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ULAM STABILITY OF LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER USING ABOODH TRANSFORM

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Abstract. Using the Aboodh transform, we prove the Hyers-Ulam stability of the nth order linear differential equation

$$x^{(n)}(v) + \sum_{\kappa=0}^{n-1} a_\kappa x^{(\kappa)}(v) = \phi(v).$$

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1. Introduction

The theory of stability is an important branch of the qualitative theory of differential equations. Alsina and Ger [4] investigated the stability of differential equation $x' = x$. They proved the following celebrated theorem.

Theorem 1.1. ([4]) *Let $\epsilon > 0$ and $f : I \to \mathbb{R}$ be a differentiable function, which satisfies the following differential inequality $|x'(t) - x(t)| \le \epsilon$ for all $t \in I$, where I is an open interval of $\mathbb{R}$. Then there is a solution $g : I \to \mathbb{R}$ of $x'(t) = x(t)$ such that for all $t \in I$, we have $|f(t) - g(t)| \le 3\epsilon$.*

This result was generalized by Takahasi *et al.* [21], who proved the Hyers-Ulam stability for the Banach space valued differential equation $y'(t) = \lambda y(t)$. Moreover, the Hyers-Ulam stability was proved for the first order linear differential equations in more general settings [7, 8, 9] and higher orders in [12, 14]. See [5, 6, 11] for more information on the stability of functional equations and applications.

In 2014, Alqifiary and Jung [2] investigated the Hyers-Ulam stability of

$$x^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k \, x^{(k)}(t) \;=\; f(t),$$

by using the Laplace transform method (see also [15, 20]).

In 2019, Murali and Ponmana Selvan [17] investigated the Hyers-Ulam stability of the linear differential equation using Fourier transform method (see also [13]). Recently, Rassias, Murali and Ponmana Selvan [19] established the Mittag-Leffler-Hyers-Ulam stability of the first and second order linear differential equations by applying Fourier transforms.

Recently, the Hyers-Ulam stability of general first order linear differential equations with constant coefficients of homogeneous and non-homogeneous types was proved with the help of Mahgoub integral transform in [10].

Very recently, Murali, Ponmana Selvan, Park and Lee [18] proved different forms of Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam stability of the second order differential equation of the form $u'' + \mu^2 u = q(t)$ by using Aboodh (integral) transform. By applying Aboodh transform, various forms of Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam stability of the first order differential equation were proved in [16].

The Hyers-Ulam stability of differential equations has been given attention and it was established by many authors. Motivated and connected by the above results, our main aim is to more efficiently prove the Hyers-Ulam stability of the higher order linear differential equation. That is, we establish the Hyers-Ulam stability of the higher order linear differential equations of the form

$$x^{(n)}(v) + \sum_{\kappa=0}^{n-1} a_\kappa x^{(\kappa)}(v) = \phi(v) \tag{1.1}$$

for all $v \in I$, $x(v) \in C^n(I)$ and $\phi(v) \in C(I)$, $I = [a, b]$, $-\infty < a < b < \infty$, by using the Aboodh transform method.

2. Preliminaries

In this section, we introduce some standard notations and definitions which will be very useful to obtain our main results.

A function $f : (0, \infty) \to \mathbb{R}$ is said to be of exponential order if there exist constants $A, B \in \mathbb{R}$ such that $|f(t)| \le Ae^{tB}$ for all $t > 0$.

Let f and g be Lebesgue integrable functions on $(-\infty, +\infty)$. Let S denote the set of x for which the Lebesgue integral

$$h(x) = \int_{-\infty}^{\infty} f(t)\; g(x-t)\; dt$$

exists. This integral defines a function h on S called the *convolution* of f and g. We also write $h = f * g$ to denote this function.

Definition 2.1. ([1, 3]) The Aboodh (integral) transform is defined, for a function of exponential order $f(t)$, by

$$\mathcal{A}\{f(t)\} = \frac{1}{v}\int_{0}^{\infty} f(t)\; e^{-vt}\; dt = F(v),$$

provided that the integral exists for some v, where $v \in (k_1, k_2)$. $\mathcal{A}$ is called the Aboodh (integral) transform operator. Here $0 < k_1 < +\infty$ and k_2 may be finite or infinite.

Theorem 2.2. ([1, 3]) *Let $F(v)$ be the Aboodh transform of $f(t)$. Then,*

(i) $\mathcal{A}\{f'(t)\} = v\; F(v) - \dfrac{f(0)}{v}$.

(ii) $\mathcal{A}\{f''(t)\} = v^2\; F(v) - \dfrac{f'(0)}{v} - f(0)$.

(iii) $\mathcal{A}\{f^{(n)}(t)\} = v^n\; F(v) - \sum\limits_{k=0}^{n-1} \dfrac{f^{(k)}(0)}{v^{2-n+k}}$.

Now, we give a definition of Hyers-Ulam stability of the differential equation (1.1).

Definition 2.3. ([17]) We say that the differential equation (1.1) has the Hyers-Ulam stability if there exists a constant $L > 0$ satisfying the following condition: For every $\epsilon > 0$ and an n times continuously differentiable function $x(v)$ satisfying the inequality

$$\left| x^{(n)}(v) + \sum_{\kappa=0}^{n-1} a_\kappa x^{(\kappa)}(v) - \phi(v) \right| \leq \epsilon$$

for all $v \in I$, there exists some $x_a \in C^n(I)$ satisfying

$$x_a^{(n)}(v) + \sum_{\kappa=0}^{n-1} a_\kappa x_a^{(\kappa)}(v) = \phi(v)$$

and $|x(v) - x_a(v)| \leq L\epsilon$ for all $v \in I$. We call such L as the Hyers-Ulam stability constant for (1.1).

3. Hyers-Ulam stability

Lemma 3.1. *Let $P_1(u) = a_0 + a_1 u + a_2 u^2 + \cdots + a_n u^n$ and $P_2(u) = b_0 + b_1 u + b_2 u^2 + \cdots + b_m u^m$ where m and n are nonnegative integers with $m < n$ and a_j, b_j are scalars. Then there exists an infinitely differentiable mapping $\psi : (0, \infty) \to \mathbb{R}$ such that*

$$\mathcal{A}(\psi) = \frac{P_2(u)}{P_1(u)} \qquad (R(u) > \rho_{P_1})$$

and

$$\psi^{(i)}(0) = \begin{cases} 0, & i = 0, 1, \ldots, n - m - 3, \\ \frac{b_m}{a_n}, & i = n - m - 2, \end{cases}$$

where $\rho_{P_1} = \max\{R(u) : P_1(u) = 0\}$. Here $R(u)$ denotes the real part of u.

Proof. Let

$$P_1(u) = a_n (u - u_1)^{n_1} (u - u_2)^{n_2} \cdots (u - u_\kappa)^{n_\kappa},$$

where u_i are complex numbers for $i = 1, 2. \ldots, \kappa$ and n_j is an integer with $n = n_1 + \cdots + n_\kappa$. Then we have

$$\frac{P_2(u)}{P_1(u)} = \sum_{i=1}^{\kappa} \sum_{j=1}^{n_i} \frac{\delta_{ij}}{(u - u_i)^j},$$

where δ_{ij} are scalars. Let

$$\chi_{ij}(v) = \frac{v^{j-2}}{(j-2)!} e^{u_i v},$$

where $i = 1, 2, \dots, \kappa$ and $j = 1, 2, \dots, n_i$. Let

$$\psi(v) = \sum_{i=1}^{\kappa} \sum_{j=1}^{n_i} \delta_{ij} \chi_{ij}(v).$$

Applying the Aboodh transform to $\psi(v)$, we get

$$\mathcal{A}(\psi) = \frac{P_2(u)}{P_1(u)}$$

for all u with $R(u) > \rho$, where $\rho = \max\{R(u_i) : i = 1, 2, \dots, \kappa\}$. By Maclaurin's series, we have

$$\psi(v) = \psi(0) + \psi'(0)v + \cdots + \frac{\psi^{(n-1)}(0)}{(n-1)!} v^{n-1} + \chi(v),$$

where

$$\chi(v) = \sum_{i=n}^{\infty} \frac{\psi^{(i)}(0)}{i!} v^i.$$

Note that $\mathcal{A}(\chi) = \dfrac{\Omega(u)}{u^{n+2}}$, where Ω is a complex function. Then

$$\mathcal{A}(\psi) = \frac{\psi(0)}{u^2} + \frac{\psi'(0)}{u^3} + \frac{\psi''(0)}{u^4} + \cdots + \frac{\psi^{(n-1)(0)}}{\psi^{n+1}} + \frac{\Omega(u)}{u^{n+2}}.$$

Hence,

$$\frac{\psi(0)}{u^2} + \frac{\psi'(0)}{u^3} + \frac{\psi''(0)}{u^4} + \cdots + \frac{\psi^{(n-1)(0)}}{\psi^{n+1}} + \frac{\Omega(u)}{u^{n+2}} = \frac{b_0 + b_1 u + \cdots + b_m u^m}{a_0 + a_1 u + \cdots + a_{m+\tau} u^{m+\tau}},$$

where $\tau = n - m$. Assume that $\tau \geq 3$. Multiplying both sides of the above equation by $u^2, u^3, \cdots, u^\tau$ and taking $u \to \infty$, we get

$$\psi(0) = \psi'(0) = \cdots = \psi^{(\tau-3)}(0) = 0 \text{ and } \psi^{\tau-2}(0) = \frac{b_m}{a_n}.$$

This completes the proof. □

Lemma 3.2. *Let $n > 1$ be an integer and $\phi : (0, \infty) \to \mathbb{R}$ be a continuous function and $P_1(u)$ be a complex polynomial of degree n. Then there exists an n times continuously differentiable function $\chi : (0, \infty) \to \mathbb{R}$ such that*

$$\mathcal{A}(\chi) = \frac{\mathcal{A}(\phi)}{P_1(u)} \quad (R(u) > \max\{\rho_{P_1}, \rho_j\}),$$

where $\rho_{P_1} = \max\{R(u) : P_1(u) = 0\}$ and ρ_j is the abscissa of convergence for ϕ. In particular, it holds that $\chi^{(i)}(0) = 0$ for all $i = 0, 1, 2, \dots, n-2$.

Proof. Let $P_2(u) = \dfrac{1}{u}$ and $P_1(u) = a_0 + a_1 u + \cdots + a_n u^n$. Then there exists an infinitely differentiable function $\psi : (0,\infty) \to \mathbb{R}$ such that

$$\mathcal{A}(\psi) = \frac{\frac{1}{u}}{P_1(u)} \quad (R(u) > \rho_{P_1})$$

and $\psi^{(i)}(0) = 0$ if $i = 0, 1, 2, \ldots, n-3$ and $\psi^{(n-2)}(0) = \frac{1}{a_n}$. Now we define $\chi = \psi * \phi$. Then we obtain $\mathcal{A}(\chi) = \frac{\mathcal{A}(\phi)}{P_1(u)}$ and

$$\chi'(t) = \psi(0)\phi(v) + \int_0^v \psi'(v-\nu)\phi(\nu)d\nu = \int_0^v \psi'(v-\nu)\phi(\nu)d(\nu),$$

$$\chi^{(i)}(v) = \psi^{(i-1)}(0)\phi(v) + \int_0^v \psi^{(i)}(v-\nu)\phi(\nu)d\nu = \int_0^v \psi^{(i)}(v-\nu)\phi(\nu)d\nu$$

for all $i = 1, 2, \ldots, n-2$ with $\chi(0) = \chi'(0) = \cdots = \chi^{(n-2)} = 0$. □

Theorem 3.3. *Let a be a scalar. If a function $x : (0,\infty) \to \mathbb{R}$ satisfies*

$$|x'(v) + ax(v) - \phi(v)| \le \epsilon \tag{3.1}$$

for all $v > 0$ and for each $\epsilon > 0$, then there exists a solution $x_a : (0,\infty) \to \mathbb{R}$ of the differential equation

$$x'(v) + ax(v) = \phi(v) \tag{3.2}$$

such that

$$|x_a(v) - x(v)| \le \begin{cases} \epsilon v, & R(a) = 0 \\ \frac{(1-e^{-R(a)v})\epsilon}{R(a)}, & R(a) \ne 0 \end{cases}$$

for all $v > 0$.

Proof. Let $\zeta(v) = x'(v) + ax(v) - \phi(v)$ for all $v > 0$. Then we get

$$\mathcal{A}(\zeta) = u\mathcal{A}(x) - \frac{x(0)}{u} + a\mathcal{A}(x) - \mathcal{A}(\phi)$$

and so

$$\mathcal{A}(x) - \frac{\frac{x(0)}{u} + \mathcal{A}(\phi)}{u+a} = \frac{\mathcal{A}(\zeta)}{u+a}. \tag{3.3}$$

Let $x_a(v) = x(0)e^{-av} + (E_{-a} * \phi)v$, where $E_{-a}(v) = e^{-av}$. Then $x_a(0) = x(0)$ and

$$\mathcal{A}(x_a) = \frac{\frac{x(0)}{u} + \mathcal{A}(\phi)}{u+a} = \frac{\frac{x_a(0)}{u} + \mathcal{A}(\phi)}{u+a}, \tag{3.4}$$

$$\mathcal{A}[x_a'(v) + ax_a(v)] = u\mathcal{A}(x_a) - \frac{x_a(0)}{u} + a\mathcal{A}(x_a) = \mathcal{A}(\phi).$$

Since $\mathcal{A}$ is injective, $x_a'(v)+ax_a(v)=\phi(v)$. Therefore, x_a is a solution of (3.2). Using (3.3) and (3.4), we have

$$\mathcal{A}[E_{-a}*\zeta]=\frac{\mathcal{A}(\zeta)}{u+a}.$$

We obtain $\mathcal{A}(x)-\mathcal{A}(x_a)=\mathcal{A}[E_{-a}*\zeta]$ and $x(v)-x_a(v)=(E_{-a}*\zeta)(v)$. By (3.1), we get $|\zeta(v)|\le\epsilon$ and by convolution, we obtain

$$|x(v)-x_a(v)|=|(E_{-a}*\zeta)(v)|\le\epsilon e^{-R(a)}v\int_0^v e^{R(a)\nu}d\nu.$$

This completes the proof. □

Theorem 3.4. *Let $a_0,a_1,\ldots,a_{n-1}$ be scalars, where $n>1$ is an integer. Then there exists a constant $N>0$ such that for each function $x:(0,\infty)\to\mathbb{R}$ satisfying*

$$\left|x^{(n)}(v)+\sum_{\kappa=0}^{n-1}a_\kappa x^{(\kappa)}(v)-\phi(v)\right|\le\epsilon \tag{3.5}$$

for all $v>0$ and for each $\epsilon>0$, there exists a solution $x_a:(0,\infty)\to\mathbb{R}$ of the differential equation

$$x^{(n)}(v)+\sum_{\kappa=0}^{n-1}a_\kappa x^{(\kappa)}(v)=\phi(v) \tag{3.6}$$

such that

$$|x_a(v)-x(v)|\le\epsilon N\frac{e^{av}}{a}$$

for all $v>0$ and $a>\max\{0,\rho,\rho_j\}$, where ρ_j was defined in Lemma 3.2 and $\rho=\max\{R(u_\kappa):\quad \kappa=1,2,\ldots,n\}$.

Proof. Using integration by parts repeatedly, we get

$$\mathcal{A}[x^{(n)}]=u^n\mathcal{A}(x)-\sum_{j=0}^{n-1}u^{n-2-j}x^{(j)}(0)$$

for any integer $n>0$. Let $a_n=1$. So x_0 is a solution of (3.6) if and only if

$$\mathcal{A}(\phi)=(a_0+\eta_{n,0}(u))\,\mathcal{A}(x_0)-\sum_{j=0}^{n-1}\eta_{n,j}(u)\frac{x_0^{(j)}(0)}{u^2}, \tag{3.7}$$

where $\eta_{n,j}(u)=\sum\limits_{\kappa=j+1}^{n}a_\kappa u^{\kappa-j}$ for $j=0,1,2,\ldots,n-1$. We consider

$$\chi(v)=x^n(v)+\sum_{\kappa=0}^{n-1}a_\kappa x^{(\kappa)}(v)-\phi(v)$$

for all $v > 0$. Then

$$\mathcal{A}(\chi) = (a_0 + \eta_{n,0}(u))\,\mathcal{A}(x) - \sum_{j=0}^{n-1} \eta_{n,j}(u)\frac{x^{(j)}(0)}{u^2} - \mathcal{A}(\phi).$$

Hence, we get

$$\mathcal{A}(x) - \frac{1}{a_0 + \eta_{n,0}(u)}\left[\sum_{j=0}^{n-1} \eta_{n,j}(u)\frac{x^{(j)}(0)}{u^2} + \mathcal{A}(\phi)\right] = \frac{\mathcal{A}(\chi)}{a_0 + \eta_{n,0}(u)}. \tag{3.8}$$

Let ρ_j be the abscissa of convergence for ϕ. Let $u_1, u_2, \ldots, u_n$ be the roots of the polynomial $\eta_{n,0}$ and let

$$\rho = \max\{R(u_\kappa) : \quad \kappa = 1, 2, \ldots, n\}. \tag{3.9}$$

For all u with $R(u) > \max\{\rho, \rho_j\}$, we define

$$\Delta(u) = \frac{1}{a_0 + \eta_{n,0}(u)}\left[\sum_{j=0}^{n-1} \eta_{n,j}(u)\frac{x^{(j)}(0)}{u^2} + \mathcal{A}(\phi)\right]. \tag{3.10}$$

By Lemma 3.2,

$$\mathcal{A}(\phi_0) = \frac{\mathcal{A}(\phi)}{a_0 + \eta_{n,0}(s)} \tag{3.11}$$

for all u with $R(u) > \max\{\rho, \rho_j\}$ and

$$\phi_0(0) = \phi_0'(0) = \cdots = \phi_0^{(n-2)}(0) = 0$$

for $j = 1, 2, \ldots, n-1$. So

$$\frac{\eta_{n,j}(u)}{a_0 + \eta_{n,0}(u)} = \frac{1}{u^j} - \frac{\sum_{\kappa=0}^{j} a_\kappa u^\kappa}{u^j\,(a_0 + \eta_{n,o}(u))} \tag{3.12}$$

for all u with $R(u) > \max\{0, \rho\}$. By Lemma 3.1,

$$P_2(u) = \sum_{\kappa=0}^{j} a_\kappa u^\kappa$$

and

$$p_1(u) = u^j\,(a_0 + \eta_{n,0}(u))\,.$$

For differentiable function ψ_j, we have

$$\mathcal{A}(\psi_j) = \frac{\sum\limits_{\kappa=0}^{j} a_\kappa u^\kappa}{u^j\,(a_0 + \eta_{n,o}(u))} \tag{3.13}$$

and $\psi_j(0) = \psi_j'(0) = \cdots = \psi_j^{(n-2)} = 0$. Let

$$\phi_j(v) = \frac{v^j}{(j)!} - \frac{\psi_j(v)}{u^2} \tag{3.14}$$

for $j = 1, 2, \ldots, n-1$. Then we get

$$\phi_j^{(i)}(0) = \begin{cases} 0, & i = 0, 1, 2, \ldots, j-2, j, j+1, \ldots, n-2, \\ 1, & i = j-1. \end{cases}$$

If we define

$$x_a(v) = \sum_{j=0}^{n-1} x^j(0)\phi_j(v) + \phi_0(v),$$

then we get $x_a^{(i)}(0) = x^{(i)}(0)$, for all $i = 0, 1, 2, \ldots, n-2$.

Using (3.10), (3.11), (3.12), (3.13), (3.14) and $\mathcal{A}(x_a) = \Delta(u)$, we get

$$\mathcal{A}(x_a) = \frac{1}{a_0 + \eta_{n,0}(u)} \left[\sum_{j=0}^{n-1} \eta_{n,j}(u) \frac{x_a^{(j)}(0)}{u^2} + \mathcal{A}(\psi) \right] \tag{3.15}$$

for all u with $R(u) > \max\{0, \rho, \rho_j\}$.

Now using (3.7) we get x_0 is a solution of (3.6). Again by (3.8) and (3.15), we get

$$\mathcal{A}(x) - \mathcal{A}(x_a) = \frac{\mathcal{A}(\chi)}{a_0 + \eta_{n,0}(u)}$$

and so

$$|x(v) - x_a(v)| = \left| \mathcal{A}^{-1} \left(\frac{\mathcal{A}(\chi)}{a_0 + \eta_{n,0}(u)} \right) \right|$$

for $v > 0$. By the definition of χ and (3.5), it holds that $|\chi(v)| \le \epsilon$ for all $v > 0$ and so

$$|\mathcal{A}(\chi)| \le \int_0^\infty \left| e^{-uv} \right| |\chi(v)| dv \le \frac{\epsilon}{R(u)} \tag{3.16}$$

for all u with $R(u) > 0$.

Finally, it follows from the formula for the inverse Aboodh transform that

$$\begin{aligned}|x(v)-x_a(v)| &= \left|\mathcal{A}^{-1}\left(\frac{\mathcal{A}(\chi)}{a_0+\eta_{n,0}(u)}\right)\right| \\ &= \frac{1}{2\pi}\left|\int_{-\infty}^{\infty}\frac{(a+ix)e^{(a+ix)t}\mathcal{A}(\chi)(a+ix)}{a_0+\eta_{n,0}(a+ix)}dx\right| \\ &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{av}\frac{\epsilon}{a}\frac{|a+ix|}{|a_0+\eta_{n,0}(a+ix)|}dx \\ &\le \frac{\epsilon e^{av}}{2\pi a}\int_{-\infty}^{\infty}\frac{|a+ix|}{|a_0+\eta_{n,0}(a+ix)|}dx \\ &\le \epsilon N\frac{e^{av}}{a}\end{aligned}$$

for all $v>0$ and any $a>\max\{0,\rho,\rho_j\}$, where

$$N=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|a+ix|}{|a_0+\eta_{n,0}(a+ix)|}dx<\infty$$

since $n>1$ is an integer. □

4. Conclusion

Using the Aboodh transform, we proved the Hyers-Ulam stability of the nth order linear differential equation

$$x^{(n)}(v)+\sum_{\kappa=0}^{n-1}a_\kappa x^{(\kappa)}(v)=\phi(v).$$

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