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A STRONGLY COUPLED SUB-LAPLACIAN SYSTEM ON THE HEISENBERG GROUP $\mathbb{H}_1$

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Abstract. In this work, we study the following sub-elliptic system involving strongly coupled critical terms and concave nonlinearities:

$$\begin{cases} -\Delta_{\mathbb{H}_1} u = \dfrac{\eta_1\alpha_1}{2^*}|u|^{\alpha_1-2}|v|^{\beta_1}u + \dfrac{\eta_2\alpha_2}{2^*}|u|^{\alpha_2-2}|v|^{\beta_2}u + \lambda g(z)|u|^{q-2}u, & z\in\Omega, \\ -\Delta_{\mathbb{H}_1} v = \dfrac{\eta_1\beta_1}{2^*}|u|^{\alpha_1}|v|^{\beta_1-2}v + \dfrac{\eta_2\beta_2}{2^*}|u|^{\alpha_2}|v|^{\beta_2-2}v + \mu h(z)|v|^{q-2}v, & z\in\Omega, \\ u=v=0, & z\in\partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{H}_1$ with smooth boundary, $-\Delta_{\mathbb{H}_1}$ is the sub-Laplacian on Heisenberg group $\mathbb{H}_1, \eta_1, \eta_2, \lambda, \mu$, are positive, $\alpha_1+\beta_1=2^*,\ \alpha_2+\beta_2=2^*,\ 1<q<2,\ 2^*=\frac{2Q}{Q-2}$ is the critical Sobolev exponent on the Heisenberg group with $Q=4$ the homogeneous dimension of $\mathbb{H}_1$. By exploiting the Nehari manifold and variational methods, we prove that the system has at least two positive solutions.

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1. Introduction

In this paper, we are concerned with the sub-Laplacian system involving strongly coupled critical terms and concave nonlinearities on the Heisenberg group $\mathbb{H}_1$ given below

$$\begin{cases} -\Delta_{\mathbb{H}_1} u = \dfrac{\eta_1\alpha_1}{4}|u|^{\alpha_1-2}|v|^{\beta_1}u + \dfrac{\eta_2\alpha_2}{4}|u|^{\alpha_2-2}|v|^{\beta_2}u + \lambda g(z)|u|^{q-2}u, & z \in \Omega, \\ -\Delta_{\mathbb{H}_1} v = \dfrac{\eta_1\beta_1}{4}|u|^{\alpha_1}|v|^{\beta_1-2}v + \dfrac{\eta_2\beta_2}{4}|u|^{\alpha_2}|v|^{\beta_2-2}v + \mu h(z)|v|^{q-2}v, & z \in \Omega, \\ u = v = 0, \quad z \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{H}_1$ with smooth boundary, $-\Delta_{\mathbb{H}_1}$ is the sub-Laplacian on $\mathbb{H}_1$. λ, μ, are positive, $2^* = 4$ is the critical Sobolev exponent, and $Q = 4$ is the homogeneous dimension of $\mathbb{H}_1$ and $g, h : \Omega \longrightarrow \mathbb{R}$ are positive continuous functions.

We consider the following conditions:

- $(\mathcal{A}_0)$ $1 < q < 2,\ 0 < \eta_i < \infty, \alpha_i, \beta_i > 1$ and $\alpha_i + \beta_i = 2^*(i = 1, 2)$,
- $(\mathcal{A}_1)$ $g, h \in L^{\frac{4}{4-q}}(\Omega)$,
- $(\mathcal{A}_2)$ there exist $a_0, r_0 > 0$ such that $B_d(0, r_0) \subset \Omega$ and $g(z),\ h(z) \geq a_0$ for all $z \in B_d(0, r_0)$,

where $B_d(z, r)$ denotes the quasi-ball with center at z and radius r with respect to the gauge d. $|u|^{\alpha_i-2}u|v|^{\beta_i}$ and $|u|^{\alpha_i}|v|^{\beta_i-2}v,\ i = 1, 2$ are called strongly coupled terms.

We now recall some known results concerning the elliptic system involving the strongly coupled critical terms. In the case of Euclidean space $(\mathbb{R}^n, +)$, $\eta_1 = \eta_2 = 1$, $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $g = h \equiv 1$, problem (1.1) becomes the following Laplacian elliptic system:

$$\begin{cases} -\Delta u = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}|v|^{\beta}u + \lambda|u|^{q-2}u & \text{in } \Omega, \\ -\Delta v = \frac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v + \mu|v|^{q-2}v & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The authors in [11] proved that the system (1.2) admits at least two positive solutions. Later, Hsu [10] obtained the same results for the p-Laplacian elliptic system. There are other multiplicity results or critical elliptic equations involving concave convex nonlinearities, see for example [1, 2]. Systems similar to (1.1) have been the subject of works [8, 21], where the fibering and Nehari manifold methods are applicable to obtain two positive solutions for

$$\begin{cases} \mathcal{L}u = \frac{\eta_1\alpha_1}{2^*}|u|^{\alpha_1-2}|v|^{\beta_1}u + \frac{\eta_2\alpha_2}{2^*}|u|^{\alpha_2-2}|v|^{\beta_2}u + \lambda\frac{|u|^{q-2}u}{|x|^\gamma}, & x \in \Omega, \\ \mathcal{L}v = \frac{\eta_1\beta_1}{2^*}|u|^{\alpha_1}|v|^{\beta_1-2}v + \frac{\eta_2\beta_2}{2^*}|u|^{\alpha_2}|v|^{\beta_2-2}v + \mu\frac{|v|^{q-2}v}{|x|^\gamma}, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where, $\mathcal{L} = -\Delta$ or $\mathcal{L}$ is $(-\Delta)^s$, the spectral fractional Laplacian operator. Contrary to the nonlinear elliptic problems in Euclidean space that have been widely investigated, the situation seems to be in a developing state for the sub-Laplacian problem on Heisenberg groups. Recently, great attention has been devoted to nonlinear elliptic problems involving critical nonlinearities, in the context of Heisenberg group, see for example [7, 14, 15] and references therein.

We look for weak solutions of (1.1) in the product space $\mathcal{H} := S_0^1(\Omega)\times S_0^1(\Omega)$, endowed with the norm

$$\|(u,v)\|_{\mathcal{H}} = \left(\|u\|^2_{S_0^1(\Omega)} + \|v\|^2_{S_0^1(\Omega)}\right)^{\frac{1}{2}} \quad \text{for} \quad (u,v) \in \mathcal{H},$$

where the Folland-Stein space $S_0^1(\Omega) = \{u \in L^4(\Omega) : \int_\Omega |\nabla_{\mathbb{H}_1} u|^2 dz < \infty\}$, is defined in [9] as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{S_0^1(\Omega)} = \left(\int_\Omega |\nabla_{\mathbb{H}_1} u|^2 dz\right)^{\frac{1}{2}}, \ \ \forall\ u \in S_0^1(\Omega).$$

By using the Nehari manifold and fibering map analysis, we establish the existence of at least two positive solutions for a sub-elliptic system (1.1)when (λ,μ) belongs to certain subset of $\mathbb{R}^2_+$. Since the embedding $S_0^1(\Omega) \hookrightarrow L^4(\Omega)$ is not compact, then the corresponding energy functional does not satisfy the Palais-Smale condition in general. Therefore, it is difficult to obtain the critical points of energy functional by simple arguments, which are based on the compactness of the Sobolev embedding. To overcome this difficulty, we extract a Palais-Smale sequence in the Nehari manifold and show that the weak limit of this sequence is the required solution of problem (1.1).

We consider the following scalar critical equation:

$$-\Delta_{\mathbb{H}_1} u = |u|^2 u \quad \text{in} \quad \mathbb{H}_1. \tag{1.4}$$

For equation (1.4), it is well known (see e.g. [4, 12]) that positive solutions have the following decay:

$$U(z) \sim \frac{C}{d(z)^2} \text{ as } d(z) \to \infty, \tag{1.5}$$

where d is the gauge norm on $\mathbb{H}_1$. This result applies, in particular, to the extremals of the Sobolev inequality on Heisenberg groups (whose existence was proved in [17], that is, to the functions U that achieve the best constant for the embedding $S_0^1(\mathbb{H}_1) \hookrightarrow L^4(\mathbb{H}_1)$, that is,

$$S_{\mathbb{H}_1} := \inf_{u \in S_0^1(\mathbb{H}_1)\backslash\{0\}} \frac{\int_{\mathbb{H}_1} |\nabla_{\mathbb{H}_1} u|^2 dz}{\left(\int_{\mathbb{H}_1} |u|^4 dz\right)^{\frac{1}{2}}} = \frac{\int_{\mathbb{H}_1} |\nabla_{\mathbb{H}_1} U|^2 dz}{\left(\int_{\mathbb{H}_1} |U|^4 dz\right)^{\frac{1}{2}}}.$$

We underline that the knowledge of the exact asymptotic behavior of Sobolev minimizers turns out to be a crucial ingredient in order to obtain existence results for Brezis-Nirenberg type problems, whenever the explicit form of Sobolev minimizers is not known, as in the present Heisenberg case. The knowledge of the behavior of Sobolev minimizers turn out to be crucial also for the system, due to the relation between the extremals for the best constant $S_{\eta,\alpha,\beta}$ associated to the system and the Sobolev constant $S_{\mathbb{H}_1}$ (see Theorem 2.1 below).

The energy functional $I_{\eta,\alpha,\beta} : \mathcal{H} \longrightarrow \mathbb{R}$ associated to (1.1) is given by

$$I_{\eta,\alpha,\beta}(u,v) = \frac{1}{2}\|(u,v)\|^2_{\mathcal{H}} - \frac{1}{4}K_\eta(u,v) - \frac{1}{q}\Psi_{\lambda,\mu}(u,v), \quad \forall (u,v) \in \mathcal{H},$$

where

$$K_\eta(u,v) = \int_\Omega \left(\eta_1|u|^{\alpha_1}|v|^{\beta_1} + \eta_2|u|^{\alpha_2}|v|^{\beta_2}\right) dz,$$
$$\Psi_{\lambda,\mu}(u,v) = \int_\Omega \left(\lambda g(z)|u|^q + \mu h(z)|v|^q\right) dz.$$

It is easy to check that $I_{\eta,\alpha,\beta} \in C^1(\mathcal{H}, \mathbb{R})$, and the critical point of $I_{\eta,\alpha,\beta}$ is the weak solution of (1.1). We call a solution (u,v) positive if both u and v are positive, (u,v) is nontrivial if $u \not\equiv 0$ or $v \not\equiv 0$.

Definition 1.1. A pair of functions $(u,v) \in \mathcal{H}$ is said to be a weak solution of problem (1.1) if

$$\begin{aligned}\int_\Omega (\nabla u \nabla \phi + \nabla v \nabla \psi)\ \mathrm{d}x &= \int_\Omega \Big(\frac{\eta_1\alpha_1}{4}|u|^{\alpha_1-2}|v|^{\beta_1}u\phi + \frac{\eta_2\alpha_2}{4}|u|^{\alpha_2-2}|v|^{\beta_2}u\phi\Big)dx \\ &\quad + \int_\Omega \Big(\frac{\eta_1\beta_1}{4}|u|^{\alpha_1}|v|^{\beta_1-2}v\psi + \frac{\eta_2\beta_2}{4}|u|^{\alpha_2}|v|^{\beta_2-2}v\psi\Big)dx \\ &\quad + \int_\Omega \left(\lambda f(x)|u|^{q-2}u\phi + \mu g(x)|v|^{q-2}v\psi\right) dx\end{aligned} \tag{1.6}$$

for all $(\phi,\psi) \in \mathcal{H}$.

Define the set

$$\mathfrak{D}_\sigma := \{(\lambda,\mu) \in \mathbb{R}^+ \times \mathbb{R}^+\backslash\{(0,0)\} : 0 < \lambda\|g\|_{L^{\frac{4}{4-q}}} + \mu\|h\|_{L^{\frac{4}{4-q}}} < \sigma\}.$$

Let

$$\Lambda := \frac{2}{4-q}\left(\frac{2-q}{(\eta_1+\eta_2)(4-q)}\right)^{\frac{2-q}{2}} S_{\mathbb{H}_1}^{\frac{4-q}{2}}. \tag{1.7}$$

The main result of this paper can be included in the following theorem.

Theorem 1.2. *Assume that* $(\mathcal{A}_0)$, $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$ *hold. Then, we have the following results:*

(i) *If* $(\lambda,\mu)\in\mathfrak{D}_\Lambda$, *then* (1.1) *has at least one positive solution in* $\mathcal{H}$.

(ii) *There exists a constant* $\Lambda_*>0$ *such that the system* (1.1) *has at least two distinct positive solutions in* $\mathcal{H}$ *for all* $(\lambda,\mu)\in\mathfrak{D}_{\Lambda_*}$.

The paper is organized as follows. In Section 2, we recall some basic definitions of Heisenberg groups and we give some useful auxiliary lemmas. In Section 3, we investigate the Palais-Smale condition for the energy functional $I_{\eta,\alpha,\beta}$. Finally, the proof of Theorem 1.2 is given in Sections 4 and 5.

2. The Heisenberg group and preliminaries lemmas

Let us recall some briefs on the Heisenberg group (see [3]). The Heisenberg group $\mathbb{H}_1=(\mathbb{R}^3,\circ)$ is the space $\mathbb{R}^3$ with the noncommutative law of product

$$(x,y,t)\circ(x',y',t')=(x+x',y+y',t+t'+2(yx'-xy')),$$

where $x,x',y,y',t,t'\in\mathbb{R}$. This operation endows $\mathbb{H}_1$ with the structure of a Lie group. The Lie algebra of $\mathbb{H}_1$ is generated by the left-invariant vector fields

$$T=\frac{\partial}{\partial t},\quad X=\frac{\partial}{\partial x}+2y\frac{\partial}{\partial t},\quad Y=\frac{\partial}{\partial y}-2x\frac{\partial}{\partial t}.$$

These generators satisfy the noncommutative formula

$$[X,Y]=-4T,\quad [X,X]=[Y,Y]=[X,T]=[Y,T]=0.$$

Let $\xi=(x,y)\in\mathbb{R}^2$ and $z=(\xi,t)\in\mathbb{H}_1$. The parabolic dilation

$$\delta_\lambda z=(\lambda x,\lambda y,\lambda^2 t)$$

satisfies

$$\delta_\lambda(z_0\circ z)=\delta_\lambda z_0\circ\delta_\lambda z$$

and

$$|z|_{\mathbb{H}_1}=\left(|\xi|^4+t^2\right)^{1/4}=\left(\left(x^2+y^2\right)^2+t^2\right)^{1/4},$$

is a norm with respect to the parabolic dilation which is known as Kornyi gauge norm $d(z)$. The Heisenberg distance between two points (ξ,t) and (ξ',t') is given by

$$\rho(\xi,t;\xi',t')=\left|(\xi',t')^{-1}\circ(\xi,t)\right|_{\mathbb{H}_1}.$$

Clearly, the vector fields X, Y are homogeneous of degree 1 under the norm $|.|_{\mathbb{H}_1}$ and T is homogeneous of degree 2. The Kornyi ball of centre z_0 and radius r is defined by

$$B_d(z_0, r) = \left\{z : \left|z^{-1} \circ z_0\right|_d \leq r\right\}$$

and it satisfies

$$|B_d(z_0, r)| = |B_d(0, r)| = r^4 |B_d(0, 1)| .$$

The Heisenberg gradient and the Kohn-Laplacian (the Heisenberg Laplacian) operator on $\mathbb{H}_1$ are given by

$$\nabla_{\mathbb{H}_1} u = (Xu)X + (Yu)Y$$

and

$$\Delta_{\mathbb{H}_1} = X^2 + Y^2,$$

respectively.

We will give some results which will be used to prove the existence in multiple critical case. Let U be a fixed positive minimizer for the best constant $S_{\mathbb{H}_1}$ and define the family

$$U_\varepsilon(z) = \varepsilon^{-1} U\left(\delta_{\frac{1}{\varepsilon}}(z)\right), \quad \forall\, \varepsilon > 0. \tag{2.1}$$

The functions U_ε are also minimizers for $S_{\mathbb{H}_1}$ and, up to a normalization, they satisfy

$$\int_{\mathbb{H}_1} |\nabla_{\mathbb{H}_1} U_\varepsilon|^2 dz = \int_{\mathbb{H}_1} |U_\varepsilon(z)|^4 dz = S_{\mathbb{H}_1}^2, \quad \forall\, \varepsilon > 0.$$

For any $0 < \eta_i < \infty$ $(i = 1, 2), \alpha_i, \beta_i > 1$ with $\alpha_i + \beta_i = 2^*$, by the Young inequality, the following best Sobolev-type constants are well defined and crucial for the study of (1.1):

$$\begin{aligned} S_{\eta,\alpha,\beta} :&= \inf_{(u,v)\in\mathcal{H}^2\setminus\{(0,0)\}} \frac{\int_{\mathbb{H}_1}\left(|\nabla_{\mathbb{H}_1} u|^2 + |\nabla_{\mathbb{H}_1} v|^2\right) dz}{\left(\int_{\mathbb{H}_1}\left(\eta_1 |u|^{\alpha_1}|v|^{\beta_1} + \eta_2 |u|^{\alpha_2}|v|^{\beta_2}\right) dx\right)^{1/2}} \\ &= \inf_{(u,v)\in\mathcal{H}^2\setminus\{(0,0)\}} \|(u,v)\|^2 \left(\int_{\mathbb{H}_1}\left(\eta_1 |u|^{\alpha_1}|v|^{\beta_1} + \eta_2 |u|^{\alpha_2}|v|^{\beta_2}\right) dx\right)^{-1/2}. \end{aligned} \tag{2.2}$$

For any $t \geq 0$, we define the function

$$\mathfrak{h}(t) := \frac{1 + t^2}{\left(\eta_1 t^{\beta_1} + \eta_2 t^{\beta_2}\right)^{\frac{1}{2}}}. \tag{2.3}$$

Since $\mathfrak{h}$ is continuous on $(0,\infty)$ such that $\lim_{t\to 0^+}\mathfrak{h}(t)=\lim_{t\to+\infty}\mathfrak{h}(t)=+\infty$, there exists $t_0>0$ a minimal point of function $\mathfrak{h}$, that is,

$$\mathfrak{h}\left(t_0\right)=\min_{t\geq 0}\mathfrak{h}(t)>0. \tag{2.4}$$

Summarizing, we have the following relationship between $S_{\mathbb{H}_1}$ and $S_{\eta,\alpha,\beta}$.

Theorem 2.1. *Assume that* $(\mathcal{A}_0)$ *hold. Then*

(i) $S_{\eta,\alpha,\beta}=\mathfrak{h}\left(t_0\right)S_{\mathbb{H}_1}$*;*

(ii) $S_{\eta,\alpha,\beta}$ *has the minimizers* $\left(U_\varepsilon(z),t_0U_\varepsilon(z)\right),\varepsilon>0$*, where* $U_\varepsilon(z)$ *are defined as in* (2.1).

Proof. Suppose $\kappa\in S_0^1(\mathbb{H}_1)$. Choosing $(u,v)=(\kappa,t_0\kappa)$ in (2.2), we have

$$\frac{1+t_0^2}{\left(\eta_1t_0^{\beta_1}+\eta_2t_0^{\beta_2}\right)^{\frac{1}{2}}}\frac{\int_{\mathbb{H}_1}|\nabla_{\mathbb{H}_1}\kappa|^2dz}{\left(\int_{\mathbb{H}_1}|\kappa|^4dz\right)^{1/2}}\geq S_{\eta,\alpha,\beta}. \tag{2.5}$$

Taking the infimum as $\kappa\in S_0^1(\mathbb{H}_1)$ in (2.5), we have

$$\mathfrak{h}\left(t_0\right)S_{\mathbb{H}_1}\geq S_{\eta,\alpha,\beta}. \tag{2.6}$$

Let $\{(u_n,v_n)\}\subset\mathcal{H}$ be a minimizing sequence of $S_{\eta,\alpha,\beta}$ and define $w_n=s_nv_n$, where

$$s_n:=\left(\left(\int_{\mathbb{H}_1}|v_n|^4\,\mathrm{d}z\right)^{-1}\int_{\mathbb{H}_1}|u_n|^4\,\mathrm{d}z\right)^{\frac{1}{4}}.$$

Then

$$\int_{\mathbb{H}_1}|w_n|^4\,\mathrm{d}z=\int_{\mathbb{H}_1}|u_n|^4\,\mathrm{d}z. \tag{2.7}$$

From the Young inequality and (2.6) it follows that

$$\begin{aligned}\int_{\mathbb{H}_1}|u_n|^{\alpha_i}|w_n|^{\beta_i}\,\mathrm{d}z&\leq\frac{\alpha_i}{4}\int_{\mathbb{H}_1}|u_n|^4\,\mathrm{d}z+\frac{\beta_i}{4}\int_{\mathbb{H}_1}|w_n|^4\,\mathrm{d}z\\&=\int_{\mathbb{H}_1}|u_n|^4\,\mathrm{d}z=\int_{\mathbb{H}_1}|w_n|^4\,\mathrm{d}z,\quad i=1,2.\end{aligned} \tag{2.8}$$

Consequently,

$$\begin{aligned}
&\frac{\|(u_n,v_n)\|^2}{\left(\int_{\mathbb{H}_1}\left(\eta_1|u_n|^{\alpha_1}|v_n|^{\beta_1}+\eta_2|u_n|^{\alpha_2}|v_n|^{\beta_2}\right)dx\right)^{1/2}}\\
&\quad\geq \frac{\int_{\mathbb{H}_1}|\nabla_{\mathbb{H}_1}u_n|^2dz}{\left(\left(\eta_1 s_n^{-\beta_1}+\eta_2 s_n^{-\beta_2}\right)\int_{\mathbb{H}_1}|u_n|^4\right)^{\frac{1}{2}}}+\frac{s_n^{-2}\int_{\mathbb{H}_1}|\nabla_{\mathbb{H}_1}z_n|^2dz}{\left(\left(\eta_1 s_n^{-\beta_1}+\eta_2 s_n^{-\beta_2}\right)\int_{\mathbb{H}_1}|w_n|^4\,\mathrm{d}z\right)^{\frac{1}{2}}}\\
&\quad\geq \mathfrak{h}\left(s_n^{-1}\right)S_{\mathbb{H}_1}\\
&\quad\geq \mathfrak{h}\left(t_0\right)S_{\mathbb{H}_1}.
\end{aligned}$$

As $n\to\infty$, we have

$$S_{\eta,\alpha,\beta}\geq \mathfrak{h}\left(t_0\right)S_{\mathbb{H}_1},$$

which together with (2.6) implies that

$$S_{\eta,\alpha,\beta}= \mathfrak{h}\left(t_0\right)S_{\mathbb{H}_1}.$$

By (2.2) and (2.1), $S_{\eta,\alpha,\beta}$ has the minimizers $(U_\varepsilon(x),t_0U_\varepsilon(x))$. □

Let $R>0$ be such that $B_d(0,R)\subset\Omega$ (we can suppose $0\in\Omega$, due to the group translation invariance) and let a cut-off function $\varphi\in C_0^\infty\left(B_d(0,R)\right)$, $0\leq\varphi\leq 1$, $\varphi=1$ in $B_d\left(0,\frac{R}{2}\right)$ and $\varphi=0$ in $\mathbb{H}_1\backslash B_d(0,R)$. Set

$$u_\varepsilon(z)=\varphi(z)U_\varepsilon(z).$$

Then, from [[12], Lemma 3.3], we obtain the required results.

Lemma 2.2. *The functions u_ε satisfy the following estimates, as $\varepsilon\to 0$:*

$$\int_\Omega|\nabla_{\mathbb{H}_1}u_\varepsilon|^2dz=S_{\mathbb{H}_1}^2+O\left(\varepsilon^2\right),$$
$$\int_\Omega|u_\varepsilon|^4dz=S_{\mathbb{H}_1}^2+O\left(\varepsilon^4\right)$$

and

$$\int_\Omega|u_\varepsilon|^2dz=C\varepsilon^2|\ln\varepsilon|+O\left(\varepsilon^2\right).$$

Moreover, similarly as the proof of [[13], Lemma 6.1], we get the following results.

Lemma 2.3. *The following estimates hold as $\varepsilon\to 0$:*

$$\int_\Omega|u_\varepsilon|^qdz\geq\begin{cases}O\left(\varepsilon^2|\ln(\varepsilon)|\right), & \textit{if } q=2,\\ O\left(\varepsilon\right), & \textit{if } 1\leq q<2.\end{cases}$$

3. The Palais-Smale condition

In this section, we use the second concentration-compactness principle and concentration-compactness principle at infinity to prove that the $(PS)_c$ condition holds.

Definition 3.1. Let $c \in \mathbb{R}$ and $I_{\eta,\alpha,\beta} \in C^1(\mathcal{H}, \mathbb{R})$.

(i) A sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is a Palais-Smale sequence at the level c ($(PS)_c$-sequence in short) for the functional $I_{\eta,\alpha,\beta}$ if $I_{\eta,\alpha,\beta}(u_n, v_n) \to c$ and $I'_{\eta,\alpha,\beta}(u_n, v_n) \to 0$ as $n \to \infty$.

(ii) We say that $I_{\eta,\alpha,\beta}$ satisfies the $(PS)_c$ condition if any $(PS)_c$-sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ for $I_{\eta,\alpha,\beta}$ has a convergent subsequence in $\mathcal{H}$.

Since $g, h \in L^{\frac{4}{4-q}}(\Omega)$, we obtain from the Hölder and Sobolev inequalities that, for all $u \in S_0^1(\Omega)$,

$$\begin{aligned}\int_\Omega g(z)|u|^q dz &\leq \left(\int_\Omega |g(z)|^{\frac{4}{4-q}} dz\right)^{\frac{4-q}{4}} \left(\int_\Omega |u|^4 dz\right)^{\frac{q}{4}} \\ &\leq \|g\|_{L^{\frac{4}{4-q}}} S_{\mathbb{H}_1}^{-\frac{q}{2}} \|u\|^q_{S_0^1(\Omega)}. \end{aligned} \tag{3.1}$$

Similarly, we get

$$\begin{aligned}\int_\Omega h(z)|v|^q dz &\leq \left(\int_\Omega |h(z)|^{\frac{4}{4-q}} dz\right)^{\frac{4-q}{4}} \left(\int_\Omega |v|^4 dz\right)^{\frac{q}{4}} \\ &\leq \|h\|_{L^{\frac{4}{4-q}}} S_{\mathbb{H}_1}^{-\frac{q}{2}} \|v\|^q_{S_0^1(\Omega)}. \end{aligned} \tag{3.2}$$

Hence, from (3.1) and (3.2), we have

$$\Psi_{\lambda,\mu}(u, v) \leq \left(\lambda\|g\|_{L^{\frac{4}{4-q}}} + \mu\|h\|_{L^{\frac{4}{4-q}}}\right) S_{\mathbb{H}_1}^{-\frac{q}{2}} \|(u, v)\|^q_{\mathcal{H}}. \tag{3.3}$$

Moreover, the Young inequality and (3.1), (3.2) imply that

$$\begin{aligned}\Psi_{\lambda,\mu}(u, v) \leq & \frac{q}{4-q}\|(u, v)\|^2_{\mathcal{H}} + \frac{2-q}{2} S_{\mathbb{H}_1}^{-\frac{q}{2-q}} \left(\frac{4-q}{2}\right)^{\frac{q}{2-q}} \\ & \times \left[\left(\lambda\|g\|_{L^{\frac{4}{4-q}}}\right)^{\frac{2}{2-q}} + \left(\mu\|h\|_{L^{\frac{4}{4-q}}}\right)^{\frac{2}{2-q}}\right]. \end{aligned} \tag{3.4}$$

Lemma 3.2. *Let $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ be a $(PS)_c$-sequence of $I_{\eta,\alpha,\beta}$ with $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $\mathcal{H}$. Then $I'_{\eta,\alpha,\beta}(u, v) = 0$ and*

$$I_{\eta,\alpha,\beta}(u,v) \geq -\frac{(4-q)(2-q)}{8q} S_{\mathbb{H}_1}^{-\frac{q}{2-q}} \left(\frac{4-q}{2}\right)^{\frac{q}{2-q}} \times \left[\left(\lambda \|g\|_{L^{\frac{4}{4-q}}}\right)^{\frac{2}{2-q}} + \left(\mu \|h\|_{L^{\frac{4}{4-q}}}\right)^{\frac{2}{2-q}}\right].$$

Proof. Since $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is a (PS) $)_c$-sequence of $I_{\eta,\alpha,\beta}$ with $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $\mathcal{H}$, it is easy to check that $I'_{\eta,\alpha,\beta}(u, v) = 0$, and then

$$\left\langle I'_{\eta,\alpha,\beta}(u,v), (u,v)\right\rangle = 0,$$

that is,

$$\|(u,v)\|_{\mathcal{H}}^2 = K_\eta(u,v) + \Psi_{\lambda,\mu}(u,v).$$

Then from (3.4), we have

$$\begin{aligned} I_{\eta,\alpha,\beta}(u,v) =& \frac{1}{4}\|(u,v)\|_{\mathcal{H}}^2 - \frac{4-q}{4q}\Psi_{\lambda,\mu}(u,v) \\ \geq& -\frac{(4-q)(2-q)}{8q} S_{\mathbb{H}_1}^{-\frac{q}{2-q}} \left(\frac{4-q}{2}\right)^{\frac{q}{2-q}} \\ &\times \left[\left(\lambda \|g\|_{L^{\frac{4}{4-q}}}\right)^{\frac{2}{2-q}} + \left(\mu \|h\|_{L^{\frac{4}{4-q}}}\right)^{\frac{2}{2-q}}\right]. \end{aligned}$$

This ends the proof. □

Lemma 3.3. *Assume that $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is a $(PS)_c$-sequence of $I_{\eta,\alpha,\beta}$ and the condition $(\mathcal{A}_1)$ holds. Then $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$.*

Proof. Assume by contradiction that $\|(u_n, v_n)\|_{\mathcal{H}} \to +\infty$. Set

$$(\tilde{u}_n, \tilde{v}_n) = \left(\frac{u_n}{\|(u_n,v_n)\|_{\mathcal{H}}}, \frac{v_n}{\|(u_n,v_n)\|_{\mathcal{H}}}\right).$$

Then, $\|(\tilde{u}_n, \tilde{v}_n)\|_{\mathcal{H}} = 1$, and

$$\begin{cases} (\tilde{u}_n, \tilde{v}_n) \rightharpoonup (u,v) \text{ weakly in } \mathcal{H}, \\ (\tilde{u}_n, \tilde{v}_n) \to (u,v) \text{ strongly in } \left(L^r(\Omega)\right)^2 \text{ for all } r \in [1,4), \\ (\tilde{u}_n(z), \tilde{v}_n(z)) \to (u(z), v(z)) \text{ a.e. in } \Omega. \end{cases} \tag{3.5}$$

Set $\bar{u}_n := \tilde{u}_n - u, \bar{v}_n := \tilde{v}_n - v$, then there exists a positive constant $C > 0$ such that

$$\int_\Omega |\bar{u}_n|^4 dz < C, \quad \int_\Omega |\bar{v}_n|^4 dz < C \tag{3.6}$$

and by (3.5), one has that for any $\varepsilon > 0$, there exists $r_0 > 0$ such that

$$\int_{B_d(0,r_0)} |\bar{u}_n|^4 dz < \varepsilon \quad \text{and} \quad \int_{B_d(0,r_0)} |\bar{v}_n|^4 dz < \varepsilon, \tag{3.7}$$

for n large enough, where $B_d(0,r_0) = \{z \in \mathbb{H}_1 : d(0,z) \leq r_0\}$ is a ball with center at 0 and radius r_0 with respect to the gauge d.

Moreover, since $g, h \in L^{\frac{4}{4-q}}(\Omega)$, for the above constant r_0, we have

$$\int_{\Omega \setminus B_d(0,r_0)} |g(z)|^{\frac{4}{4-q}} dz < \varepsilon \quad \text{and} \quad \int_{\Omega \setminus B_d(0,r_0)} |h(z)|^{\frac{4}{4-q}} dz < \varepsilon. \tag{3.8}$$

Then, by (3.6), (3.7), (3.8) and Hölder inequality, we get

$$\begin{aligned}
\Psi_{\lambda,\mu}(\bar{u}_n, \bar{u}_n) = &\int_{\Omega \setminus B_d(0,r_0)} (\lambda g(z)|\bar{u}_n|^q + \mu h(z)|\bar{v}_n|^q)\, dz \\
&+ \int_{B_d(0,r_0)} (\lambda g(z)|\bar{u}_n|^q + \mu h(z)|\bar{v}_n|q^q)\, dz \\
\leq &\lambda \left(\int_{\Omega \setminus B_d(0,r_0)} |g|^{\frac{4}{4-q}} dz\right)^{\frac{4-q}{4}} \left(\int_{\Omega \setminus B_d(0,r_0)} |\bar{u}_n|^4 dz\right)^{\frac{q}{4}} \\
&+ \mu \left(\int_{\Omega \setminus B_d(0,r_0)} |h|^{\frac{4}{4-q}} dz\right)^{\frac{4-q}{4}} \left(\int_{\Omega \setminus B_d(0,u_0)} |\bar{v}_n|^4 dz\right)^{\frac{q}{4}} \\
&+ \lambda \left(\int_{B_d(0,r_0)} |g|^{\frac{2^*}{2^*-q}} dz\right)^{\frac{4-q}{4}} \left(\int_{B_d(0,r_0)} |\bar{u}_n|^4 dz\right)^{\frac{q}{4}} \\
&+ \mu \left(\int_{B_d(0,r_0)} |h|^{\frac{4}{4-q}} dz\right)^{\frac{4-q}{4}} \left(\int_{B_d(0,u_0)} |\bar{v}_n|^4 dz\right)^{\frac{q}{4}} \\
\leq &C_1 \varepsilon^{\frac{4-q}{4}} + C_2 \varepsilon^{\frac{q}{4}},
\end{aligned}$$

which yields that $\Psi_{\lambda,\mu}(\bar{u}_n, \bar{v}_n) \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} \Psi_{\lambda,\mu}(\tilde{u}_n, \tilde{v}_n) = \lim_{n\to\infty} \Psi_{\lambda,\mu}(\bar{u}_n, \bar{v}_n) + \Psi_{\lambda,\mu}(u,v) = \Psi_{\lambda,\mu}(u,v). \tag{3.9}$$

On the other hand, since $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is a $(PS)_c$-sequence of $I_{\eta,\alpha,\beta}$ and $u_n = \|(u_n,v_n)\|_{\mathcal{H}} \cdot \tilde{u}_n,\ v_n = \|(u_n,v_n)\|_{\mathcal{H}} \cdot \tilde{v}_n$, we deduce that

$$\begin{aligned}
\frac{1}{2}\|(u_n,v_n)\|_{\mathcal{H}}^2 \|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}}^2 = &\frac{1}{4}\|(u_n,v_n)\|_{\mathcal{H}}^4 K_\eta(\tilde{u}_n,\tilde{v}_n) \\
&+ \frac{1}{q}\|(u_n,v_n)\|_{\mathcal{H}}^q \Psi_{\lambda,\mu}(\tilde{u}_n,\tilde{v}_n) + o_n(1)
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\|(u_n,v_n)\|_{\mathcal{H}}^2\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}}^2 =& \|(u_n,v_n)\|_{\mathcal{H}} K_\eta(\tilde{u}_n,\tilde{v}_n)\\ &+\|(u_n,v_n)\|_{\mathcal{H}}^q \Psi_{\lambda,\mu}(\tilde{u}_n,\tilde{v}_n)+o_n(1).\end{aligned} \tag{3.11}$$

From (3.9), (3.10), (3.11), $1<q<2$ and $\|(u_n,v_n)\|_{\mathcal{H}} \to +\infty$, we have

$$\lim_{n\to\infty}\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}}^2 = \frac{4-q}{q}\lim_{n\to\infty}\frac{\Psi_{\lambda,\mu}(\bar{u}_n,\bar{v}_n)}{\|(u_n,v_n)\|_{\mathcal{H}}^{2-q}}=0,$$

which contradicts $\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}}=1$. Therefore, $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$. □

Lemma 3.4. *$\mathcal{I}_{\eta,\alpha,\beta}$ satisfies the $(PS)_c$ condition in $\mathcal{H}$, with c satisfying*

$$0<c<c_\infty := \frac{1}{4}S_{\eta,\alpha,\beta}^2 - C_0\left[\left(\lambda\|g\|_{L^{\frac{4}{4-q}}}\right)^{\frac{2}{2-q}}+\left(\mu\|h\|_{L^{\frac{4}{4-q}}}\right)^{\frac{2}{2-q}}\right], \tag{3.12}$$

where $C_0=C_0(q):=\frac{(4-q)(2-q)}{8q}S_{\mathbb{H}_1}^{-\frac{q}{2-q}}\left(\frac{4-q}{2}\right)^{\frac{q}{2-q}}$ is a positive constant depending only on q and $S_{\mathbb{H}_1}$.

Proof. Let $\{(u_n,v_n)\}_{n\in\mathbb{N}}\subset\mathcal{H}$ be a $(PS)_c$-sequence for $I_{\eta,\alpha,\beta}$ with $c\in(0,c_\infty)$. It follows from Lemma 3.3 that $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$. Then, there exists a subsequence still denoted by $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ and $(u,v)\in\mathcal{H}$ such that $(u_n,v_n)\rightharpoonup(u,v)$ weakly in $\mathcal{H}$, and

$$\begin{cases} u_n\rightharpoonup u, \quad v_n\rightharpoonup v \text{ weakly in } L^4(\Omega),\\ u_n\to u, \quad v_n\to v \text{ strongly in } L^r(\Omega) \text{ for all } 1\le r<4,\\ u_n(z)\to u(z), \quad v_n(z)\to v(z) \text{ a.e. in } \Omega.\end{cases} \tag{3.13}$$

Hence, from (3.13), it is easy to verify that

$$I'_{\eta,\alpha,\beta}(u,v)=0 \quad\text{and}\quad \lim_{n\to\infty}\Psi_{\lambda,\mu}(u_n,v_n)=\Psi_{\lambda,\mu}(u,v). \tag{3.14}$$

Set $\tilde{u}_n=u_n-u, \tilde{v}_n=v_n-v$. By Brzis-Lieb lemma [18], we get

$$\|(u_n,v_n)\|_{\mathcal{H}}^2=\|(u,v)\|_{\mathcal{H}}^2+\|(\tilde{u}_n,\tilde{v}_n)\|_{\mathcal{H}}^2+o_n(1), \tag{3.15}$$

$$\int_\Omega |u_n|^4dz=\int_\Omega|u|^4dz+\int_\Omega|\tilde{u}_n|^4dz+o_n(1), \tag{3.16}$$

$$\int_\Omega |v_n|^4dz=\int_\Omega|v|^4dz+\int_\Omega|\tilde{v}_n|^4dz+o_n(1) \tag{3.17}$$

and

$$\int_\Omega|u_n|^{\alpha_i}|v_n|^{\beta_i}dz=\int_\Omega|u|^{\alpha_i}|v|^{\beta_i}dz+\int_\Omega|\tilde{u}_n|^{\alpha_i}|\tilde{v}_n|^{\beta_i}dz+o_n(1). \tag{3.18}$$

So, (3.16), (3.17) and (3.18) yield

$$K_\eta(u_n,v_n)=K_\eta(u,v)+K_\eta(\tilde{u}_n,\tilde{v}_n)+o_n(1). \tag{3.19}$$

Then, using (3.14), (3.15) and (3.19), we have

$$c = \frac{1}{2}\left\|(\tilde{u}_n, \tilde{v}_n)\right\|_{\mathcal{H}}^2 - \frac{1}{4}K_\eta\left(\tilde{u}_n, \tilde{v}_n\right) + I_{\eta,\alpha,\beta}(u,v) + o_n(1) \tag{3.20}$$

and

$$o_n(1) = \left\|(\bar{u}_n, \bar{v}_n)\right\|_{\mathcal{H}}^2 - K_\eta\left(\bar{u}_n, \bar{v}_n\right). \tag{3.21}$$

We may assume that

$$\left\|(\tilde{u}_n, \tilde{v}_n)\right\|_{\mathcal{H}}^2 \to l, \quad K_\eta\left(\tilde{u}_n, \tilde{v}_n\right) \to l \geq 0 \quad \text{as } n \to \infty.$$

If $l = 0$, the proof is completed. Assume that $l > 0$, then from (3.21), we have

$$S_{\eta,\alpha,\beta} l^{\frac{1}{2}} = S_{\eta,\alpha,\beta}\left(\lim_{n\to\infty} K_\eta\left(\tilde{u}_n, \tilde{v}_n\right)\right)^{\frac{1}{2}} \leq \lim_{n\to\infty}\left\|(\tilde{u}_n, \tilde{v}_n)\right\|_{\mathcal{H}}^2 = l,$$

which implies that $l \geq S_{\eta,\alpha,\beta}^2$. Hence, from (3.20) and Lemma 3.2, we have

$$\begin{aligned} c &= I_{\eta,\alpha,\beta}(u_n, v_n) + o_n(1) \\ &= \left(\frac{1}{2} - \frac{1}{4}\right) l + I_{\eta,\alpha,\beta}(u,v) + o_n(1) \\ &\geq \frac{1}{4} S_{\eta,\alpha,\beta}^2 - C_0\left[\left(\lambda\|g\|_{L^{\frac{4}{4-q}}}\right)^{\frac{2}{2-q}} + \left(\mu\|h\|_{L^{\frac{4}{4-q}}}\right)^{\frac{2}{2-q}}\right], \end{aligned} \tag{3.22}$$

which contradicts $c < c_\infty$. The proof is completed. □

4. Nehari manifold

Now we focus our attention on Problem (1.1) by using the Nehari manifold approach. For this reason, we introduce the Nehari manifold

$$\mathcal{N}_{\eta,\alpha,\beta} = \left\{w \in \mathcal{H}\backslash\{0\} : \left\langle I'_{\eta,\alpha,\beta}(w), w\right\rangle = 0\right\},$$

where $w = (u, v)$ and $\|w\|_{\mathcal{H}} = \|(u,v)\|_{\mathcal{H}}$. Note that $\mathcal{N}_{\eta,\alpha,\beta}$ contains all nonzero solution of (1.1), and $w \in \mathcal{N}_{\eta,\alpha,\beta}$ if and only if

$$\|w\|_{\mathcal{H}}^2 = K_\eta(w) + \Psi_{\lambda,\mu}(w). \tag{4.1}$$

Lemma 4.1. *$I_{\eta,\alpha,\beta}$ is coercive and bounded below on $\mathcal{N}_{\eta,\alpha,\beta}$.*

Proof. Let $w \in \mathcal{N}_{\eta,\alpha,\beta}$ by (3.3) and (4.1). We find

$$\begin{aligned} I_{\eta,\alpha,\beta}(w) =& \frac{1}{4}\|w\|_{\mathcal{H}}^2 - \frac{4-q}{4q}\Psi_{\lambda,\mu}(w) \\ \geq& \frac{1}{4}\|w\|_{\mathcal{H}}^2 - \frac{4-q}{4q}\left(\lambda\|g\|_{L^{\frac{4}{4-q}}} + \mu\|h\|_{L^{\frac{4-q}{4}}}\right) S_{\mathbb{H}_1}^{-\frac{q}{2}}\|w\|_{\mathcal{H}}^q. \end{aligned} \tag{4.2}$$

Since $1 < q < 2$, then $I_{\eta,\alpha,\beta}$ is coercive and bounded below on $\mathcal{N}_{\eta,\alpha,\beta}$. This achieves the proof. □

Define $\Phi(w) := \left\langle I'_{\eta,\alpha,\beta}(w), w \right\rangle$, then for all $w = (u, v) \in \mathcal{N}_{\eta,\alpha,\beta}$, we have

$$\begin{aligned} \left\langle \Phi'(w), w \right\rangle &= 2\|w\|_{\mathcal{H}}^2 - 4K_\eta(w) - q\Psi_{\lambda,\mu}(w) \\ &= (2-q)\|w\|_{\mathcal{H}}^2 - (4-q)\, K_\eta(w) \\ &= 2\|w\|_{\mathcal{H}}^2 + (4-q)\, \Psi_{\lambda,\mu}(w). \end{aligned} \tag{4.3}$$

Now, similar to the method used in [16], we split $\mathcal{N}_{\eta,\alpha,\beta}$ into three disjoint parts:

$$\begin{aligned} \mathcal{N}^+_{\eta,\alpha,\beta} &:= \left\{ w \in \mathcal{N}_{\eta,\alpha,\beta} : \left\langle \Phi'(w), w \right\rangle > 0 \right\}, \\ \mathcal{N}^0_{\eta,\alpha,\beta} &:= \left\{ w \in \mathcal{N}_{\eta,\alpha,\beta} : \left\langle \Phi'(w), w \right\rangle = 0 \right\}, \\ \mathcal{N}^-_{\eta,\alpha,\beta} &:= \left\{ w \in \mathcal{N}_{\eta,\alpha,\beta} : \left\langle \Phi'(w), w \right\rangle < 0 \right\}. \end{aligned} \tag{4.4}$$

Note that $\mathcal{N}_{\eta,\alpha,\beta}$ contains every nonzero solution of problem (1.1). In order to study the properties of Nehari manifolds. We now present some properties of $\mathcal{N}^+_{\eta,\alpha,\beta}$, $\mathcal{N}^0_{\eta,\alpha,\beta}$ and $\mathcal{N}^-_{\eta,\alpha,\beta}$ to state our main results.

Lemma 4.2. *Assume that $w_0 = (u_0, v_0)$ is a local minimizer for $I_{\eta,\alpha,\beta}$ on the set $\mathcal{N}_{\eta,\alpha,\beta} \setminus \mathcal{N}^0_{\eta,\alpha,\beta}$. Then $I'_{\eta,\alpha,\beta}(w_0) = 0$ in $\mathcal{H}^{-1}$, where $\mathcal{H}^{-1}$ denotes the dual space of the space $\mathcal{H}$.*

Proof. The proof is similar as that of [20, Lemma 3.4] and the details are omitted. □

Lemma 4.3. *$\mathcal{N}^0_{\eta,\alpha,\beta} = \emptyset$ for all $(\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+$ with*

$$0 < \lambda \|g\|_{L^{\frac{4}{4-q}}} + \mu \|h\|_{L^{\frac{4}{4-q}}} < \Lambda,$$

where Λ is given in (1.7).

Proof. We argue by contradiction. Assume that there exist $\lambda, \mu \in (0, +\infty)$ with

$$0 < \lambda \|g\|_{L^{\frac{4}{4-q}}} + \mu \|h\|_{L^{\frac{4}{4-q}}} < \Lambda$$

such that $\mathcal{N}^0_{\eta,\alpha,\beta} \neq \emptyset$. Then, for $w \in \mathcal{N}^0_{\eta,\alpha,\beta}$, by (4.3), we have

$$\|w\|_{\mathcal{H}}^2 = \frac{4-q}{2-q} K_\eta(w) \tag{4.5}$$

and

$$\|w\|_{\mathcal{H}}^2 = \frac{4-q}{2} \Psi_{\lambda,\mu}(w). \tag{4.6}$$

From the Young inequality, we have that

$$K_\eta(w) \le (\eta_1 + \eta_2)\, S_{\mathbb{H}_1}^{-2} \|w\|_{\mathcal{H}}^4$$

and (4.5) yields

$$\|w\|_{\mathcal{H}} \geq \left(\frac{2-q}{(\eta_1+\eta_2)(4-q)} S_{\mathbb{H}_1}^2\right)^{\frac{1}{2}}. \tag{4.7}$$

On the other hand, from (3.3) and (4.6), it follows that

$$\|w\|_{\mathcal{H}} \leq \left(\frac{4-q}{2}\left(\lambda\|g\|_{L^{\frac{4}{4-q}}} + \mu\|h\|_{L^{\frac{4}{4-q}}}\right) S_{\mathbb{H}_1}^{\frac{-q}{2}})\right)^{\frac{1}{2-q}}. \tag{4.8}$$

Therefore, in view of (4.7) and (4.8), we obtain

$$\lambda\|g\|_{L^{\frac{4}{4-q}}} + \mu\|h\|_{L^{\frac{4}{4-q}}} \geq \frac{2}{4-q}\left(\frac{2-q}{(\eta_1+\eta_2)(4-q)}\right)^{\frac{2-q}{4-2}} S_{\mathbb{H}_1}^{\frac{4-q}{2}} := \Lambda,$$

which is a contradiction. This completes the proof. □

By Lemmas 4.2 and 4.3, for $(\lambda,\mu) \in \mathfrak{D}_\Lambda$, we can write $\mathcal{N}_{\eta,\alpha,\beta} = \mathcal{N}_{\eta,\alpha,\beta}^+ \cup \mathcal{N}_{\eta,\alpha,\beta}^-$ and define

$$\begin{aligned} c_{\eta,\alpha,\beta} &= \inf_{w\in\mathcal{N}_{\eta,\alpha,\beta}} I_{\eta,\alpha,\beta}(w), \\ c_{\eta,\alpha,\beta}^+ &= \inf_{w\in\mathcal{N}_{\eta,\alpha,\beta}^+} I_{\eta,\alpha,\beta}(w), \\ c_{\eta,\alpha,\beta}^- &= \inf_{w\in\mathcal{N}_{\eta,\alpha,\beta}^-} I_{\eta,\alpha,\beta}(w). \end{aligned}$$

Lemma 4.4. *Assume that $(\mathcal{A}_0)$ hold. Then, we have the following results:*

(i) $c_{\eta,\alpha,\beta} \leq c_{\eta,\alpha,\beta}^+ < 0$ *for all* $(\lambda,\mu) \in \mathfrak{D}_\Lambda$.

(ii) *There exists a constant* $C_0 = C_0(\lambda,\mu,q,S_{\mathbb{H}_1},\Lambda) > 0$ *such that* $c_{\eta,\alpha,\beta}^- \geq C_0 > 0$ *for all* $(\lambda,\mu) \in \mathfrak{D}_{\frac{q}{2}\Lambda}$.

Proof. (i) For $w \in \mathcal{N}_{\eta,\alpha,\beta}^+ \subset \mathcal{N}_{\eta,\alpha,\beta}$, by (4.3), we have

$$\|w\|_{\mathcal{H}}^2 > \frac{4-q}{2-q} K_\eta(w)$$

and so

$$\begin{aligned} I_{\eta,\alpha,\beta}(w) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|w\|_{\mathcal{H}}^2 - \left(\frac{1}{4}-\frac{1}{q}\right)K_\eta(w) \\ &\leq \left(\frac{q-2}{2q} + \frac{4-q}{4q}\frac{2-q}{4-q}\right)\|w\|_{\mathcal{H}}^2 \\ &= -\frac{2-q}{4q}\|w\|_{\mathcal{H}}^2 < 0. \end{aligned}$$

Thus, from the definition of $c_{\eta,\alpha,\beta}$ and $c_{\eta,\alpha,\beta}^+$, we can deduce that $c_{\eta,\alpha,\beta} \leq c_{\eta,\alpha,\beta}^+ < 0$.

(ii) For $w \in \mathcal{N}^{-}_{\eta,\alpha,\beta}$, similar to (4.7), we have

$$\|w\|_{\mathcal{H}} > \left(\frac{2-q}{(\eta_1+\eta_2)(4-q)} S^2_{\mathbb{H}_1} \right)^{\frac{1}{2}}. \tag{4.9}$$

In view of (4.2) and (4.9), we get

$$\begin{aligned} I_{\eta,\alpha,\beta}(w) \geq & \|w\|^q_{\mathcal{H}} \left(\frac{1}{4}\|w\|^{2-q}_{\mathcal{H}} - \frac{4-q}{4q}\left(\lambda\|g\|_{L^{\frac{4}{4-q}}} + \mu\|h\|_{L^{\frac{4}{4-q}}}\right) S^{-\frac{q}{2}}_{\mathbb{H}_1} \right) \\ \geq & \|w\|^q_{\mathcal{H}} \left(\frac{1}{4}\left(\frac{2-q}{(\eta_1+\eta_2)(4-q)} \right)^{\frac{2-q}{2}} S^{2-q}_{\mathbb{H}_1} \right. \\ & \left. - \frac{4-q}{4q}\left(\lambda\|g\|_{L^{\frac{4}{4-q}}} + \mu\|h\|_{L^{\frac{4}{4-q}}}\right) S^{-\frac{q}{2}}_{\mathbb{H}_1} \right). \end{aligned}$$

So, if namely,

$$0 < \lambda\|g\|_{L^{\frac{4}{4-q}}} + \mu\|h\|_{L^{\frac{4}{4-q}}} < \frac{q}{4-q}\left(\frac{2-q}{(\eta_1+\eta_2)(4-q)} \right)^{\frac{2-q}{2}} S^{\frac{4-q}{2}}_{\mathbb{H}_1} = \frac{q}{2}\Lambda,$$

we get

$$\begin{aligned} I_{\eta,\alpha,\beta}(w) \geq & \left(\frac{2-q}{(\eta_1+\eta_2)(4-q)} S^2_{\mathbb{H}_1} \right)^{\frac{q}{2}} \left(\frac{1}{4}\left(\frac{2-q}{(\eta_1+\eta_2)(4-q)} \right)^{\frac{2-q}{2}} S^{2-q}_{\mathbb{H}_1} \right. \\ & \left. - \frac{4-q}{4q}\left(\lambda\|g\|_{L^{\frac{4}{4-q}}} + \mu\|h\|_{L^{\frac{4}{4-q}}}\right) S^{-\frac{q}{2}}_{\mathbb{H}_1} \right) \\ & := C_0\left(\lambda,\mu,q,S_{\mathbb{H}_1},\Lambda\right) > 0, \end{aligned}$$

and this completes the proof. □

For each $w \in \mathcal{H}\backslash\{0\}$, we have $K_\eta(w) > 0$ and let

$$t_{\max} = \left(\frac{(2-q)\|w\|^2_{\mathcal{H}}}{(4-q)\,K_\eta(w)} \right)^{\frac{1}{2}} > 0.$$

So, we get the following result.

Lemma 4.5. *Let $(\lambda,\mu) \in \mathfrak{D}_\Lambda$. For every $w \in \mathcal{H}$ with $K_\eta(w) > 0$, the following results hold:*

(i) *If $\Psi_{\lambda,\mu}(w) \leq 0$, then there is a unique $t^- > t_{\max}$ such that $(t^- w) \in \mathcal{N}^{-}_{\eta,\alpha,\beta}$ and*

$$I_{\eta,\alpha,\beta}\left(t^- w\right) = \sup_{t\geq 0} I_{\eta,\alpha,\beta}(tw).$$

(ii) *If* $\Psi_{\lambda,\mu}(w) > 0$*, then there are unique* t^{+} *and* t^{-} *with* $0 < t^{+} < t_{\max} < t^{-}$ *such that* $(t^{+}w) \in \mathcal{N}^{+}_{\eta,\alpha,\beta}\,(t^{-}w) \in \mathcal{N}^{-}_{\eta,\alpha,\beta}$ *and*

$$I_{\eta,\alpha,\beta}\left(t^{+}w\right) = \inf_{0\leq t\leq I_{\max}} I_{\eta,\alpha,\beta}(tw), \quad I_{\eta,\alpha,\beta}\left(t^{-}w\right) = \sup_{t\geq 0} I_{\eta,\alpha,\beta}(tw).$$

Proof. The proof is similar to [[6], Lemma 2.6] and is omitted here. □

5. Proof of the main results

In this section, we provide the proofs of the main results of this work. Before giving the proof of Theorem 1, we need the following lemma.

Lemma 5.1. *Assume that* $(\mathcal{A}_0)$ *hold. Then, we have the following results:*

(i) *If* $(\lambda,\mu) \in \mathfrak{D}_{\Lambda}$*, then there exists a* $(PS)_{c_{\eta,\alpha,\beta}}$*-sequence* $\{(u_n,v_n)\}_{n\in\mathbb{N}} \subset \mathcal{N}_{\eta,\alpha,\beta}$ *for* $I_{\eta,\alpha,\beta}$*.*

(ii) *If* $(\lambda,\mu) \in \mathfrak{D}_{\frac{q}{2}\Lambda}$*, then there exists a* $(PS)_{c^{-}_{\eta,\alpha,\beta}}$*-sequence* $\{(u_n,v_n)\}_{n\in\mathbb{N}} \subset \mathcal{N}^{-}_{\eta,\alpha,\beta}$ *for* $I_{\eta,\alpha,\beta}$*.*

Proof. The proof is almost the same as Proposition 9 in [19]. □

Now we establish the existence of a local minimizer of $I_{\eta,\alpha,\beta}$ on $\mathcal{N}^{+}_{\eta,\alpha,\beta}$.

Theorem 5.2. *Assume that* $(\mathcal{A}_0)$ *hold. If* $(\lambda,\mu) \in \mathfrak{D}_{\Lambda}$*, then* $I_{\eta,\alpha,\beta}$ *has a minimizer* $(u_1,v_1) \in \mathcal{N}^{+}_{\eta,\alpha,\beta}$ *such that* (u_1,v_1) *is a nonnegative solution of* (1.1) *and*

$$I_{\eta,\alpha,\beta}\left(u_1,v_1\right) = c_{\eta,\alpha,\beta} = c^{+}_{\eta,\alpha,\beta} < 0.$$

Proof. In view of the Lemma 5.1, (i), there exists a minimizing sequence $\{(u_n,v_n)\}_{n\in\mathbb{N}} \subset \mathcal{N}_{\eta,\alpha,\beta}$ such that

$$\lim_{n\to\infty} I_{\eta,\alpha,\beta}\left(u_n,v_n\right) = c_{\eta,\alpha,\beta} \text{ and } \lim_{n\to\infty} I'_{\eta,\alpha,\beta}\left(u_n,v_n\right) = 0. \tag{5.1}$$

Since $I_{\eta,\alpha,\beta}$ is coercive on $\mathcal{N}_{\eta,\alpha,\beta}$, we get that $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$. Passing to a subsequence, still denoted by $\{(u_n,v_n)\}_{n\in\mathbb{N}}$, we can assume that there exists $(u_1,v_1) \in \mathcal{H}$ such that $(u_n,v_n) \rightharpoonup (u_1,v_1)$ weakly in $\mathcal{H}$ and

$$\begin{cases} u_n \rightharpoonup u_1, \quad v_n \rightharpoonup v_1 \text{ weakly in } L^4(\Omega), \\ u_n \to u_1, \quad v_n \to v_1 \text{ strongly in } L^r(\Omega) \text{ for all } r \in [1,4), \\ u_n(z) \to u_1(z), \quad v_n(z) \to v_1(z) \text{ a.e. in } \Omega. \end{cases} \tag{5.2}$$

By the proof of Lemma 3.3 and (5.2), we get

$$\lim_{n\to\infty} \Psi_{\lambda,\mu}\left(u_n,v_n\right) = \Psi_{\lambda,\mu}\left(u_1,v_1\right). \tag{5.3}$$

From (5.1), (5.2) and (5.3), it is easy to prove that (u_1, v_1) is a weak solution of (1.1). Moreover, the fact that $(u_n, v_n) \in \mathcal{N}_{\eta,\alpha,\beta}$ implies that

$$\Psi_{\lambda,\mu}(u_n, v_n) = \frac{q}{4-q}\|(u_n, v_n)\|_{\mathcal{H}}^2 - \frac{4q}{4-q} I_{\eta,\alpha,\beta}(u_n, v_n). \tag{5.4}$$

Let $n \to \infty$ in (5.4), by (5.3) and $c_{\eta,\alpha,\beta} < 0$, we deduce that

$$\Psi_{\lambda,\mu}(u_1, v_1) \geq -\frac{4q}{4-q} c_{\eta,\alpha,\beta} > 0,$$

which implies that $(u_1, v_1) \in \mathcal{H}$ is a nontrivial solution of (1.1).

Now, we prove that $(u_n, v_n) \to (u_1, v_1)$ strongly in $\mathcal{H}$ and that $I_{\eta,\alpha,\beta}(u_1, v_1) = c_{\eta,\alpha,\beta}$. By applying Fatou's lemma and $(u_1, v_1) \in \mathcal{N}_{\eta,\alpha,\beta}$, we have

$$\begin{aligned} c_{\eta,\alpha,\beta} &\leq I_{\eta,\alpha,\beta}(u_1, v_1) \\ &= \frac{1}{4}\|(u_1, v_1)\|_{\mathcal{H}}^2 - \frac{4-q}{4q}\Psi_{\lambda,\mu}(u_1, v_1) \\ &\leq \liminf_{n\to\infty}\left[\frac{1}{4}\|(u_n, v_n)\|_{\mathcal{H}}^2 - \frac{4-q}{4q}\Psi_{\lambda,\mu}(u_n, v_n)\right] \\ &\leq \lim_{n\to\infty} I_{\eta,\alpha,\beta}(u_n, v_n) \\ &= c_{\eta,\alpha,\beta}. \end{aligned}$$

This yields $I_{\eta,\alpha,\beta}(u_1, v_1) = c_{\eta,\alpha,\beta}$ and $\lim_{n\to\infty}\|(u_n, v_n)\|_{\mathcal{H}}^2 = \|(u_1, v_1)\|_{\mathcal{H}}^2$. The standard argument shows that $(u_n, v_n) \to (u_1, v_1)$ strongly in $\mathcal{H}$.

Next, we claim that $(u_1, v_1) \in \mathcal{N}_{\eta,\alpha,\beta}^+$. In fact, if $(u_1, v_1) \in \mathcal{N}_{\eta,\alpha,\beta}^-$, by Lemma 4.5 (ii), there are unique $t_1^+$and $t_1^- > 0$ such that $\left(t_1^+ u_1, t_1^+ v_1\right) \in \mathcal{N}_{\eta,\alpha,\beta}^+$, $\left(t_1^- u_1, t_1^- v_1\right) \in \mathcal{N}_{\eta,\alpha,\beta}^-$ and $t_1^+ < t_1^- = 1$. Since $\frac{d}{dt} I_{\eta,\alpha,\beta}\left(t_1^+ u_1, t_1^+ v_1\right) = 0$ and $\frac{d^2}{dt^2} I_{\eta,\alpha,\beta}\left(t_1^+ u_1, t_1^+ v_1\right) > 0$, there exists $t_1^* \in \left(t_1^+, t_1^-\right)$ such that

$$I_{\eta,\alpha,\beta}\left(t_1^+ u_1, t_1^+ v_1\right) < I_{\eta,\alpha,\beta}\left(t_1^* u_1, t_1^* v_1\right).$$

By Lemma 4.5, it follows that

$$I_{\eta,\alpha,\beta}\left(t_1^+ u_1, t_1^+ v_1\right) < I_{\eta,\alpha,\beta}\left(t_1^* u_1, t_1^* v_1\right) \leq I_{\eta,\alpha,\beta}\left(t_1^- u_1, t_1^- v_1\right) = I_{\eta,\alpha,\beta}(u_1, v_1),$$

which contradicts $I_{\eta,\alpha,\beta}(u_1, v_1) = c_{\eta,\alpha,\beta}$. Moreover, since $I_{\eta,\alpha,\beta}(u_1, v_1) = I_{\eta,\alpha,\beta}(|u_1|, |v_1|)$ and $(|u_1|, |v_1|) \in \mathcal{N}_{\eta,\alpha,\beta}^+$, we may assume that (u_1, v_1) is a nonnegative nontrivial solution of system (1.1). By means of Bony's maximum principle [5], such solution turn out to be strictly positive. $\square$

Now we establish the existence of a local minimizer of $I_{\eta,\alpha,\beta}$ on $\mathcal{N}_{\eta,\alpha,\beta}^-$.

Lemma 5.3. *Assume that $\mathcal{A}_0$ hold. Then, there exist $(u_0, v_0) \in \mathcal{H}\backslash\{(0,0)\}$ and $\Lambda_* > 0$ such that for all $(\lambda, \mu) \in \mathfrak{D}_{\Lambda_*}$, the following holds:*

$$\sup_{t\geq 0} I_{\eta,\alpha,\beta}(tu_0, tv_0) < c_\infty, \tag{5.5}$$

where c_∞ is a constant given in (3.12). *In particular, $c^-_{\eta,\alpha,\beta} < c_\infty$ for all $(\lambda, \mu) \in \mathfrak{D}_{\Lambda_*}$.*

Proof. Without loss of generality, we assume that $0 \in \Omega$. Let $R \in (0, r_0)$ be such that the quasi-ball $B_d(0,R) \subset \Omega$, and let a cut-off function $\varphi \in C_0^\infty(B_d(0,R))$ satisfying $0 \leq \varphi \leq 1, \varphi = 1$ in $B_d\left(0, \frac{R}{2}\right)$ and $\varphi = 0$ in $\mathbb{H}_1\backslash B_d(0,R)$. Here r_0 is given in $(\mathcal{A}_2)$.

Now, let $u_\varepsilon(z) = \varphi(z)U_\varepsilon(z)$ and consider the function

$$J_\eta(t) = \frac{t^2}{2}\left(1+t_0^2\right)\|u_\varepsilon\|^2_{S_0^1(\Omega)} - \frac{t^4}{4}\left(\eta_1 t_0^{\beta_1} + \eta_2 t_0^{\beta_2}\right)\int_\Omega |u_\varepsilon|^4 dz, \tag{5.6}$$

where t_0 be given in Theorem 2.1. By Lemma 2.2 and the definition of $S_{\eta,\alpha,\beta}$, we obtain that

$$\begin{aligned}
\sup_{t\geq 0} J_\eta(t) &\leq \frac{1}{4}\left(\frac{\left(1+t_0^2\right)\|u_\varepsilon\|^2_{S_0^1(\Omega)}}{\left(\eta_1 t_0^{\beta_1} + \eta_2 t_0^{\beta_2}\right)^{\frac{1}{2}}\left(\int_\Omega |u_\varepsilon|^4 dz\right)^{\frac{1}{2}}}\right)^2 \\
&\leq \frac{1}{4}\left(\mathfrak{h}(t_0)\frac{\|u_\varepsilon\|^2_{S_0^1(\Omega)}}{\left(\int_\Omega |u_\varepsilon|^4 dz\right)^{\frac{1}{2}}}\right)^2 \\
&= \frac{1}{4}\left(\mathfrak{h}(t_0)\frac{S^2_{\mathbb{H}_1} + +O(\varepsilon^2)}{\left(S^2_{\mathbb{H}_1} + O(\varepsilon^4)\right)^{\frac{1}{2}}}\right)^2 \\
&= \frac{1}{4}\left(\mathfrak{h}(t_0)S_{\mathbb{H}_1}\right)^2 + c_1\varepsilon^2 \\
&= \frac{1}{4}S^2_{\eta,\alpha,\beta} + c_1\varepsilon^2,
\end{aligned} \tag{5.7}$$

where c_1 is a positive constant and the following fact has been used:

$$\sup_{t\geq 0}\left(\frac{t^2}{2}A - \frac{t^4}{4}B\right) = \frac{1}{4}\frac{A^2}{B} \quad \text{for all } A, B > 0.$$

Choosing $\Lambda_1 > 0$ such that $0 < \lambda\|g\|_{L^{\frac{4}{4-q}}} + \mu\|h\|_{L^{\frac{4}{4-q}}} < \Lambda_1$, by the definitions of $I_{\eta,\alpha,\beta}$, there exists $t_m \in (0,1)$ such that

$$I_{\eta,\alpha,\beta}(tu_\varepsilon, tt_0u_\varepsilon) \leq \frac{t^2}{2}\left(1+t_0^2\right)\|u_\varepsilon\|^2_{S_0^1(\Omega)} < c_\infty \quad \text{for all } t < t_m,$$

and we have

$$\sup_{0\le t<t_m} I_{\eta,\alpha,\beta}\left(tu_\varepsilon, tt_0u_\varepsilon\right) < c_\infty \tag{5.8}$$

for all $\lambda,\mu\in(0,+\infty)$ with

$$0<\lambda\|g\|_{L^{\frac{4}{4-q}}}+\mu\|h\|_{L^{\frac{4}{4-q}}}<\Lambda_1.$$

Moreover, by the definitions of $I_{\eta,\alpha,\beta}$ and $(u_\varepsilon, t_0u_\varepsilon)$, using the condition $(\mathcal{A}_2)$, Lemma 2.3 and (5.7), we have

$$\begin{aligned}\sup_{t\ge t_m} I_{\eta,\alpha,\beta}\left(tu_\varepsilon, tt_0u_\varepsilon\right) &= \sup_{t\ge t_m}\left(J_\lambda(t)-\frac{t^q}{q}\int_\Omega\left(\lambda g(z)+\mu h(z)t_0^q\right)|u_\varepsilon|^q dz\right)\\ &\le \frac{1}{4}S^2_{\eta,\alpha,\beta}+c_1\varepsilon^2-\frac{t_m^q}{q}a_0\left(\lambda+\mu t_0^q\right)\int_\Omega |u_\varepsilon|^q dz\\ &\le \frac{1}{4}S^2_{\eta,\alpha,\beta}+c_1\varepsilon^2\\ &\quad -\frac{t_m^q}{q}a_0\left(\lambda+\mu t_0^q\right)\begin{cases} c_2\varepsilon^2|\ln\varepsilon|, & \text{if } q=2,\\ c_3\varepsilon^q, & \text{if } q<2,\end{cases}\end{aligned} \tag{5.9}$$

where c_2, c_3 are positive constants.

For $1<q<2$ and $\varepsilon>0$ small enough, we can choose $\Lambda_2>0$ such that

$$0<\sup_{t\ge t_m} I_{\eta,\alpha,\beta}\left(tu_\varepsilon, tt_0u_\varepsilon\right)\le \frac{1}{4}S^{\frac{q}{2}}_{\eta,\alpha,\beta}+c_1\varepsilon^2-\frac{t_0^q}{q}a_0c_4\varepsilon^q<c_\infty \tag{5.10}$$

for all $\lambda,\mu\in(0,+\infty)$ with $0<\lambda\|g\|_{L^{\frac{4}{4-q}}}+\mu\|h\|_{L^{\frac{4}{4-q}}}<\Lambda_2$.

Thus, taking $\Lambda_3=\min\{\Lambda_1,\Lambda_2\}$, (5.8) and (5.10) induce that $\sup_{t\ge 0} I_{\eta,\alpha,\beta}\left(tu_\varepsilon, tt_0u_\varepsilon\right)<c_\infty$ holds for all $\lambda,\mu\in(0,+\infty)$ with

$$0<\lambda\|g\|_{L^{\frac{4}{4-q}}}+\mu\|h\|_{L^{\frac{4}{4-q}}}<\Lambda_3.$$

Finally, we prove that $c^-_{\eta,\alpha,\beta}<c_\infty$ for all $\lambda,\mu\in(0,+\infty)$ with $(\lambda,\mu)\in\mathfrak{D}_{\Lambda_3}$. Recall that $(u_0,v_0):=(u_\varepsilon,t_0u_\varepsilon)$. It is easy to see that $K_\eta\left(u_\varepsilon,t_0u_\varepsilon\right)>0$. Then, combining (5.5) with Lemma 4.5, and using the definition of $c^-_{\eta,\alpha,\beta}$, we obtain that there exists $t_2^->0$ such that $\left(t_2^-u_0,t_2^-v_0\right)\in\mathcal{N}^-_{\eta,\alpha,\beta}$ and

$$c^-_{\eta,\alpha,\beta}\le I_{\eta,\alpha,\beta}\left(t_2^-u_0,t_2^-v_0\right)\le\sup_{t\ge0}I_{\eta,\alpha,\beta}\left(tu_0,tv_0\right)<c_\infty$$

for all $\lambda,\mu\in(0,+\infty)$ with $(\lambda,\mu)\in\mathfrak{D}_{\Lambda_3}$. The proof is completed. □

Theorem 5.4. *Under the assumptions of Theorem* 1.2. *If* $(\lambda,\mu)\in\mathfrak{D}_{\Lambda_*}$, *then the functional* $I_{\eta,\alpha,\beta}$ *has a minimizer* $(u_2,v_2)\in\mathcal{N}^-_{\eta,\alpha,\beta}$ *and it satisfies* $I_{\eta,\alpha,\beta}\left(u_2,v_2\right)=c^-_{\eta,\alpha,\beta}$, *and* (u_2,v_2) *is a positive solution of* (1.1), *where* $\Lambda_*=\min\left\{\Lambda_3,\frac{q}{2}\Lambda\right\}$.

Proof. By Lemma 5.1(ii), there exists a minimizing sequence $\{(u_n, v_n)\} \subset \mathcal{N}^-_{\eta,\alpha,\beta}$in $\mathcal{H}$ for $I_{\eta,\alpha,\beta}$, for all $(\lambda,\mu) \in \mathbb{R}^+ \times \mathbb{R}^+$satisfying

$$0 < \lambda \|g\|_{L^{\frac{4}{4-q}}} + \mu \|h\|_{L^{\frac{4}{4-q}}} < \frac{q}{2}\Lambda.$$

In the light of Lemmas 5.3, 3.4 and 5.1(ii), for $0 < \lambda \|g\|_{L^{\frac{4}{4-q}}} + \mu \|h\|_{L^{\frac{4}{4-q}}} < \Lambda_*$, the functional $I_{\eta,\alpha,\beta}$ satisfies $(PS)_{c^-_{\eta,\alpha,\beta}}$condition for $c^-_{\eta,\alpha,\beta} > 0$. Since $I_{\eta,\alpha,\beta}$ is coercive on $\mathcal{N}_{\eta,\alpha,\beta}$, we can deduce that $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{N}_{\eta,\alpha,\beta}$ and $\mathcal{H}$. So, there exists a subsequence still denoted by $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ and $(u_2, v_2) \in \mathcal{N}^-_{\eta,\alpha,\beta}$ such that $(u_n, v_n) \to (u_2, v_2)$ strongly in $\mathcal{H}$, and $I_{\eta,\alpha,\beta}(u_2, v_2) = c^-_{\eta,\alpha,\beta} > 0, I'_{\eta,\alpha,\beta}(u_2, v_2) = 0$ for all $(\lambda,\mu) \in \mathbb{R}^+ \times \mathbb{R}^+$with

$$0 < \lambda \|g\|_{L^{\frac{4}{4-q}}} + \mu \|h\|_{L^{\frac{4}{4-q}}} < \Lambda_*.$$

Finally, arguing as in the proof of Theorem 5.2 , we have that (u_2, v_2) is a positive solution of the system (1.1). □

Proof of Theorem 1.2: By Theorem 5.2, we obtain that for all $(\lambda,\mu) \in \mathfrak{D}_\Lambda$, Problem (1.1) has a positive solution $(u_1, v_1) \in \mathcal{N}^+_{\eta,\alpha,\beta}$. By Theorem 5.4 , we obtain a second positive solution $(u_2, v_2) \in \mathcal{N}^-_{\eta,\alpha,\beta}$ for all $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_*} \subset \mathfrak{D}_\Lambda$. Since $\mathcal{N}^+_{\eta,\alpha,\beta} \cap \mathcal{N}^-_{\eta,\alpha,\beta} = \emptyset$, this implies that (u_1, v_1) and (u_2, v_2) are distinct.

References

[1] CO. Alves and YH. Ding, *Multiplicity of positive solutions for the p-Laplacian equation involving critical nonlinearity,* J. Math. Anal. Appl., **279** (2003), 508–521.

[2] S. Benmouloud and R. Echarghaouiand SM. Sba, *Multiplicity of positive solutions for a critical quasilinear elliptic system with concave and convex nonlinearities,* J. Math. Anal. Appl., **396**(1) (2012), 375–385.

[3] I. Birindelli, F. Ferrari and E. Valdinoci, *Semilinear PDEs in the Heisenberg group: the role of the right invariant vector fields,* Nonlinear Anal., **72** (2010), 987–997.

[4] A. Bonfiglioli and F. Uguzzoni, *Nonlinear Liouville theorems for some critical problems on H-type groups,* J. Funct. Anal., **207** (2004), 161–215.

[5] JM. Bony, *Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés,* Ann. Inst. Fourier (Geenobles), **19** (1969), 277–304.

[6] KJ. Brown and TF. Wu, *A semilinear elliptic system involving nonlinear boundary condition and sign changing weight function,* J. Math. Anal. Appl., **337** (2008), 1326–1336.

[7] L. Capogna, D. Danielli, SD. Pauls and JT. Tyson, *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem,* Birkhuser-Verlag, Berlin, 2007.

[8] R. Echarghaoui, A. Hatimi and M. Hatimi, *A Fractional Elliptic System With Strongly Coupled Critical Terms and Concave-Convex Nonlinearities,* Int. J. Anal. Appl., **22** (2024), 107.

[9] GB. Folland, *Sub-elliptic estimates and function spaces on nilpotent Lie groups,* Ark. Mat., **13** (1975), 161–207.

[10] TS. Hsu, *Multiple positive solutions for a critical quasilinear elliptic system with concave-convex nonlinearities,* Nonlinear Anal., **71** (2009), 2688–2698.

[11] TS. Hsu and HL. Lin, *Multiple positive solutions for a critical elliptic system with concave-convex nonlinearities,* Proc. Roy. Soc. Edinburgh., **139A** (2009), 1163–1177.

[12] A. Loiudice, *Semilinear subelliptic problems with critical growth on Carnot groups,* Manuscript Math., **124** (2007), 247–259.

[13] A. Loiudice, *Critical growth problems with singular nonlinearities on Carnot groups,* Nonlinear Anal., **126** (2015), 415–436.

[14] P. Pucci, *Existence and multiplicity results for quasilinear equations in the Heisenberg group,* Opuscula Math., **39**(2) (2019), 247–257.

[15] F. Safari and A. Razani, *Existence of positive radial solutions for Neumann problem on the Heisenberg group,* Bound. value prob., 2020; No. 88.

[16] G. Tarantello, *On nonhomogeneous elliptic involving critical Sobolev exponent,* Ann. Inst. H. Poincar Anal. Non Linaire, **9** (1992), 281–304.

[17] D. Vassilev, *Existence of solutions and regularity near the characteristic boundary for subLaplacian equation on Carnot Groups,* Pacific J. Math.,**227**(2) (2007), 361–397.

[18] W. Willem, *Minimax Theorems. Progress in Nonlinear Differential Equations and their Applications,* **24**, 1996 (Boston, MA: Birkhuser Boston Inc.).

[19] TF. Wu, *On semilinear elliptic equations involving concave-convex nonlinearities and sign changing weight function,* J. Math. Anal. Appl., **318** (2006), 253–270.

[20] J. Zhang and TS. Hsu, *Nonlocal elliptic systems involving critical Sobolev-Hardy exponents and concave-convex nonlinearities,* Taiwanese J. Math., **23**(6) (2019), 1479–1510.

[21] X. Zhou, HY. Li and JF. Liao, *Multiplicity of Positive Solutions for a Semilinear Elliptic System with Strongly Coupled Critical Terms and Concave Nonlinearities,* Qualitative Theory of Dyna. Syst., **22**(4) (2023):126.