u is monotonic on $[\delta,\Delta]$, for $s\in(\delta,\Delta)$. If ϕ is symmetric about $\frac{\delta+\Delta}{2}$, then it follows that*

(1) *If u is increasing, then*

$$\begin{aligned}&\frac{u(\delta)}{2h\left(\frac{1}{2}\right)}\left[\int_a^b(t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1}+(\Delta-t)^{\mu_2}\left(\ln\frac{\Delta}{t}\right)^{\nu_2-1}dt\right]\phi\left(\frac{\delta+\Delta}{2}\right)\\&\leq\frac{\Gamma(\mu_1+1)F^{\mu_1+1,\nu_1}_{u,\omega;\Delta-}\phi(\delta)}{v(\delta)}+\frac{\Gamma(\mu_2+1)F^{\mu_2+1,\nu_2}_{u,\omega;\delta+}\phi(\Delta)}{v(\Delta)}\\&\leq u(\Delta)\left((\Delta-\delta)^{\mu_1+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_1-1}+(\Delta-\delta)^{\mu_2+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_2-1}\right)\\&\quad\times\left(\phi(\delta)\int_0^1h(z)+\phi(\Delta)\int_0^1h(1-z)dz\right)\end{aligned}$$

holds (3.36)

(2) *If u is decreasing, then*

$$\frac{u(\Delta)}{2h\left(\frac{1}{2}\right)}\left[\int_a^b (t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1}+(\Delta-t)^{\mu_2}\left(\ln\frac{\Delta}{t}\right)^{\nu_2-1}dt\right]\phi\left(\frac{\delta+\Delta}{2}\right)$$
$$\leq \frac{\Gamma(\mu_1+1)F^{\mu_1+1,\nu_1}_{u,\omega;\Delta-}\phi(\delta)}{\omega(\delta)}+\frac{\Gamma(\mu_2+1)F^{\mu_2+1,\nu_2}_{u,\omega;\delta+}\phi(\Delta)}{\omega(\Delta)}$$
$$\leq u(\delta)\left((\Delta-\delta)^{\mu_1+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_1-1}+(\Delta-\delta)^{\mu_2+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_2-1}\right)$$
$$\times\left(\phi(\delta)\int_0^1 h(z)+\phi(\Delta)\int_0^1 h(1-z)dz\right)$$

is valid. (3.37)

Proof. (1) We start by the case u is increasing. For $t\in[\delta,\Delta],\mu_1>0,\nu_1\geq 1$, we have

$$(t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1}u(t)\leq(\Delta-\delta)^{\mu_1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_1-1}u(\Delta) \tag{3.38}$$

and

$$\phi(t)\leq h\left(\frac{t-\delta}{\Delta-\delta}\right)\phi(\delta)+h\left(\frac{\Delta-t}{\Delta-\delta}\right)\phi(\Delta). \tag{3.39}$$

Multiplying inequalities (3.38),(3.39) side to side, and integrating the result over $[\delta,\Delta]$. It follows that

$$\int_\delta^\Delta (t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1}u(t)\phi(t)dt$$
$$\leq(\Delta-\delta)^{\mu_1+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_1-1}u(\Delta)\left(\phi(\delta)\int_0^1 h(z)+\phi(\Delta)\int_0^1 h(1-z)dz\right). \tag{3.40}$$

From which, we have

$$\frac{\Gamma(\mu_1+1)\mathrm{F}^{\mu_1+1,\nu_1}_{u,\omega;\Delta-}\phi(\delta)}{\omega(\delta)} \tag{3.41}$$
$$\leq u(\Delta)(\Delta-\delta)^{\mu_1+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_1-1}\left(\phi(\delta)\int_0^1 h(z)+\phi(\Delta)\int_0^1 h(1-z)dz\right).$$

On the other hand for $t\in[\delta,\Delta]$, we have

$$(\Delta-t)^{\mu_2}\left(\ln\frac{\Delta}{t}\right)^{\nu_2-1}u(t)\leq(\Delta-\delta)^{\mu_2}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_2-1}u(\Delta). \tag{3.42}$$

By multiplying (3.39) and (3.42) and integrating the result over $[\delta, \Delta]$, we get

$$\frac{\Gamma(\mu_2+1)\mathrm{F}^{\mu_2+1,\nu_2}_{u,\omega;\delta+}\phi(\Delta)}{\omega(\Delta)} \leq u(\Delta)(\Delta-\delta)^{\mu_2+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_2-1}\left(\phi(\delta)\int_0^1 h(z)+\phi(\Delta)\int_0^1 h(1-z)dz\right). \tag{3.43}$$

By adding (3.41) and (3.43), it results that

$$\begin{aligned}\frac{\Gamma(\mu_1+1)\mathrm{F}^{\mu_1+1,\nu_1}_{u,\omega;\Delta-}\phi(\delta)}{\omega(\delta)}&+\frac{\Gamma(\mu_2+1)\mathrm{F}^{\mu_2+1,\nu_2}_{u,\omega;\delta+}\phi(\Delta)}{\omega(\Delta)}\\ &\leq u(\Delta)\left((\Delta-\delta)^{\mu_1+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_1-1}+(\Delta-\delta)^{\mu_2+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_2-1}\right)\\ &\quad\times\left(\phi(\delta)\int_0^1 h(z)+\phi(\Delta)\int_0^1 h(1-z)dz\right).\end{aligned} \tag{3.44}$$

Using Lemma 3.13, we have

$$\begin{aligned}\phi\left(\frac{\delta+\Delta}{2}\right)&u(\delta)(t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1}\\ &\leq 2h\left(\frac{1}{2}\right)\phi(t)u(t)(t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1},\end{aligned} \tag{3.45}$$

integrating (3.45) over $[\delta, \Delta]$, we get

$$\begin{aligned}u(\delta)\phi\left(\frac{\delta+\Delta}{2}\right)&\int_\delta^\Delta (t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1}dt\\ &\leq 2h\left(\frac{1}{2}\right)\frac{\Gamma(\mu_1+1)\,\mathrm{F}^{\mu_1+1,\nu_1}_{u,v;\Delta-}\phi(\delta)}{v(\delta)}.\end{aligned} \tag{3.46}$$

Similarly, we have

$$\begin{aligned}\phi\left(\frac{\delta+\Delta}{2}\right)&u(\delta)(\Delta-t)^{\mu_2}\left(\ln\frac{\Delta}{t}\right)^{\nu_2-1}\\ &\leq 2h\left(\frac{1}{2}\right)\phi(t)u(t)(\Delta-t)^{\mu_2}\left(\ln\frac{\Delta}{t}\right)^{\nu_2-1},\end{aligned} \tag{3.47}$$

integrating (3.47) with respect to t over $[\delta, \Delta]$, we get

$$u(\delta)\phi\left(\frac{\delta+\Delta}{2}\right)\int_{\delta}^{\Delta}(\Delta-t)^{\mu_2}\left(\ln\frac{\Delta}{t}\right)^{\nu_2-1}dt \leq 2h\left(\frac{1}{2}\right)\frac{\Gamma(\mu_2+1)\mathrm{F}_{u,v;\delta+}^{\mu_2+1,\nu_2}\phi(\Delta)}{v(\Delta)}. \tag{3.48}$$

Adding (3.46) and (3.48), we obtain

$$u(\delta)\phi\left(\frac{\delta+\Delta}{2}\right)\left[\int_{\delta}^{\Delta}(\Delta-t)^{\mu_2}\left(\ln\frac{\Delta}{t}\right)^{\nu_2-1}+(t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1}dt\right] \leq 2h\left(\frac{1}{2}\right)\left[\frac{\Gamma(\mu_1+1)\mathrm{F}_{u,v;\Delta-}^{\mu_1+1,\nu_1}\phi(\delta)}{v(\delta)}+\frac{\Gamma(\mu_2+1)\mathrm{F}_{u,v;\delta+}^{\mu_2+1,\nu_2}\phi(\Delta)}{v(\Delta)}\right], \tag{3.49}$$

combining (3.44) and (3.49), we have

$$\begin{aligned}
&\frac{u(\delta)}{2h\left(\frac{1}{2}\right)}\left[\int_{\delta}^{\Delta}(\Delta-t)^{\mu_2}\left(\ln\frac{\Delta}{t}\right)^{\nu_2-1}+(t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1}dt\right]\phi\left(\frac{\Delta+\delta}{2}\right)\\
&\leq \frac{\Gamma(\mu_1+1)\mathrm{F}_{u,\omega;\Delta-}^{\mu_1+1,\nu_1}\phi(\delta)}{\omega(\delta)}+\frac{\Gamma(\mu_2+1)\mathrm{F}_{u,\omega;\delta+}^{\mu_2+1,\nu_2}\phi(\Delta)}{\omega(\Delta)}\\
&\leq u(\Delta)\left((\Delta-\delta)^{\mu_1+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_1-1}+(\Delta-\delta)^{\mu_2+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_2-1}\right)\\
&\quad\times\left(\phi(\delta)\int_0^1 h(z)+\phi(\Delta)\int_0^1 h(1-z)dz\right).
\end{aligned}$$

(2) Similar proof for the case u decreasing. □

Corollary 3.15. *By setting $\mu_1=\mu_2=\mu$ and $\nu_1=\nu_2=\nu$, we obtain*

$$\begin{aligned}
&\frac{u(\delta)}{2h\left(\frac{1}{2}\right)}\left[\int_{\delta}^{\Delta}(\Delta-t)^{\mu}\left(\ln\frac{\Delta}{t}\right)^{\nu-1}+(t-\delta)^{\mu}\left(\ln\frac{t}{\delta}\right)^{\nu-1}dt\right]\phi\left(\frac{\delta+\Delta}{2}\right)\\
&\leq \Gamma(\mu+1)\left(\frac{F_{u,\omega;\Delta-}^{\mu+1,\nu}\phi(\delta)}{\omega(\delta)}+\frac{F_{u,\omega;\delta+}^{\mu+1,\nu}\phi(\Delta)}{\omega(\Delta)}\right) \qquad (3.50)\\
&\leq 2u(\Delta)(\Delta-\delta)^{\mu+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu-1}\left(\phi(\delta)\int_0^1 h(z)+\phi(\Delta)\int_0^1 h(1-z)dz\right).
\end{aligned}$$

Example 3.16. The following example illustrates the validity of estimates. Let $\phi : [\delta; \Delta] \to \mathbb{R}_+$, $\phi(t) = 1$ and $h_k(t) = t^k, k \leq 1, t > 0$. Let $\mu > 0, \nu \geq 1$, $u = 1, \omega = 1$. We verify that

(1) ϕ is h_k-convex.

(2) ϕ is symmetric about $\dfrac{\delta + \Delta}{2}$.

Hence from Corollary 3.15, we get the estimates

$$\begin{aligned} &\frac{1}{2^{1-k}} \left[\int_\delta^\Delta (\Delta - t)^\mu \left(\ln \frac{\Delta}{t} \right)^{\nu-1} + (t-\delta)^\mu \left(\ln \frac{t}{\delta} \right)^{\nu-1} dt \right] \\ &\leq \Gamma(\mu+1) \left(\mathrm{F}^{\mu+1,\nu}_{1,1;\Delta-} 1(\delta) + \mathrm{F}^{\mu+1,\nu}_{1,1;\delta+} 1(\Delta) \right) \\ &\leq 2(\Delta-\delta)^{\mu+1} \left(\ln \frac{\Delta}{\delta} \right)^{\nu-1} \left(\int_0^1 z^k + \int_0^1 (1-z)^k dz \right). \end{aligned} \tag{3.51}$$

Or

$$\begin{aligned} \frac{1}{2^{1-k}} \int_\delta^\Delta (\Delta - t)^\mu \left(\ln \frac{\Delta}{t} \right)^{\nu-1} &\leq \int_\delta^\Delta (\Delta - t)^\mu \left(\ln \frac{\Delta}{t} \right)^{\nu-1} dt \\ &\leq \frac{2(\Delta-\delta)^{\mu+1}}{k+1} \left(\ln \frac{\Delta}{\delta} \right)^{\nu-1} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2^{1-k}} \int_\delta^\Delta (t-\delta)^\mu \left(\ln \frac{t}{\delta} \right)^{\nu-1} &\leq \int_\delta^\Delta (t-\delta)^\mu \left(\ln \frac{t}{\delta} \right)^{\nu-1} dt \\ &\leq \frac{2(\Delta-\delta)^{\mu}}{k+1} + \left(\ln \frac{\Delta}{\delta} \right)^{\nu-1}. \end{aligned}$$

Take the change of variables $t = \delta + (\Delta - \delta)e^{-x}$ and $t = \Delta - (\Delta - \delta)e^{-x}$, we get

$$\begin{aligned} &\frac{1}{2^{1-k}} \int_0^\infty e^{-(\mu+1)x} \left[\ln \left(1 + \frac{\Delta-\delta}{\delta} e^{-x} \right) \right]^{\nu-1} dx \\ &\leq \int_0^\infty e^{-(\mu+1)x} \left[\ln \left(1 + \frac{\Delta-\delta}{\delta} e^{-x} \right) \right]^{\nu-1} dx \\ &\leq \frac{2}{k+1} \left(\ln \frac{\Delta}{\delta} \right)^{\nu-1} \end{aligned} \tag{3.52}$$

and

$$\begin{aligned}&\frac{1}{2^{1-k}}\int_0^\infty e^{-(\mu+1)x}\left[\ln\frac{\Delta e^x}{\Delta\, e^x-\Delta+\delta}\right]^{\nu-1}dx\\&\le\int_0^\infty e^{-(\mu+1)x}\left[\ln\frac{\Delta e^x}{\Delta\, e^x-\Delta+\delta}\right]^{\nu-1}dx\\&\le\frac{2}{k+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu-1}.\end{aligned}\tag{3.53}$$

For $\nu=2$,

$$\begin{aligned}\frac{1}{2^{1-k}}\int_0^\infty e^{-(\mu+1)x}\ln\frac{\delta\, e^x+\Delta-\delta}{\delta\, e^x}\,dx&\le\int_0^\infty e^{-(\mu+1)x}\ln\frac{\delta\, e^x+\Delta-\delta}{\delta\, e^x}\,dx\\&\le\frac{2}{k+1}\ln\frac{\Delta}{\delta}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2^{1-k}}\int_0^\infty e^{-(\mu+1)x}\ln\frac{\Delta e^x}{\Delta\, e^x-\Delta+\delta}\,dx&\le\int_0^\infty e^{-(\mu+1)x}\ln\frac{\Delta e^x}{\Delta\, e^x-\Delta+\delta}\,dx\\&\le\frac{2}{k+1}\ln\frac{\Delta}{\delta}.\end{aligned}$$

Taking $k=1$, we get

$$F(\lambda):=\int_0^\infty e^{-\lambda x}\ln\frac{\delta\, e^x+\Delta-\delta}{\delta\, e^x}\,dx\le\ln\frac{\Delta}{\delta}\ (\lambda>0)$$

and

$$G(\lambda):=\int_0^\infty e^{-\lambda x}\ln\frac{\Delta e^x}{\Delta\, e^x-\Delta+\delta}\,dx\le\ln\frac{\Delta}{\delta}\ (\lambda>0).$$

The functions F, G are the Laplace transforms of $f(x)=\frac{\delta\, e^x+\Delta-\delta}{\delta\, e^x}$, $g(x)=\frac{\Delta e^x}{\Delta\, e^x-\Delta+\delta}$, respectively.

Corollary 3.17. *If we choose $u=1,\omega=1,\nu=1$ and taking $h(x)=x$ in* (3.15)*, then the inequality*

$$\begin{aligned}\frac{1}{(\mu+1)}\,\phi\left(\frac{\delta+\Delta}{2}\right)&\le\frac{\Gamma(\mu+1)}{2(\Delta-\delta)^{\mu+1}}\left(\mathbf{J}^{\mu+1}_{\Delta-}\phi(\delta)+\mathbf{J}^{\mu+1,}_{\delta+}\phi(\Delta)\right)\\&\le\frac{\phi(\delta)+\phi(\Delta)}{2}\end{aligned}\tag{3.54}$$

holds.

Remark 3.18. On letting $\mu \to 0$ in (3.54), we get the inequality of Hermite-Hadamard

$$\phi\left(\frac{\Delta+\delta}{2}\right) \leq \frac{1}{\Delta-\delta}\int_{\delta}^{\Delta}\phi(t)dt \leq \frac{\phi(\Delta)+\phi(\delta)}{2}. \tag{3.55}$$

4. Conclusions

In this study, we have introduced a new fractional integral operator with a logarithmic kernel and two parameters, incorporating two non-negative weight functions. Focusing on h-convex functions in fractional integral inequalities, we deepen our understanding and extend the utility of h-convex functions in fractional calculus. By looking at these functions in fractional calculus. Notably, we derive some estimates and bounds for Laplace transform of functions, (see examples), also are obtained integral inequalities involving Riemann-Liouville integrals and the classical Hermite-Hadamard inequality.

In conclusion, our research contributes to mathematical analysis by addressing challenges in h-convexity and fractional calculus, opening avenues for exploration at the intersection of integral and classical inequalities. We expect that the ideas and techniques of the paper may stimulate further research in this field.

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