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HYERS-ULAM STABILITY OF ANTIDERIVATION IN BANACH ALGEBRA VIA FIXED POINT THEOREM

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Abstract. In this article, we introduce the concept of antiderivation in complex algebras and investigate the Hyers-Ulam stability of antiderivation in Banach algebras A, associated to the ρ-functional inequality:

$$\begin{aligned}&\|f(\sigma+\tau+\nu)-f(\sigma+\nu)-f(\sigma+\tau-\nu)+f(\sigma-\nu)\|\\&\quad\leq\|\rho\left(f(\sigma-\tau+\nu)-f(\sigma+\nu)-f(\sigma-\tau-\nu)+f(\sigma-\nu)\right)\|\end{aligned}$$

for all $\sigma,\tau,\nu\in A$ with $|\rho|<1$ by using the fixed point method.

1. Introduction

The concept of stability was started of Ulam [33], followed by Hyers [18, 19] and it has been widely studied in mathematics and functional equations(see [19, 25, 26, 27, 29, 32]).

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In the past years, a number of articles and research have been published on several extensions and applications of the stability to a number of mappings and functional equations, for instance, additive mappings, quadratic mappings, homomorphism and derivation mappings and system of functional equations(see [1, 3, 4, 9, 12, 15, 31]).

In old years, the stability of various functional equations have been extensively established by many of reaserchers and there are many interesting results, containing Hadamard homomorphism, ternary antiderivation additive mappingsand system of functional equations concerning this problem(see [2, 5, 6, 7, 8, 13, 21, 28, 30]).

The fixed point technique is one of the methods that can be used to study the stability of functional equations, system of functional equations(see for instance [10, 11, 12, 16, 23]).

Assume that $\mathcal{A}$ is a complex Banach algebra. A $\mathbb{C}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ is a derivation if D satisfies

$$D(\sigma\tau) = D(\sigma)\tau + \sigma D(\tau)$$

for all $\sigma, \tau \in \mathcal{A}$.

For $\sigma, \tau, \nu \in \mathcal{A}$, we consider the following inequality

$$\begin{aligned} &\|f(\sigma+\tau+\nu) - f(\sigma+\nu) - f(\sigma+\tau-\nu) + f(\sigma-\nu))\| \\ &\quad \le \|\rho(f(\sigma-\tau+\nu) - f(\sigma+\nu) - f(\sigma-\tau-\nu) + f(\sigma-\nu))\|, \end{aligned} \tag{1.1}$$

where $|\rho| < 1$.

In this paper, suppose that $\mathcal{A}$ is a complex Banach algebra and ρ is fixed complex number with $0 < |\rho| < 1$.

2. Stability of (1.1)

Applying the fixed point method, we show the stability of (1.1).

Lemma 2.1. *If a mapping $f : \mathcal{A} \to \mathcal{A}$ satisfies $f(0) = 0$ and*

$$\begin{aligned} &\|f(\sigma+\tau+\nu) - f(\sigma+\nu) - f(\sigma+\tau-\nu) + f(\sigma-\nu)\| \\ &\quad \le \|\rho(f(\sigma-\tau+\nu) - f(\sigma+\nu) - f(\sigma-\tau-\nu) + f(\sigma-\nu))\| \end{aligned} \tag{2.1}$$

for all $\sigma, \tau, \nu \in \mathcal{A}$, then the mapping $f : \mathcal{A} \to \mathcal{A}$ is additive.

Proof. Suppose that $f : \mathcal{A} \to \mathcal{A}$ satisfies (2.1). Replacing τ by $-\tau$ in (2.1), we get

$$\begin{aligned} &\|f(\sigma-\tau+\nu) - f(\sigma+\nu) - f(\sigma-\tau-\nu) + f(\sigma-\nu)\| \\ &\quad \le \|\rho(f(\sigma+\tau+\nu) - f(\sigma+\nu) - f(\sigma+\tau-\nu) + f(\sigma-\nu))\| \end{aligned} \tag{2.2}$$

for all $\sigma, \tau, \nu \in \mathcal{A}$. Using (2.1) and (2.2) yields

$$\begin{aligned}&\|f(\sigma+\tau+\nu)-f(\sigma+\nu)-f(\sigma+\tau-\nu)+f(\sigma-\nu)\|\\ &\leq \|\rho^2(f(\sigma+\tau+\nu)-f(\sigma+\nu)-f(\sigma+\tau-\nu)+f(\sigma-\nu))\|\end{aligned}$$

and so

$$f(\sigma+\tau+\nu)-f(\sigma+\nu)-f(\sigma+\tau-\nu)+f(\sigma-\nu)=0 \tag{2.3}$$

for all $\sigma, \tau, \nu \in \mathcal{A}$, since $|\rho| < 1$.

Letting $\nu = \sigma$ in (2.3), then

$$f(2\sigma+\tau)-f(2\sigma)-f(\tau)=0, \qquad \sigma, \tau \in \mathcal{A}.$$

Thus, f is additive. □

Theorem 2.2. *Assume that $\Psi : \mathcal{A}^3 \to [0,\infty)$ is a function such that there is an $L < 1$ with*

$$\Psi\left(\frac{\sigma}{2}, \frac{\tau}{2}, \frac{\nu}{2}\right) \leq \frac{L}{2}\Psi(\sigma,\tau,\nu) \tag{2.4}$$

for all $\sigma, \tau, \nu \in \mathcal{A}$. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping satisfying $f(0) = 0$ and

$$\begin{aligned}&\|f(\sigma+\tau+\nu)-f(\sigma+\nu)-f(\sigma+\tau-\nu)+f(\sigma-\nu)\|+|\rho|\Psi(\sigma,-\tau,\nu)\\ &\leq \|\rho(f(\sigma-\tau+\nu)-f(\sigma+\nu)-f(\sigma-\tau-\nu)+f(\sigma-\nu))\|+\Psi(\sigma,\tau,\nu)\end{aligned} \tag{2.5}$$

for all $\sigma, \tau, \nu \in \mathcal{A}$. Then there is a unique additive mapping $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(\sigma)-\delta(\sigma)\| \leq \frac{L}{2(1-L)}\Psi\left(\frac{\sigma}{2}, \sigma, \frac{\sigma}{2}\right) \tag{2.6}$$

for all $\sigma \in \mathcal{A}$.

Proof. Replace τ by $-\tau$ in (2.5), we have

$$\begin{aligned}&\|f(\sigma-\tau+\nu)-f(\sigma+\nu)-f(\sigma-\tau-\nu)+f(\sigma-\nu)\|+|\rho|\Psi(\sigma,\tau,\nu)\\ &\leq \|\rho(f(\sigma+\tau+\nu)-f(\sigma+\nu)-f(\sigma+\tau-\nu)+f(\sigma-\nu))\|+\Psi(\sigma,-\tau,\nu)\end{aligned} \tag{2.7}$$

for all $\sigma, \tau, \nu \in \mathcal{A}$. From (2.5) and (2.7), we arrive at

$$\|f(\sigma+\tau+\nu)-f(\sigma+\nu)-f(\sigma+\tau-\nu)+f(\sigma-\nu)\| \leq \Psi(\sigma,\tau,\nu) \tag{2.8}$$

for all $\sigma, \tau, \nu \in \mathcal{A}$.

Letting $\sigma = \nu = \frac{u}{2}$ and $\tau = u$ in (2.8), then we have

$$\|f(2u)-2f(u)\| \leq \Psi\left(\frac{u}{2}, u, \frac{u}{2}\right) \tag{2.9}$$

for all $u \in \mathcal{A}$.

Setting

$$\Omega := \{h : \mathcal{A} \to \mathcal{A} : h(0) = 0\}$$

and define $d : \Omega \times \Omega \to \mathbb{R}$ by

$$d(g,h) = \inf\left\{\eta \in \mathbb{R}_+ : \|g(\sigma) - h(\sigma)\| \leq \eta\Psi\left(\frac{\sigma}{2}, \sigma, \frac{\sigma}{2}\right), \forall \sigma \in \mathcal{A}\right\},$$

and taking $\inf \emptyset = +\infty$. It is easy to show that (Ω, d) is complete(see [22]).

W define the linear mapping $\mathcal{J} : \Omega \to \Omega$ by

$$\mathcal{J}g(\sigma) := 2g\left(\frac{\sigma}{2}\right)$$

for all $\sigma \in \mathcal{A}$.

Let $g, h \in \Omega$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(\sigma) - h(\sigma)\| \leq \varepsilon\Psi\left(\frac{\sigma}{2}, \sigma, \frac{\sigma}{2}\right)$$

for all $\sigma \in \mathcal{A}$. Hence

$$\begin{aligned}\|\mathcal{J}g(\sigma) - \mathcal{J}h(\sigma)\| &= \left\|2g\left(\frac{\sigma}{2}\right) - 2h\left(\frac{\sigma}{2}\right)\right\| \\ &\leq 2\varepsilon\Psi\left(\frac{\sigma}{4}, \frac{\sigma}{2}, \frac{\sigma}{4}\right) \\ &\leq L\varepsilon\Psi\left(\frac{\sigma}{2}, \sigma, \frac{\sigma}{2}\right)\end{aligned}$$

for all $\sigma \in \mathcal{A}$. Thus $d(g,h) = \varepsilon$ implies that $d(\mathcal{J}g(\sigma), \mathcal{J}h(\sigma)) \leq L\varepsilon$. Hence

$$d(\mathcal{J}g(\sigma), \mathcal{J}h(\sigma)) \leq Ld(g,h)$$

for all $g, h \in \Omega$. From (2.9), we see that

$$\begin{aligned}\left\|f(\sigma) - 2f\left(\frac{\sigma}{2}\right)\right\| &\leq \Psi\left(\frac{\sigma}{4}, \frac{\sigma}{2}, \frac{\sigma}{4}\right) \\ &\leq \frac{L}{2}\Psi\left(\frac{\sigma}{2}, \sigma, \frac{\sigma}{2}\right)\end{aligned}$$

for all $\sigma \in \mathcal{A}$. So $d(f, \mathcal{J}f) \leq \frac{L}{2}$.

By alternative fixed point theorem in [14] there exists a mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying the following:

(1) δ is a fixed point of $\mathcal{J}$, that is,

$$\delta(\sigma) = 2\delta\left(\frac{\sigma}{2}\right) \tag{2.10}$$

for all $\sigma \in \mathcal{A}$. The mapping δ is a unique fixed point of $\mathcal{J}$ in the set

$$\Theta = \{g \in \Omega : d(f,g) < \infty\}.$$

It follows that δ is a unique mapping satisfying (2.10) such that there is an $\eta \in (0, \infty)$ satisfying

$$\|f(\sigma) - \delta(\sigma)\| \leq \eta \Psi\left(\frac{\sigma}{2}, \sigma, \frac{\sigma}{2}\right)$$

for all $\sigma \in \mathcal{A}$.

(2) Since $\lim_{n\to\infty} d(\mathcal{J}^n f, \delta) = 0$,

$$\lim_{n\to\infty} 2^n f\left(\frac{\sigma}{2^n}\right) = \delta(\sigma) \tag{2.11}$$

for all $\sigma \in \mathcal{A}$.

(3) $d(f, \delta) \leq \frac{1}{1-L} d(f, \mathcal{J}f)$, which implies

$$\|f(\sigma) - \delta(\sigma)\| \leq \frac{L}{2(1-L)} \Psi\left(\frac{\sigma}{2}, \sigma, \frac{\sigma}{2}\right)$$

for all $\sigma \in \mathcal{A}$.

It follows from (2.4) and (2.5) that

$$\begin{aligned}
&\|\delta(\sigma+\tau+\nu) - \delta(\sigma+\nu) - \delta(\sigma+\tau-\nu) + \delta(\sigma-\nu)\| \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\frac{\sigma+\tau+\nu}{2^n}\right) - f\left(\frac{\sigma+\nu}{2^n}\right) - f\left(\frac{\sigma+\tau-\nu}{2^n}\right) + f\left(\frac{\sigma-\nu}{2^n}\right) \right\| \\
&\leq \lim_{n\to\infty} 2^n |\rho| \left\| f\left(\frac{\sigma-\tau+\nu}{2^n}\right) - f\left(\frac{\sigma+\nu}{2^n}\right) - f\left(\frac{\sigma-\tau-\nu}{2^n}\right) + f\left(\frac{\sigma-\nu}{2^n}\right) \right\| \\
&\quad + \lim_{n\to\infty} 2^n \left(\Psi\left(\frac{\sigma}{2^n}, \frac{\tau}{2^n}, \frac{\nu}{2^n}\right) - \rho \Psi\left(\frac{\sigma}{2^n}, \frac{-\tau}{2^n}, \frac{\nu}{2^n}\right) \right) \\
&\leq \|\rho(\delta(\sigma-\tau+\nu) - \delta(\sigma+\nu) - \delta(\sigma-\tau-\nu) + \delta(\sigma-\nu))\|
\end{aligned}$$

for all $\sigma, \tau, \nu \in \mathcal{A}$. Therefore by Lemma 2.1, the mapping δ is additive. □

Corollary 2.3. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&\|f(\sigma+\tau+\nu) - f(\sigma+\nu) - f(\sigma+\tau-\nu) + f(\sigma-\nu)\| \\
&\quad \leq \|\rho(f(\sigma-\tau+\nu) - f(\sigma+\nu) - f(\sigma-\tau-\nu) + f(\sigma-\nu))\| \\
&\qquad + (1 - |\rho|)\left(\|\sigma^2\| + \|\tau^2\| + \|\nu^2\|\right)
\end{aligned}$$

for all $\sigma, \tau, \nu \in \mathcal{A}$. Then there exists a unique additive mapping $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(\sigma) - \delta(\sigma)\| \leq \|\sigma\|^2$$

for all $\sigma \in \mathcal{A}$.

Proof. The proof follows from Theorem 2.2 by taking $\Psi(\sigma,\tau,\nu) = \|\sigma^2\| + \|\tau^2\| + \|\nu^2\|$ for all $\sigma,\tau,\nu \in \mathcal{A}$. Choosing $L = \frac{4}{7}$, we gain the desired result. □

Corollary 2.4. *If a mapping $f : \mathcal{A} \to \mathcal{A}$ fulfills $f(0) = 0$ and*

$$\begin{aligned}&\|f(\sigma+\tau+\nu)-f(\sigma+\nu)-f(\sigma+\tau-\nu)+f(\sigma-\nu)\|\\ &\quad\le \|\rho(f(\sigma-\tau+\nu)-f(\sigma+\nu)-f(\sigma-\tau-\nu)+f(\sigma-\nu))\|+(1-|\rho|)\|\sigma\tau\nu\|\end{aligned}$$

for all $\sigma,\tau,\nu \in \mathcal{A}$, then there is a unique additive mapping $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(\sigma)-\delta(\sigma)\| \le \|\sigma\|^3$$

for all $\sigma \in \mathcal{A}$.

Proof. This proof follows from Theorem 2.2 by setting $\Psi(\sigma,\tau,\nu) = \|\sigma\tau\nu\|$ for all $\sigma,\tau,\nu \in \mathcal{A}$. Choosing $L = \frac{8}{9}$, we gain the desired result. □

Corollary 2.5. *Assume that $f : \mathcal{A} \to \mathcal{A}$ is an odd mapping satisfying*

$$\begin{aligned}&\|f(\sigma+\tau+\nu)-f(\sigma+\nu)-f(\sigma+\tau-\nu)+f(\sigma-\nu)\|\\ &\le \|\rho(f(\sigma-\tau+\nu)-f(\sigma+\nu)-f(\sigma-\tau-\nu)+f(\sigma-\nu))\|+(1-|\rho|)\|\sigma\tau\nu\|\end{aligned} \tag{2.12}$$

for all $\sigma,\tau,\nu \in \mathcal{A}$. Then f is additive.

Proof. Putting $\sigma = 0$ in (2.12), we deduce that

$$\begin{aligned}&\|f(\tau+\nu)-f(\nu)-f(\tau-\nu)+f(-\nu)\|\\ &\quad\le \|\rho(f(-\tau+\nu)-f(\nu)-f(-\tau-\nu)+f(-\nu))\|\end{aligned} \tag{2.13}$$

for all $\tau,\nu \in \mathcal{A}$. Replace τ by $-\tau$ in (2.13) to get

$$\begin{aligned}&\|f(-\tau+\nu)-f(\nu)-f(-\tau-\nu)+f(-\nu)\|\\ &\quad\le \|\rho(f(\tau+\nu)-f(\nu)-f(\tau-\nu)+f(-\nu))\|\end{aligned} \tag{2.14}$$

for all $\tau,\nu \in \mathcal{A}$. From (2.13) and (2.14), it follows that

$$f(\tau+\nu)-f(\nu)-f(\tau-\nu)+f(-\nu) = 0$$

for all $\tau,\nu \in \mathcal{A}$. Applying the oddness of the mapping f we see that

$$f(\nu+\tau)+f(\nu-\tau)-2f(\nu) = 0$$

for all $\tau,\nu \in \mathcal{A}$. This means that the mapping f is additive. □

3. Stability of antiderivation in Banach algebras

First, we introduce concept antiderivation in algebras and by applying the fixed point technique, we study the stability of antiderivation related to (1.1) in Banach algebras.

Definition 3.1. Let $\mathcal{A}$ be a complex algebra. A $\mathbb{C}$-linear mapping $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ is named an antiderivation if

$$\mathcal{I}(\sigma)\mathcal{I}(\tau) = \mathcal{I}(\mathcal{I}(\sigma)\tau) + \mathcal{I}(\sigma\mathcal{I}(\tau)), \qquad \sigma, \tau \in \mathcal{A}.$$

Example 3.2. Let $\mathbf{P}_n$ be the set of all polynomials of degree n with complex coefficients and

$$\mathcal{H} = \{p \in \mathbf{P}_n | p(0) = 0,\ n \in \mathbb{N}\}.$$

Define $\mathcal{I} : \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{I}\left(\sum_{\kappa=1}^{n} \vartheta_\kappa x^\kappa\right) = \sum_{\kappa=0}^{n} \frac{\vartheta_\kappa}{\kappa+1} x^{\kappa+1}$$

and $\mathcal{I}(0) = 0$. Then $\mathcal{I}$ is an antiderivation.

Example 3.3. Let $C(\mathbb{R})$ be the set of every continuous functions on $\mathbb{R}$.

Define $\mathcal{I} : C(\mathbb{R}) \to C(\mathbb{R})$ by

$$\mathcal{I}(f(x)) = \int_0^x f(t)dt$$

for all $x \in \mathbb{R}$. Then $\mathcal{I}$ is an antiderivation.

Lemma 3.4. ([24]) *Assume that $\mathcal{A}$ is a complex Banach algebra and $f : \mathcal{A} \to \mathcal{A}$ is an additive mapping such that $f(\lambda\sigma) = \lambda f(\sigma)$ for each $\lambda \in \mathbb{T}^1 := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ and every $\sigma \in \mathcal{A}$. Then f is $\mathbb{C}$-linear.*

Definition 3.5. ([17]) A double sequence $\{\gamma_{n,m}\}$ converges to L and we write $\lim_{n,m\to\infty} \gamma_{n,m} = L$ if for each $\epsilon > 0$ there exists an integer N such that for every $n, m \geq N$,

$$|\gamma_{n,m} - L| < \epsilon.$$

If no such number L exists, we say that $\{\gamma_{n,m}\}$ diverges.

Theorem 3.6. *Suppose that $\Psi : \mathcal{A}^3 \to [0, \infty)$ is a function such that there is an $L < 1$ with*

$$\Psi(\sigma, \tau, \nu) \leq 2L\Psi\left(\frac{\sigma}{2}, \frac{\tau}{2}, \frac{\nu}{2}\right) \tag{3.1}$$

for all $\sigma, \tau, \nu \in \mathcal{A}$. Let $f : \mathcal{A} \to \mathcal{A}$ be a continuous function satisfying $f(0) = 0$ and

$$\begin{aligned} &\|\lambda f(\sigma+\tau+\nu) - f(\lambda(\sigma+\nu)) - f(\lambda(\sigma+\tau-\nu)) + \lambda f(\sigma-\nu)\| \\ &\quad \le \|\rho(\lambda f(\sigma-\tau+\nu) - f(\lambda(\sigma+\nu)) - f(\lambda(\sigma-\tau-\nu)) + \lambda f(\sigma-\nu))\| \\ &\qquad + \Psi(\sigma,\tau,\nu) - |\rho|\Psi(\sigma,-\tau,\nu), \end{aligned} \tag{3.2}$$

$$\|f(\sigma)f(\tau) - f(f(\sigma)\tau) - f(\sigma f(\tau))\| \le \Psi(\sigma,\tau,\sigma) \tag{3.3}$$

for all $\lambda \in \mathbb{T}^1$ and all $\sigma, \tau, \nu \in \mathcal{A}$, and, in addition, $\{f_n(\sigma)\} := \left\{\frac{1}{2^n} f(2^n\sigma)\right\}$ converges uniformly for all $\sigma \in \mathcal{A}$, double sequences $\left\{\frac{1}{2^{n+m}} f(f(2^n\sigma)2^m\tau)\right\}$ and $\left\{\frac{1}{2^{n+m}} f(2^n\sigma f(2^m\tau))\right\}$ are convergent for all $\sigma, \tau \in \mathcal{A}$. Then there is a unique continuous antiderivation $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(\sigma) - \mathcal{I}(\sigma)\| \le \frac{1}{2(1-L)} \Psi\left(\frac{\sigma}{2}, \sigma, -\frac{\sigma}{2}\right) \tag{3.4}$$

for all $\sigma \in \mathcal{A}$.

Proof. Letting τ by $-\tau$ in (3.2), we obtain

$$\begin{aligned} &\|\lambda f(\sigma-\tau+\nu) - f(\lambda(\sigma+\nu)) - f(\lambda(\sigma-\tau-\nu)) + \lambda f(\sigma-\nu)\| + |\rho|\Psi(\sigma,\tau,\nu) \\ &\quad \le \|\rho(\lambda f(\sigma+\tau+\nu) - f(\lambda(\sigma+\nu)) - f(\lambda(\sigma+\tau-\nu)) + \lambda f(\sigma-\nu))\| \\ &\qquad + \Psi(\sigma,-\tau,\nu) \end{aligned} \tag{3.5}$$

for all $\lambda \in \mathbb{T}^1$ and all $\sigma, \tau, \nu \in \mathcal{A}$. From (3.2) and (3.5), we deduce that

$$\begin{aligned} &\|\lambda f(\sigma+\tau+\nu) - f(\lambda(\sigma+\nu)) - f(\lambda(\sigma+\tau-\nu)) + \lambda f(\sigma-\nu)\| \\ &\quad \le \Psi(\sigma,\tau,\nu) \end{aligned} \tag{3.6}$$

for all $\lambda \in \mathbb{T}^1$ and all $\sigma, \tau, \nu \in \mathcal{A}$.

Setting $\sigma = \frac{u}{2}, \tau = u$, $\nu = -\frac{u}{2}$ and $\lambda = 1$ in (3.6), we have

$$\|f(2u) - 2f(u)\| \le \Psi\left(\frac{u}{2}, u, -\frac{u}{2}\right) \tag{3.7}$$

for all $u \in \mathcal{A}$.

Define the function $d : \Omega \times \Omega \to \mathbb{R}$ by

$$d(\theta,\omega) = \inf\left\{\eta \in \mathbb{R}_+ : \|\theta(\sigma) - \omega(\sigma)\| \le \eta\Psi\left(\frac{\sigma}{2}, \sigma, -\frac{\sigma}{2}\right),\ \forall\sigma \in \mathcal{A}\right\},$$

where

$$\Omega := \{\omega : \mathcal{A} \to \mathcal{A} : \omega(0) = 0\}$$

and $\inf \emptyset = +\infty$. Then it is easy to prove that (Ω, d) is complete(see [22]).

Define the linear mapping $\mathcal{J} : \Omega \to \Omega$ as follows:

$$\mathcal{J}\theta(\sigma) = \frac{1}{2}\theta(2\sigma), \quad \sigma \in \mathcal{A}.$$

Assume that $\omega, \theta \in \Omega$ is given so that $d(\omega,\theta)=\varepsilon$. Then

$$\|\omega(\sigma)-\theta(\sigma)\| \le \varepsilon\Psi\left(\frac{\sigma}{2},\sigma,-\frac{\sigma}{2}\right)$$

for all $\sigma \in \mathcal{A}$. Thus,

$$\begin{aligned}\|\mathcal{J}\theta(\sigma)-\mathcal{J}\omega(\sigma)\| &= \left\|\frac{1}{2}\theta(2\sigma)-\frac{1}{2}\omega(2\sigma)\right\| \\ &\le \frac{\varepsilon}{2}\Psi(\sigma,2\sigma,-\sigma) \\ &\le L\varepsilon\Psi\left(\frac{\sigma}{2},\sigma,-\frac{\sigma}{2}\right)\end{aligned}$$

for all $\sigma \in \mathcal{A}$. Therefore $d(\theta,\omega)=\varepsilon$, it follows that $d(\mathcal{J}\theta(\sigma),\mathcal{J}\omega(\sigma)) \le L\varepsilon$. Hence,

$$d(\mathcal{J}\theta(\sigma),\mathcal{J}\omega(\sigma)) \le Ld(\omega,\theta)$$

for all $\theta,\omega \in \Omega$. Using (3.7) yields

$$\left\|f(\sigma)-\frac{1}{2}f(2\sigma)\right\| \le \frac{1}{2}\Psi\left(\frac{\sigma}{2},\sigma,-\frac{\sigma}{2}\right)$$

for all $\sigma \in \mathcal{A}$ and so $d(\mathcal{J}f,f) \le \frac{1}{2}$.

By alternative fixed point theorem(see [14]), there exists a mapping $\mathcal{I}:\mathcal{A}\to\mathcal{A}$ satisfying the following:

(1) $\mathcal{I}$ is a fixed point of $\mathcal{J}$, that is,

$$\mathcal{I}(\sigma)=\frac{1}{2}\mathcal{I}(2\sigma) \tag{3.8}$$

for all $\sigma \in \mathcal{A}$. The mapping $\mathcal{I}$ is a unique fixed point of $\mathcal{J}$ in the set

$$\Theta=\{\theta \in E : d(f,\theta)<\infty\}.$$

This shows that $\mathcal{I}$ is a unique mapping fulfilling (3.8) such that there is an $\eta \in (0,\infty)$ satisfying

$$\|f(\sigma)-\mathcal{I}(\sigma)\| \le \eta\Psi\left(\frac{\sigma}{2},\sigma,-\frac{\sigma}{2}\right), \quad \sigma \in \mathcal{A};$$

(2) Since $\lim\limits_{n\to\infty} d(\mathcal{J}^n f,\mathcal{I})=0$,

$$\lim_{n\to\infty}\frac{1}{2^n}f(2^n\sigma)=\mathcal{I}(\sigma), \qquad \sigma \in \mathcal{A}; \tag{3.9}$$

(3) $d(f,\mathcal{I}) \le \frac{1}{1-L}d(f,\mathcal{J}f)$, it follows that

$$\|f(\sigma)-\mathcal{I}(\sigma)\| \le \frac{1}{2(1-L)}\Psi\left(\frac{\sigma}{2},\sigma,-\frac{\sigma}{2}\right), \quad \sigma \in \mathcal{A}.$$

From (3.1) and (3.6), we gain

$$\begin{aligned}
&\|\lambda\mathcal{I}(\sigma+\tau+\nu)-\mathcal{I}(\lambda(\sigma+\nu))-\mathcal{I}(\lambda(\sigma+\tau-\nu))+\lambda\mathcal{I}(\sigma-\nu)\| \\
&\quad=\lim_{n\to\infty}\frac{1}{2^n}\big\|\lambda f(2^n(\sigma+\tau+\nu))-f(\lambda(2^n(\sigma+\nu))) \\
&\qquad\qquad -f(\lambda(2^n(\sigma+\tau-\nu)))+\lambda f(2^n(\sigma-\nu))\big\| \\
&\quad\leq\lim_{n\to\infty}\frac{1}{2^n}\Psi\left(2^n\sigma,2^n\tau,2^n\nu\right) \\
&\quad\leq\lim_{n\to\infty}L^n\Psi(\sigma,\tau,\nu)
\end{aligned}$$

for all $\lambda\in\mathbb{T}^1$ and all $\sigma\in\mathcal{A}$. Since $L<1$,

$$\lambda\mathcal{I}(\sigma+\tau+\nu)-\mathcal{I}(\lambda(\sigma+\nu))-\mathcal{I}(\lambda(\sigma+\tau-\nu))+\lambda\mathcal{I}(\sigma-\nu)=0 \qquad (3.10)$$

for all $\lambda\in\mathbb{T}^1$ and all $\sigma\in\mathcal{A}$. Set $\lambda=1$ in (3.10). Then by Lemma 2.1, $\mathcal{I}$ is additive.

Now, taking $\tau=\nu=0$ in (3.10), we see that

$$\mathcal{I}(\lambda\sigma)=\lambda\mathcal{I}(\sigma)$$

for every $\lambda\in\mathbb{T}^1$ and every $\sigma\in\mathcal{A}$. By Lemma 3.4, the mapping $\mathcal{I}$ is $\mathbb{C}$-linear.

Since $\{f_n\}$ converges uniformly and f is continuous, $\mathcal{I}$ is continuous. From (3.1) and (3.3), we deduce that

$$\begin{aligned}
&\|\mathcal{I}(\sigma)\mathcal{I}(\tau)-\mathcal{I}(\mathcal{I}(\sigma)\tau)-\mathcal{I}(\sigma\mathcal{I}(\tau))\| \\
&\quad=\lim_{n\to\infty}\frac{1}{4^n}\|f(2^n\sigma)f(2^n\tau)-f(f(2^n\sigma)2^n\tau)-f(2^n\sigma f(2^n\tau))\| \\
&\quad\leq\lim_{n\to\infty}\frac{1}{4^n}\Psi(2^n\sigma,2^n\tau,2^n\sigma) \\
&\quad\leq\lim_{n\to\infty}\left(\frac{L}{2}\right)^n\Psi(\sigma,\tau,\sigma) \\
&\quad\leq\lim_{n\to\infty}\frac{1}{2^n}\Psi(\sigma,\tau,\sigma)
\end{aligned}$$

for all $\sigma,\tau\in\mathcal{A}$. Hence the mapping $\mathcal{I}$ is an antiderivation, since $L<1$. □

Corollary 3.7. *Assume that* $f:\mathcal{A}\to\mathcal{A}$ *is a continuous mapping fulfilling* $f(0)=0$,

$$\begin{aligned}
&\|\lambda f(\sigma+\tau+\nu)-f(\lambda(\sigma+\nu))-f(\lambda(\sigma+\tau-\nu))+\lambda f(\sigma-\nu)\| \\
&\quad\leq\|\rho(\lambda f(\sigma-\tau+\nu)-f(\lambda(\sigma+\nu))-f(\lambda(\sigma-\tau-\nu))+\lambda f(\sigma-\nu))\| \\
&\qquad+\|\sigma+\tau+\nu\|^{\frac{1}{2}}-|\rho|\|\sigma-\tau+\nu\|^{\frac{1}{2}}
\end{aligned}$$

and

$$\|f(\sigma)f(\tau)-f(f(\sigma)\tau)-f(\sigma f(\tau))\|\leq\|2\sigma+\tau\|^{\frac{1}{2}}$$

for all $\sigma, \tau, \nu \in \mathcal{A}$ *and all* $\lambda \in \mathbb{T}^1$. *If the sequence* $\{f_n(\sigma)\} := \left\{\frac{1}{2^n} f(2^n\sigma)\right\}$ *converges uniformly and double sequences* $\left\{\frac{1}{2^{n+m}} f(f(2^n\sigma)2^m\tau)\right\}$ *and* $\left\{\frac{1}{2^{n+m}} f(2^n\sigma f(2^m\tau))\right\}$ *are convergent for all* $\sigma, \tau \in \mathcal{A}$, *then there is a unique continuous antiderivation* $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ *such that*

$$\|f(\sigma) - \mathcal{I}(\sigma)\| \leq \frac{7}{4}\|\sigma\|^{\frac{1}{2}}$$

for all $\sigma \in \mathcal{A}$.

Proof. The proof follows from Theorem 3.6 by letting $\Psi(\sigma, \tau, \nu) = \|\sigma+\tau+\nu\|^{\frac{1}{2}}$ for all $\sigma, \tau, \nu \in \mathcal{A}$. Setting $L = \frac{5}{7}$, we get the desired result. □

Theorem 3.8. *Let* E *be a subset of* $\mathbb{C}$ *with* $\mathbb{T}^1 \subseteq \overline{E}$. *Let* $\Psi : \mathcal{A}^3 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\Psi(\sigma, \tau, \nu) \leq 2L\Psi\left(\frac{\sigma}{2}, \frac{\tau}{2}, \frac{\nu}{2}\right) \tag{3.11}$$

for all $\sigma, \tau, \nu \in \mathcal{A}$. *Let* $f : \mathcal{A} \to \mathcal{A}$ *be a continuous mapping satisfying* $f(0) = 0$, (3.3) *and*

$$\begin{aligned} &\|\lambda f(\sigma+\tau+\nu) - f(\lambda(\sigma+\nu)) - f(\lambda(\sigma+\tau-\nu)) + \lambda f(\sigma-\nu)\| \\ &\quad \leq \|\rho(\lambda f(\sigma-\tau+\nu) - f(\lambda(\sigma+\nu)) - f(\lambda(\sigma-\tau-\nu)) + \lambda f(\sigma-\nu))\| \\ &\qquad + \Psi(\sigma, \tau, \nu) - |\rho|\Psi(\sigma, -\tau, \nu) \end{aligned} \tag{3.12}$$

for all $\sigma, \tau, \nu \in \mathcal{A}$ *and all* $\lambda \in E$ *and, moreover,* $\{f_n(\sigma)\} := \left\{\frac{1}{2^n} f(2^n\sigma)\right\}$ *converges uniformly for all* $\sigma \in \mathcal{A}$, *double sequences* $\left\{\frac{1}{2^{n+m}} f(f(2^n\sigma)2^m\tau)\right\}$ *and* $\left\{\frac{1}{2^{n+m}} f(2^n\sigma f(2^m\tau))\right\}$ *are convergent for all* $\sigma, \tau \in \mathcal{A}$. *Then there is a unique continuous antiderivation* $\mathcal{I} : \mathcal{A} \to \mathcal{A}$ *with*

$$\|f(\sigma) - \mathcal{I}(\sigma)\| \leq \frac{1}{2(1-L)}\Psi\left(\frac{\sigma}{2}, \sigma, -\frac{\sigma}{2}\right) \tag{3.13}$$

for all $\sigma \in \mathcal{A}$.

Proof. Let $\lambda \in \mathbb{T}^1$. Then there exists a sequence $\{\lambda_n\}_{n=1}^{\infty} \subseteq E$ such that

$$\lim_{n\to\infty} \lambda_n = \lambda.$$

By (3.12) we get

$$\begin{aligned} &\|\lambda_n f(\sigma+\tau+\nu) - f(\lambda_n(\sigma+\nu)) - f(\lambda_n(\sigma+\tau-\nu)) + \lambda_n f(\sigma-\nu)\| \\ &\quad \leq \|\rho(\lambda_n f(\sigma-\tau+\nu) - f(\lambda_n(\sigma+\nu)) - f(\lambda_n(\sigma-\tau-\nu)) + \lambda_n f(\sigma-\nu))\| \\ &\qquad + \Psi(\sigma, \tau, \nu) - |\rho|\Psi(\sigma, -\tau, \nu) \end{aligned}$$

for all $\sigma, \tau, \nu \in \mathcal{A}$ and all positive integers n. Passing to the limit as $n \to \infty$, and applying the continuity of f, we arrive at

$$\begin{aligned}
&\|\lambda f(\sigma+\tau+\nu) - f(\lambda(\sigma+\nu)) - f(\lambda(\sigma+\tau-\nu)) + \lambda f(\sigma-\nu)\| \\
&\quad \leq \|\rho(\lambda f(\sigma-\tau+\nu) - f(\lambda(\sigma+\nu)) - f(\lambda(\sigma-\tau-\nu)) + \lambda f(\sigma-\nu))\| \\
&\qquad + \Psi(\sigma,\tau,\nu) - |\rho|\Psi(\sigma,-\tau,\nu)
\end{aligned}$$

for all $\lambda \in \mathbb{T}^1$ and all $\sigma, \tau, \nu \in \mathcal{A}$. Thus with the same argument as in the proof of Theorem 3.6, the proof is complete. □

4. Conclusions

In this note, we introduced the definition of antiderivation mapping on Banach algebra and we studied the stability of antiderivation mappings on Banach algebra by fixed point theorem.

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