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AN APPLICATION OF EL-ZAKI TRANSFORM TO VARIOUS TYPES OF ULAM STABILITY OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate the Hyers-Ulam stability, generalized Hyers-Ulam stability, Mittag-Leffler-Hyers-Ulam stability and generalized Mittag-Leffler-Hyers-Ulam stability of general linear differential equations of first order with constant coefficients by using El-Zaki transform method. Moreover, the Hyers-Ulam stability constants of these differential equations are obtained. Some examples are given to illustrate our main results.

1. Introduction

In [45], Ulam proposed the universal Ulam stability problem. When is it true that by slightly changing the hypotheses of a theorem one can still assert

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that the thesis of the theorem remains true or approximately true? In [11], Hyers gave the first confirmatory answer to the question of Ulam for additive functional equations in Banach spaces. Hyers result has since then seen many significant generalizations, both in terms of the control condition used to define the concept of approximate solution [4, 6, 9, 12, 22, 23, 24, 39, 42, 44].

A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation $\phi\left(f, x, x', x'', \cdots, x^{(n)}\right) = 0$ has the Hyers-Ulam stability if for a given $\epsilon > 0$ and a function x such that

$$\left|\phi\left(f, x, x', x'', \cdots, x^{(n)}\right)\right| \leq \epsilon,$$

there exists a solution x_a of the differential equation such that $|x(t) - x_a(t)| \leq K(\epsilon)$ and

$$\lim_{\epsilon \to 0} K(\epsilon) = 0.$$

If the preceding statement is also true when we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\varphi(t)$, where ϕ, φ are appropriate functions not depending on x and x_a explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability.

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations [35, 36]. Thereafter, in 1998, Alsina and Ger [2] investigated the Hyers-Ulam stability of differential equations. They proved in [2] the following theorem.

Theorem 1.1. *Assume that a differentiable function $f : I \to R$ is a solution of the differential inequality $\|x'(t) - x(t)\| \leq \epsilon$, where I is an open subinterval of $\mathbb{R}$. Then there exists a solution $g : I \to R$ of the differential equation $x'(t) = x(t)$ such that for any $t \in I$, we have $\|f(t) - g(t)\| \leq 3\epsilon$.*

This result of Alsina and Ger [2] has been generalized by Takahasi *et al.* [43]. They proved in [43] that the Hyers-Ulam stability holds true for the Banach space-valued differential equation $y'(t) = \lambda y(t)$. Indeed, the Hyers-Ulam stability has been proved for the first order linear differential equations in more general settings [14, 15, 16, 17, 26]. In 2006, Jung [18] investigated the Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients by using matrix method. In 2007, Wang, Zhou and Sun [46] studied the Hyers-Ulam stability of a class of first-order linear differential equations. Rus [41] discussed four types of Ulam stability: Hyers-Ulam stability, generalized Hyers-Ulam stability, Hyers-Ulam-Rassias stability and generalized Hyers-Ulam-Rassias stability of the ordinary differential equation

$$u'(t) = A(u(t)) + f(t, u(t)), t \in [a, b].$$

In 2014, Alqifiary and Jung [3] proved the generalized Hyers-Ulam stability of linear differential equation of the form

$$x^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k \; x^{(k)}(t) \;\; = \;\; f(t)$$

by using the Laplace transform method, where α_k are scalars and x and f are n times continuously differentiable function and of the exponential order, respectively. The Hyers-Ulam stability of differential equations has been investigated by many authors in [1, 5, 7, 8, 10, 13, 20, 21, 25, 27, 28, 29, 30, 32, 33, 37, 38] and the Hyers-Ulam stability of differential equations has been given attention.

Recently, Murali, Selvan and Park [34] investigated the Hyers-Ulam stability of the linear differential equation using Fourier transform method (see also [31, 40]).

Motivated and connected by the above results, our main aim is to study the Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the first order homogeneous linear differential equation of the form

$$x'(t) + l \; x(t) = 0 \tag{1.1}$$

and the non-homogeneous linear differential equation

$$x'(t) + l \; x(t) = r(t) \tag{1.2}$$

by using a new transform method, namely, the El-Zaki transform method, where l is scalar, $x(t)$ and $r(t)$ are continuously differentiable functions.

2. Preliminaries and basic notations

In this section, we introduce some standard notations and definitions which will be useful to prove our main results.

Throughout this paper, $\mathbb{F}$ denotes the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A function $f : (0, \infty) \longrightarrow \mathbb{F}$ is said to be of exponential order if there exists a constants $A, B \in \mathbb{R}$ such that $|f(t)| \le Ae^{tB}$ for all $t > 0$.

For each function $f : (0, \infty) \to \mathbb{F}$ of exponential order, let us consider the set $\mathcal{A}$, which is defined by

$$\mathcal{A} = \left\{ f(t) : \exists M, \; |f(t)| < Me^{|t|/k_j}, \; k_1 \;\text{ and }\; k_2 > 0, \; t \in (-1)^j \times [0, \infty) \right\},$$

where the constant M must be finite while k_1 and k_2 may be infinite. The El-Zaki transform is defined by

$$\mathcal{E}\{f(t)\} = \xi \int_0^\infty f(t) \; e^{-t/\xi} \; dt = F(\xi), \qquad t \ge 0, \;\; \xi \in (-k_1, k_2),$$

in which the variable ξ in the El-Zaki transform is used to factor the variable t in the argument of the function f, specially for $f(t)$ in $\mathcal{A}$.

For given Lebesgue integrable functions f and g on $(-\infty, +\infty)$, let S denote the set of x for which the Lebesgue integral

$$h(x) = \int_{-\infty}^{\infty} f(t)\ g(x-t)\ dt$$

exists. This integral defines a function h on S called the *convolution* of f and g. We also write $h = f * g$ to denote this function.

Definition 2.1. ([19]) The Mittag-Leffler function of one parameter is denoted by $E_\nu(t)$ and defined as

$$E_\nu(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\nu k + 1)},$$

where $t, \nu \in \mathbb{C}$ and $Re(\nu) > 0$.

If we put $\nu = 1$, then the above equation becomes

$$E_1(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{t^k}{k} = e^t.$$

Definition 2.2. ([19]) The generalization of $E_\nu(t)$ is defined as a function

$$E_{\nu,\vartheta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\nu k + \vartheta)},$$

where $t, \nu, \vartheta \in \mathbb{C}$, $Re(\nu) > 0$ and $Re(\vartheta) > 0$.

Now, we give the definition of Hyers-Ulam stability and generalized Hyers-Ulam stability of the differential equations (1.1) and (1.2).

Definition 2.3. ([34]) The linear differential equation (1.1) is said to have the Hyers-Ulam stability if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$ such that there exists a continuously differentiable function $x(t)$ satisfying the inequality

$$|x'(t) + l\ x(t)| \leq \epsilon, \tag{2.1}$$

there exists some $y : (0, \infty) \to \mathbb{F}$ satisfying the differential equation (1.1) such that $|x(t) - y(t)| \leq K\epsilon$, for all $t > 0$. We call such K as the Hyers-Ulam stability constant for (1.1).

Definition 2.4. ([34]) We say that the non-homogeneous linear differential equation (1.2) has the Hyers-Ulam stability if there exists a continuously differentiable function $x(t)$ satisfying the following condition: For every $\epsilon > 0$, there exists a positive constant K such that

$$|x'(t) + l\ x(t) - r(t)| \leq \epsilon, \tag{2.2}$$

there exists a solution $y : (0, \infty) \to \mathbb{F}$ satisfying the differential equation (1.2) such that $|x(t) - y(t)| \leq K\epsilon$, for all $t > 0$. We call such K as the Hyers-Ulam stability constant for the differential equation (1.2).

Definition 2.5. ([34]) We say that the homogeneous linear differential equation (1.1) has the generalized Hyers-Ulam stability if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$ and a continuously differentiable function $x(t)$ such that there exists $\phi : (0, \infty) \to (0, \infty)$ satisfying the inequality

$$|x'(t) + l\ x(t)| \leq \phi(t)\epsilon, \tag{2.3}$$

there exists some $y : (0, \infty) \to \mathbb{F}$ satisfying the differential equation (1.1) such that $|x(t) - y(t)| \leq K\ \phi(t)\epsilon$ for all $t > 0$. We call such K as generalized Hyers-Ulam stability constant for (1.1).

Definition 2.6. ([34]) The differential equation (1.2) is said to have the generalized Hyers-Ulam stability if there exists a positive constant K with the following condition: For every $\epsilon > 0$, a continuously differentiable function $x(t)$ and a function $\phi : (0, \infty) \to (0, \infty)$ satisfying the inequality

$$|x'(t) + l\ x(t) - r(t)| \leq \phi(t)\epsilon, \tag{2.4}$$

there exists a solution $y : (0, \infty) \to \mathbb{F}$ satisfies the differential equation (1.2) such that $|x(t) - y(t)| \leq K\ \phi(t)\epsilon$ for all $t > 0$. We call such K as the generalized Hyers-Ulam stability constant for the differential equation (1.2).

Finally, we give the definitions of Mittag-Leffler-Hyers-Ulam stability and generalized Mittag-Leffler-Hyers-Ulam stability of the differential equations (1.1) and (1.2).

Definition 2.7. ([34]) We say that the differential equation (1.1) has Mittag-Leffler-Hyers-Ulam stability if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$ and a continuously differentiable function $x(t)$ satisfying the inequality

$$|x'(t) + l\ x(t)| \leq \epsilon E_\nu(t), \tag{2.5}$$

where $E_\nu(t)$ is the Mittag-Leffler function, there exists some $y : (0, \infty) \to \mathbb{F}$ satisfying the differential equation (1.1) such that $|x(t) - y(t)| \leq K\epsilon E_\nu(t)$ for all $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for (1.1).

Definition 2.8. ([34]) We say that the non-homogeneous differential equation (1.2) has the Mittag-Leffler-Hyers-Ulam stability if there exists a positive constant $K > 0$ with the following condition: For every $\epsilon > 0$ and a continuously differentiable function $x(t)$ such that

$$|x'(t) + l\ x(t) - r(t)| \leq \epsilon E_\nu(t), \tag{2.6}$$

where $E_\nu(t)$ is the Mittag-Leffler function, there exists a solution $y : (0, \infty) \to \mathbb{F}$ satisfying the differential equation (1.2) such that $|x(t) - y(t)| \leq K\epsilon E_\nu(t)$ for all $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for the differential equation (1.2).

Definition 2.9. ([34]) We say that the differential equation (1.1) has the generalized Mittag-Leffler-Hyers-Ulam stability if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$ and a continuously differentiable function $x(t)$ such that there exists $\phi : (0, \infty) \to (0, \infty)$ satisfying the inequality

$$|x'(t) + l\ x(t)| \leq \phi(t)\epsilon E_\nu(t), \tag{2.7}$$

where $E_\nu(t)$ is the Mittag-Leffler function, there exists some $y : (0, \infty) \to \mathbb{F}$ satisfying the differential equation (1.1) such that $|x(t) - y(t)| \leq K\ \phi(t)\epsilon E_\nu(t)$ for all $t > 0$. We call such K as generalized Mittag-Leffler -Hyers-Ulam stability constant for (1.1).

Definition 2.10. ([34]) We say that the differential equation (1.2) has the generalized Mittag-Leffler-Hyers-Ulam stability if there exists a positive constant K with the following condition: For every $\epsilon > 0$, a continuously differentiable function $x(t)$ and a function $\phi : (0, \infty) \to (0, \infty)$ satisfying the inequality

$$|x'(t) + l\ x(t) - r(t)| \leq \phi(t)\epsilon E_\nu(t), \tag{2.8}$$

where $E_\nu(t)$ is the Mittag-Leffler function, there is a solution $y : (0, \infty) \to \mathbb{F}$ satisfying the differential equation (1.2) such that $|x(t) - y(t)| \leq K\ \phi(t)\epsilon E_\nu(t)$ for all $t > 0$. We call such K as the generalized Mittag-Leffler-Hyers-Ulam stability constant for the differential equation (1.2).

3. Hyers-Ulam stabilities of the linear differential equation (1.1)

In this section, we prove the different types of Hyers-Ulam stability of the homogeneous linear differential equations (1.1) by using El-Zaki transform.

Firstly, we investigate the Hyers-Ulam stability of first order homogeneous differential equation (1.1).

Theorem 3.1. *The homogeneous linear differential equation* (1.1) *is Hyers-Ulam stable.*

Proof. Assume that $x(t)$ is a continuously differentiable function satisfying the inequality (2.1). Let us define a function $p : (0, \infty) \longrightarrow \mathbb{F}$ such that $p(t) := x'(t) + l\ x(t)$ for each $t > 0$. In view of (2.1), we have $|p(t)| \le \epsilon$. Taking El-Zaki transform to $p(t)$, we get

$$\begin{aligned} P(\xi) := \mathcal{E}\{p(t)\} &= \mathcal{E}\{x'(t) + l\ x(t)\} = \mathcal{E}\{x'(t)\} + l\ \mathcal{E}\{x(t)\} \\ &= \frac{X(\xi)}{\xi} - \xi\ x(0) + l\ X(\xi). \end{aligned}$$

Thus

$$\mathcal{E}\{x(t)\} = X(\xi) = \frac{\xi^2\ x(0) + \xi\ P(\xi)}{1 + l\xi}. \tag{3.1}$$

Put $y(t) = e^{-lt}\ x(0)$. Then $y(0) = x(0)$. Taking El-Zaki transform to $y(t)$, we get

$$\mathcal{E}\{y(t)\} = Y(\xi) = \frac{\xi^2\ x(0)}{1 + l\xi}. \tag{3.2}$$

Thus, we have

$$\begin{aligned} \mathcal{E}\{y'(t) + l\ y(t)\} &= \mathcal{E}\{y'(t)\} + l\ \mathcal{E}\{y(t)\} \\ &= \frac{Y(\xi)}{\xi} - \xi\ y(0) + l\ Y(\xi). \end{aligned}$$

Using (3.2), we have $\mathcal{E}\{y'(t) + l\ y(t)\} = 0$. Since $\mathcal{E}$ is one-to-one operator, $y'(t) + l\ y(t) = 0$. Hence $y(t)$ is a solution of the differential equation (1.1). Set $Q(\xi) = \dfrac{\xi}{(1 + l\xi)}$. Then the equality $\mathcal{E}\{q(t)\} = \dfrac{\xi}{1 + l\xi}$ implies that $q(t) = \mathcal{E}^{-1}\left(\dfrac{\xi}{1 + l\xi}\right)$. Plugging (3.1) into (3.2), we can obtain

$$\mathcal{E}\{x(t)\} - \mathcal{E}\{y(t)\} = X(\xi) - Y(\xi) = \frac{\xi\ P(\xi)}{1 + l\xi} = P(\xi)\ Q(\xi) = \mathcal{E}\{p(t)\}\ \mathcal{E}\{q(t)\}.$$

Consequently, $\mathcal{E}\{x(t)-y(t)\} = \mathcal{E}\{p(t)*q(t)\}$ which gives $x(t)-y(t) = p(t)*q(t)$. Taking modulus on both sides, we have

$$\begin{aligned} |x(t) - y(t)| = |p(t) * q(t)| &= \left| \int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds \right| \\ &\le |p(t)| \left| \int_{-\infty}^{\infty} q(t-s)\ ds \right| \\ &\le K\epsilon, \end{aligned}$$

where $K = \left| \int_{-\infty}^{\infty} q(t-s)\ ds \right|$ and the integral exists for each value of t. This shows that the homogeneous linear differential equation (1.1) has the Hyers-Ulam stability. □

In analogy with Theorems 3.1, we bring the following generalized Hyers-Ulam stability result for the differential equation (1.1). We include some parts of the proof for the sake of completeness.

Theorem 3.2. *The differential equation* (1.1) *has the generalized Hyers-Ulam stability.*

Proof. Let $x(t)$ be a continuously differentiable function satisfying the inequality (2.3). Defining a function $p : (0,\infty) \longrightarrow \mathbb{F}$ through $p(t) =: x'(t) + l\ x(t)$ for each $t > 0$, we have $|p(t)| \le \phi(t)\epsilon$. Taking El-Zaki transform to $p(t)$, we get

$$\mathcal{E}\{x(t)\} = X(\xi) = \frac{\xi^2\ x(0) + \xi\ P(\xi)}{1 + l\xi}. \tag{3.3}$$

If $y(t) = e^{-lt}\ x(0)$, then $y(0) = x(0)$ and hence

$$\mathcal{E}\{y(t)\} = Y(\xi) = \frac{\xi^2\ x(0)}{1 + l\xi}. \tag{3.4}$$

Thus,

$$\mathcal{E}\{y'(t) + l\ y(t)\} = \mathcal{E}\{y'(t)\} + l\ \mathcal{E}\{y(t)\} = \frac{Y(\xi)}{\xi} - \xi\ y(0) + l\ Y(\xi).$$

It follows from (3.4) that $\mathcal{E}\{y'(t) + l\ y(t)\} = 0$ and so $y'(t) + l\ y(t) = 0$. This means that $y(t)$ is a solution of the differential equation (1.1).

On the other hand, for $Q(\xi) = \dfrac{\xi}{(1 + l\xi)}$, we have $\mathcal{E}\{q(t)\} = \dfrac{\xi}{1 + l\xi}$ and consequently $q(t) = \mathcal{E}^{-1}\left(\dfrac{\xi}{1 + l\xi}\right)$. The relations (3.3) and (3.4) necessitate that

$$\mathcal{E}\{x(t)\} - \mathcal{E}\{y(t)\} = X(\xi) - Y(\xi) = \frac{\xi\ P(\xi)}{1 + l\xi} = P(\xi)\ Q(\xi) = \mathcal{E}\{p(t)\}\ \mathcal{E}\{q(t)\}$$

which implies that $\mathcal{E}\{x(t)-y(t)\}=\mathcal{E}\{p(t)*q(t)\}$. Therefore,

$$x(t)-y(t)=p(t)*q(t).$$

Similar to the proof of Theorem 3.1, one can show that we have

$$|x(t)-y(t)|\le K\phi(t)\epsilon,$$

where $K=\left|\int_{-\infty}^{\infty} q(t-s)\ ds\right|$ for which the integral exists for each value of t and $\phi(t)$ is an integrable function. Therefore, we get the desired result. □

Now, we establish the Mittag-Leffler-Hyers-Ulam stability of the differential equations (1.1) by using El-Zaki transform.

Theorem 3.3. *The homogeneous differential equation* (1.1) *has the Mittag-Leffler-Hyers-Ulam stability.*

Proof. Suppose that $x(t)$ is a continuously differentiable function satisfying the inequaty (2.5). Let $p:(0,\infty)\longrightarrow\mathbb{F}$ be a function defined by $p(t):=x'(t)+l\ x(t)$ for each $t>0$. In view of (2.5), we have $|p(t)|\le\epsilon E_\nu(t)$. Taking El-Zaki transform to $p(t)$, we get

$$\begin{aligned}P(\xi):=\mathcal{E}\{p(t)\}=\mathcal{E}\{x'(t)+l\ x(t)\}&=\mathcal{E}\{x'(t)\}+l\ \mathcal{E}\{x(t)\}\\ &=\frac{X(\xi)}{\xi}-\xi\ x(0)+l\ X(\xi).\end{aligned}$$

Thus,

$$\mathcal{E}\{x(t)\}=X(\xi)=\frac{\xi^2\ x(0)+\xi\ P(\xi)}{1+l\xi}. \tag{3.5}$$

Put $y(t)=e^{-lt}\ x(0)$. Then $y(0)=x(0)$. Taking El-Zaki transform to $y(t)$, we get

$$\mathcal{E}\{y(t)\}=Y(\xi)=\frac{\xi^2\ x(0)}{1+l\xi}. \tag{3.6}$$

Thus

$$\mathcal{E}\{y'(t)+l\ y(t)\}=\mathcal{E}\{y'(t)\}+l\ \mathcal{E}\{y(t)\}=\frac{Y(\xi)}{\xi}-\xi\ y(0)+l\ Y(\xi).$$

Using (3.6), we have $\mathcal{E}\{y'(t)+l\ y(t)\}=0$. Since $\mathcal{E}$ is one-to-one operator, $y'(t)+l\ y(t)=0$. Hence $y(t)$ is a solution of the differential equation (1.1). Set $Q(\xi)=\dfrac{\xi}{(1+l\xi)}$. Then the equality $\mathcal{E}\{q(t)\}=\dfrac{\xi}{1+l\xi}$ implies that $q(t)=$

$\mathcal{E}^{-1}\left(\dfrac{\xi}{1+l\xi}\right)$. Plugging (3.5) into (3.6), we can obtain

$$\mathcal{E}\{x(t)\} - \mathcal{E}\{y(t)\} = X(\xi) - Y(\xi) = \frac{\xi\ P(\xi)}{1+l\xi} = P(\xi)\ Q(\xi) = \mathcal{E}\{p(t)\}\ \mathcal{E}\{q(t)\}.$$

Consequently, $\mathcal{E}\{x(t)-y(t)\} = \mathcal{E}\{p(t) * q(t)\}$, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides and using $|p(t)| \le \epsilon E_\nu(t)$, we have

$$|x(t)-y(t)| = |p(t) * q(t)| = \left|\int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds\right| \le |p(t)|\ \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right|.$$

Choose $K = \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right|$. Since the integral exists for each value of t,

$$|x(t) - y(t)| \le K\epsilon E_\nu(t).$$

Then by the virtue of Definition 2.7, the homogeneous linear differential equation (1.1) has the Mittag-Leffler-Hyers-Ulam stability. □

In analogy with Theorem 3.3, we bring the following generalized Mittag-Leffler-Hyers-Ulam stability result for the differential equation (1.1). We include the proof for the sake of completeness.

Corollary 3.4. *The differential equation* (1.1) *has the generalized Mittag-Leffler-Hyers-Ulam stability.*

Proof. Let $x(t)$ be a continuously differentiable function satisfying the inequality (2.7). We prove that there is a positive constant K independent of ϵ and x such that

$$|x(t) - y(t)| \le K\phi(t)\epsilon E_\nu(t)$$

for some $y(t)$ satisfying the differential equation (1.1).

Defining a function $p : (0,\infty) \longrightarrow \mathbb{F}$ through $p(t) =: x'(t) + l\ x(t)$ for each $t > 0$, we have $|p(t)| \le \phi(t)\epsilon E_\nu(t)$. Then by using the same technique as in the proof of Theorem 3.3, one can easily obtain

$$|x(t)-y(t)| = |p(t) * q(t)| = \left|\int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds\right| \le |p(t)|\ \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right|.$$

Choose $K = \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right|$. Since the integral exists for each value of t and $\phi(t)$ is an integrable function,

$$|x(t) - y(t)| \le K\phi(t)\epsilon E_\nu(t).$$

Therefore, we get the desired result. □

4. Hyers-Ulam stabilities of the linear differential equation (1.2)

In this section, we study the various types of Hyers-Ulam stability of the differential equation (1.2) by using El-Zaki transform. Firstly, we investigate the Hyers-Ulam stability for (1.2).

Theorem 4.1. *The differential equation* (1.2) *has the Hyers-Ulam stability.*

Proof. Suppose that $x(t)$ is a continuously differentiable function satisfying the inequality (2.2). Consider the function $p : (0, \infty) \longrightarrow \mathbb{F}$ defined by

$$p(t) := x'(t) + l\ x(t) - r(t)$$

for all $t > 0$ that $|p(t)| \leq \epsilon$. Taking El-Zaki transform to $p(t)$, we get

$$\mathcal{E}\{p(t)\} = \mathcal{E}\{x'(t) + l\ x(t) - r(t)\}.$$

In other words, $P(\xi) := \mathcal{E}\{x'(t)\} + l\mathcal{E}\{x(t)\} - \mathcal{E}\{r(t)\} = \frac{X(\xi)}{\xi} - \xi\ x(0) + l\ X(\xi) - R(\xi)$. The last equality implies that

$$\mathcal{E}\{x(t)\} = X(\xi) = \frac{\xi^2\ x(0) + \xi\ P(\xi) + \xi R(\xi)}{1 + l\xi}. \tag{4.1}$$

Put $Q(\xi) = \dfrac{\xi}{(1 + l\xi)}$. Then $\mathcal{E}\{q(t)\} = \dfrac{\xi}{1 + l\xi}$ and hence $q(t) = \mathcal{E}^{-1}\left(\dfrac{\xi}{1 + l\xi}\right)$.

Set $y(t) = e^{-lt}\ x(0) + (r(t) * q(t))$. Once more, by taking El-Zaki transform on both sides of the last equality, we get

$$\mathcal{E}\{y(t)\} = Y(\xi) = \frac{\xi^2\ x(0)}{1 + l\xi} + R(\xi)\ Q(\xi). \tag{4.2}$$

On the other hand,

$$\mathcal{E}\{y'(t) + l\ y(t)\} = \frac{Y(\xi)}{\xi} - \xi\ x(0) + l\ Y(\xi).$$

Then, by using (4.2), we have $\mathcal{E}\{y'(t) + l\ y(t)\} = R(\xi) = \mathcal{E}\{r(t)\}$ and thus $y'(t) + l\ y(t) = r(t)$. Hence $y(t)$ is a solution of the differential equation (1.2). In addition, by applying (4.1) and (4.2) we obtains

$$\mathcal{E}\{x(t)\} - \mathcal{E}\{y(t)\} = X(\xi) - Y(\xi) = \frac{\xi\ P(\xi)}{1 + l\xi} = P(\xi)\ Q(\xi) = \mathcal{E}\{p(t)\}\ \mathcal{E}\{q(t)\}.$$

Therefore, $\mathcal{E}\{x(t)-y(t)\} = \mathcal{E}\{p(t)*q(t)\}$ which implies $x(t)-y(t) = p(t)*q(t)$. Furthermore,

$$\begin{aligned} |x(t) - y(t)| = |p(t) * q(t)| &= \left| \int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds \right| \\ &\leq |p(t)| \left| \int_{-\infty}^{\infty} q(t-s)\ ds \right| \\ &\leq K\epsilon, \end{aligned}$$

where $K = \left| \int_{-\infty}^{\infty} q(t-s)\ ds \right|$ and the integral exists for all values of t. This completes the proof. □

Now, for the non-homogeneous linear differential equation (1.2), we have the following result.

Theorem 4.2. *The differential equation* (1.2) *is generalized Hyers-Ulam stable.*

Proof. We firstly consider a continuously differentiable function $x(t)$ satisfying (2.4). Define the function $p : (0, \infty) \longrightarrow \mathbb{F}$ via $p(t) =: x'(t) + l\ x(t) - r(t)$ for each $t > 0$. Then $|p(t)| \leq \phi(t)\epsilon$. It is not hard to check that

$$\mathcal{E}\{x(t)\} = X(\xi) = \frac{\xi^2\ x(0) + \xi\ P(\xi) + \xi R(\xi)}{1 + l\xi}. \tag{4.3}$$

For $Q(\xi) = \dfrac{\xi}{(1 + l\xi)}$, we have $\mathcal{E}\{q(t)\} = \dfrac{\xi}{1 + l\xi}$ implies that

$$q(t) = \mathcal{E}^{-1} \left(\frac{\xi}{1 + l\xi} \right).$$

Letting $y(t) = e^{-lt}\ x(0) + (r(t) * q(t))$ and taking El-Zaki transform on both sides, we get

$$\mathcal{E}\{y(t)\} = Y(\xi) = \frac{\xi^2\ x(0)}{1 + l\xi} + R(\xi)\ Q(\xi). \tag{4.4}$$

On the other hand, $\mathcal{E}\{y'(t)+l\ y(t)\} = \frac{Y(\xi)}{\xi} - \xi\ x(0) + l\ Y(\xi)$. The relation (4.4) implies that $\mathcal{E}\{y'(t) + l\ y(t)\} = R(\xi) = \mathcal{E}\{r(t)\}$ and thus $y'(t) + l\ y(t) = r(t)$, that is, $y(t)$ is a solution of the differential equation (1.2).

Applying now (4.3) and (4.4), we get

$$\mathcal{E}\{x(t)\} - \mathcal{E}\{y(t)\} = X(\xi) - Y(\xi) = \frac{\xi\ P(\xi)}{1 + l\xi} = P(\xi)\ Q(\xi) = \mathcal{E}\{p(t)\}\ \mathcal{E}\{q(t)\}.$$

$\mathcal{E}\{x(t) - y(t)\} = \mathcal{E}\{p(t) * q(t)\}$ which gives $x(t) - y(t) = p(t) * q(t)$. The rest of the proof is similar to the previous results. □

In the oncoming result, we prove the Mittag-Leffler-Hyers-Ulam stability of the non-homogeneous linear differential equation (1.2) by using El-Zaki transform method.

Theorem 4.3. *The non-homogeneous linear differential equation* (1.2) *has the Mittag-Leffler-Hyers-Ulam stability.*

Proof. Suppose $x(t)$ is a continuously differentiable function satisfying (2.6). Consider the function $p : (0,\infty) \longrightarrow \mathbb{F}$ defined by $p(t) := x'(t) + l\ x(t) - r(t)$ for all $t > 0$ that $|p(t)| \leq \epsilon E_\nu(t)$. Taking El-Zaki transform to $p(t)$, we get

$$\mathcal{E}\{p(t)\} = \mathcal{E}\{x'(t) + l\ x(t) - r(t)\}.$$

In other words,

$$P(\xi) := \mathcal{E}\{x'(t)\} + l\mathcal{E}\{x(t)\} - \mathcal{E}\{r(t)\} = \frac{X(\xi)}{\xi} - \xi\ x(0) + l\ X(\xi) - R(\xi).$$

The last equality implies that

$$\mathcal{E}\{x(t)\} = X(\xi) = \frac{\xi^2\ x(0) + \xi\ P(\xi) + \xi R(\xi)}{1 + l\xi}. \tag{4.5}$$

Put $Q(\xi) = \dfrac{\xi}{(1+l\xi)}$. Then $\mathcal{E}\{q(t)\} = \dfrac{\xi}{1+l\xi}$ and hence $q(t) = \mathcal{E}^{-1}\left(\dfrac{\xi}{1+l\xi}\right)$. Set $y(t) = e^{-lt}\ x(0) + (r(t) * q(t))$. Once more, by taking El-Zaki transform on both sides of the last equality, we get

$$\mathcal{E}\{y(t)\} = Y(\xi) = \frac{\xi^2\ x(0)}{1+l\xi} + R(\xi)\ Q(\xi). \tag{4.6}$$

On the other hand, $\mathcal{E}\{y'(t) + l\ y(t)\} = \dfrac{Y(\xi)}{\xi} - \xi\ x(0) + l\ Y(\xi)$. Then, by (4.6), we have

$$\mathcal{E}\{y'(t) + l\ y(t)\} = R(\xi) = \mathcal{E}\{r(t)\}$$

and thus $y'(t) + l\ y(t) = r(t)$. Hence $y(t)$ is a solution of the differential equation (1.2). In addition, by applying (4.5) and (4.6) we obtains

$$\mathcal{E}\{x(t)\} - \mathcal{E}\{y(t)\} = X(\xi) - Y(\xi) = \frac{\xi\ P(\xi)}{1+l\xi} = P(\xi)\ Q(\xi) = \mathcal{E}\{p(t)\}\ \mathcal{E}\{q(t)\}.$$

Therefore, $\mathcal{E}\{x(t) - y(t)\} = \mathcal{E}\{p(t) * q(t)\}$ which implies that

$$x(t) - y(t) = p(t) * q(t).$$

Furthermore,

$$|x(t) - y(t)| = |p(t) * q(t)| = \left|\int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds\right| \leq |p(t)|\ \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right|.$$

Choose $K = \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right|$. Since the integral exists for all value of t and $|p(t)| \leq \epsilon E_\nu(t)$, we have

$$|x(t) - y(t)| \leq K\epsilon E_\nu(t).$$

This completes the proof. □

By using the same methodology in Theorem 4.3, we have the following generalized Mittag-Leffler-Hyers-Ulam stability for (1.2). We include some parts of the proof for the sake of completeness.

Corollary 4.4. *The non-homogeneous differential equation* (1.2) *has the generalized Mittag-Leffler-Hyers-Ulam stability.*

Proof. Suppose that $x(t)$ is a continuously differentiable function satisfying the inequality (2.8). We prove that there is positive constant K independent of ϵ and x such that

$$|x(t) - y(t)| \leq K\phi(t)\epsilon E_\nu(t)$$

for some $y(t)$ which is a solution of the differential equation (1.2).

Defining a function $p : (0, \infty) \longrightarrow \mathbb{F}$ by $p(t) =: x'(t) + l\ x(t) - r(t)$ for each $t > 0$, we have $|p(t)| \leq \phi(t)\epsilon E_\nu(t)$. Then by using the same technique as in the proof of Theorem 4.3, we can easily get

$$|x(t)-y(t)| = |p(t) * q(t)| = \left|\int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds\right| \leq |p(t)|\ \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right|$$

and choosing $K = \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right|$ for which the integral exists for each value of t and $\phi(t)$ is an integrable function, we get

$$|x(t) - y(t)| \leq K\phi(t)\epsilon E_\nu(t).$$

Therefore, we get our desired result. □

5. Examples

We close this paper by some examples as applications to illustrate the main results.

Example 5.1. Consider the non-homogeneous differential equation:

$$x'(t) + x(t) = 2\cos t, \qquad x(0) = 1. \tag{5.1}$$

Using Theorem 4.1, we have $|x'(t)+x(t)-2\cos t| \leq \epsilon$, where x is a continuously differentiable function. Let $p(t) = x'(t)+x(t)-2\cos t$ for each $t > 0$. We have $|p(t)| \leq \epsilon$. Taking El-Zaki transform to $p(t)$, we get

$$P(\xi) = \frac{X(\xi)}{\xi} - \xi\ x(0)+X(\xi)-\mathcal{E}\{2\cos t\} = \frac{X(\xi)}{\xi} - \xi\ x(0)+X(\xi)-\frac{2\xi^2}{1+\xi^2}.$$

Consequently, $X(\xi) = \dfrac{\xi}{1+\xi}\left(P(\xi) + \xi + \dfrac{2\xi^2}{1+\xi^2}\right)$. Let $Q(\xi) = \dfrac{\xi}{(1+\xi)}$. Then $\mathcal{E}\{q(t)\} = \dfrac{\xi}{(1+\xi)}$. On the other hand, we have a solution function $y(t) = e^{-t}x(0) + [(2\cos t) * q(t)]$ with $x(0) = y(0)$. Once more, by taking El-Zaki transform, we obtain

$$\mathcal{E}\{y(t)\} = Y(\xi) = \frac{\xi^2}{(1+\xi)} + \frac{2\xi^2}{1+\xi^2}Q(\xi).$$

In addition, $\mathcal{E}\{y'(t) + y(t)\} = 2\mathcal{E}\{\cos t\}$. Since $\mathcal{E}$ is one-to-one operator, $y'(t) + y(t) = 2\cos t$ and hence $y(t)$ is a solution of the differential equation (5.1). Now, Theorem 4.1 implies that $|x(t) - y(t)| \le K\epsilon$. Therefore, the non-homogeneous differential equation (5.1) has the Hyers-Ulam stability.

Example 5.2. Let us take the first order differential equation:

$$x'(t) + 3\ x(t) = t \tag{5.2}$$

with initial condition $x(0) = 1$. Applying Theorem 4.1, we have $|x'(t) + 3\ x(t) - t| \le \epsilon$, where x is a continuously differentiable function. Let $p(t) = x'(t) + 3\ x(t) - t$ for each $t > 0$ and $|p(t)| \le \epsilon$. Now, taking El-Zaki transform to $p(t)$, we obtain

$$P(\xi) = \frac{X(\xi)}{\xi} - \xi\ x(0) + 3\ X(\xi) - \xi^3.$$

Thus $X(\xi) = \dfrac{\xi}{1+3\xi}\left(P(\xi) + \xi + \xi^3\right)$. Put $Q(\xi) = \dfrac{\xi}{(1+3\xi)}$. Then $\mathcal{E}\{q(t)\} = \dfrac{\xi}{(1+3\xi)}$. We have a solution function $y(t) = e^{-3t}x(0) + [t * q(t)]$ with $x(0) = y(0)$ and also taking El-Zaki transform, we get

$$\mathcal{E}\{y(t)\} = Y(\xi) = \frac{\xi^2}{(1+3\xi)} + \xi^3\ Q(\xi).$$

Furthermore, $\mathcal{E}\{y'(t)+y(t)\} = \mathcal{E}\{t\}$, which implies that $y'(t)+y(t) = t$. Hence $y(t)$ is a solution of the differential equation (5.2). Then, by Theorem 4.1, we obtain that $|x(t) - y(t)| \le K\epsilon$. Therefore, the differential equation (5.2) is the Hyers-Ulam stable.

Remark 5.3. We note that the above examples are also true when we replace ϵ and $K\epsilon$ with $\phi(t)\epsilon$ and $K\phi(t)\epsilon$, respectively, where $\phi(t)$ does not depend on x and y explicitly. In this case, we see that the corresponding differential equations has the generalized Hyers-Ulam stability.

Remark 5.4. We note that the above examples are also true for Mittag-Leffler function. That is, we can easily obtain that the corresponding differential equations has the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler generalized Hyers-Ulam stability.

6. Conclusion

We have proved the Hyers-Ulam stability, generalized Hyers-Ulam stability, Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler generalized Hyers-Ulam stability of the linear differential equations of first order with constant coefficient using El-Zaki transform method. That is, we established the sufficient criteria for Ulam's stability of the linear differential equation of first order with constant coefficients using El-Zaki transform method. Additionally, this paper also provides another method to study the Hyers-Ulam stability of differential equations. Also, this paper shows that the El-Zaki transform method is more convenient to study the Ulam's stability problem for the linear differential equation with constant coefficient.

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