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EQUILIBRIUM SOLUTION IN NON-COOPERATIVE QUEUING GAMES ON OPEN NETWORK WITH UN-EQUAL ARRIVAL RATES

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Abstract. This paper studies the equilibrium solution in a class of 2-player queueing games played on a network of a single server. While much research has been done on equilibrium solutions in such games, the case of unequal arrival rates for players has not been thoroughly addressed. The objective of both players is to minimize the mean expected time their clients spend in the system. Each player can choose either a single path or distribute clients over multiple paths. We demonstrate that this game may or may not have a pure strategy Nash equilibrium (PNE) under discrete strategies, using the finite improvement path property. When players can split traffic, existence and uniqueness of PNE is shown using Lagrange multipliers and the KarushKuhnTucker conditions. Pure and mixed strategy equilibria are discussed, along with the price of anarchy. We also explore the congestion game formulation of this queueing setup.

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1. Introduction

The field of operations research has a long history of using queueing theory as a methodology. It offers a crucial tool for studying the dynamics of numerous resource-constrained service systems, including traffic flow systems, computer systems, facility design, communication network systems, manufacturing systems, and scheduling. In such systems, clients often compete for scarce service resources. As a result, a game-theoretic framework emerges, and game theory provides a promising research direction for queueing theory. According to the classification of game theory, the challenge in queueing systems is often a multi-player, stochastic, and non-cooperative game.

Non-cooperative games provide a suitable framework for examining such decentralized queueing systems [5, 18, 22, 23, 26]. While it is well known that a Nash equilibrium (NE) in mixed strategies exists for finite non-cooperative games, a pure strategy Nash equilibrium (PNE) is not necessarily guaranteed. Identifying conditions under which a PNE exists is a core aim of game theory.

This paper studies a specific class of 2-player non-cooperative queueing games played on an open network, where each vertex in the network follows an $M/M/1$ queueing discipline. The arrival rates for both players may be equal or unequal. The objective of each player is to reduce the mean expected time of its clients in the system. Each player must strategically select a path through the network to route clients from a given source to a destination. First, we consider the case where each player selects a single path. This leads to a game with a finite strategy space. A weighted congestion game is a specific case of this [9, 25], and a PNE is not always guaranteed. We then examine games with continuous strategy spaces, where players may divide their clients over multiple paths.

Several works in the literature integrate queueing and game theory models. Beginning with the pioneering work of Naor in 1969 [21], significant attention has been given to game-theoretic analysis of queueing systems [1, 5, 10, 11, 13]. Queueing theory has also been applied in security games and communication networks for optimal routing [1, 16]. The Braess paradox in queueing networks is discussed in [9], and a comprehensive summary of rational behavior in queueing systems appears in [12]. Equilibrium solutions in queueing games have been widely studied [8, 7, 26, 27], and the case of equal arrival rates has been addressed in [15].

The rest of this paper is organized as follows: Section 2 provides an overview of directed graphs, $M/M/1$ queues, and non-cooperative games on networks. Section 3 describes the proposed game model. Section 4 analyzes equilibrium solutions under two cases: discrete strategies (where each player selects a single path) and continuous strategies (where clients can be split across paths). In the continuous case, we show the existence and uniqueness of a PNE using Lagrange multipliers and KKT conditions. Section 5 presents the game as a congestion game and derives the corresponding payoff functions for different arrival scenarios.

2. Preliminaries

This section introduces the framework for non-cooperative games played on networks and provides definitions for the notation used throughout the article.

Definition 2.1. ([18, Directed graph]) A directed graph $D = (V, E)$ consists of a nonempty set of vertices V and a set of directed edges E where $E \subseteq V \times V$. Each edge $e \in E$ is specified by an ordered pair of vertices $u, v \in V$.

Definition 2.2. ([18, Directed network]) A directed network is a directed graph with vertex set V and edge set E.

Definition 2.3. ([15, 18, Directed path]) A directed path in a directed graph D is a sequence $v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n$ such that $v_i \in V$, $e_i \in E$, $0 \leq i \leq n$, all v_i are distinct, and this represents a path from vertex v_0 to v_n.

Definition 2.4. ([12, 21, M/M/1]) The M/M/1 queue (Figure 1) is a classical queueing model with exponentially distributed interarrival times (Poisson arrivals with rate λ), exponentially distributed service times with rate μ, and a single server.

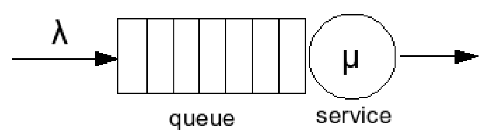

Figure 1. An M/M/1 queueing vertex

Definition 2.5. ([15, 16, 24, Games on networks]) A game played on a network is defined by three components: (a) a network, (b) a strategy space, and (c) payoff functions.

Definition 2.6. ([15, 18, Network]) A network is a directed graph $D = (V, E)$ where each vertex $i \in V = \{1, \ldots, n\}$ is an FCFS (First Come First Served) single-server queue with arrival rate λ_i and service rate μ_i.

Definition 2.7. ([14, 19, Strategy space]) The strategy space for player i is denoted by $S = \times_{i \in V} S_i$, where S_i is the finite pure strategy set for player i. A strategy profile is a vector $x = (x_1, \ldots, x_n) \in S$ such that $x_i \in S_i$ for all $i \in V$.

Definition 2.8. ([14, 19, Payoff function]) When player i competes with others in a stage of the game, he receives a payoff. Let $\pi_{ij}(x_i, x_j)$ denote the payoff that player i receives when playing strategy $x_i \in S_i$ against opponent $j \in N_i$ (the neighbors of i), who selects strategy $x_j \in S_j$.

The aggregate payoff for player i is

$$\pi_i(x_i, x_{-i}) = \sum_{j \in N_i \setminus \{i\}} w_{ij} \pi_{ij}(x_i, x_j), \tag{2.1}$$

where $w_{ij} \in \mathbb{R}$ is the weight corresponding to the local interaction between i and j. This is called a pairwise game on a network.

In contrast, if players interact in groups larger than two, the local encounters form an n-player game. The aggregate payoff becomes

$$\pi_i(x_i, x_{-i}) = \sum_{j \in N_i} w_j \pi_{ij}(x_i, x_{-i}), \tag{2.2}$$

where $w_j \in \mathbb{R}$ is the mutual weight for player j in the local n-player game. A non-cooperative game with such payoffs is called a groupwise game on a network.

The combined objective function is

$$\pi : S \to \mathbb{R}^n, \quad \pi(x) = [\pi_1(x), \ldots, \pi_n(x)]$$

(see [14, 19]).

3. Model description

This section introduces the 2-player non-cooperative game played on a network of $M/M/1$ queues.

Consider a directed network $D = (V, E)$ with n vertices and m edges. Let $K = \{1, 2\}$ be the set of two players, and for each player $j \in K$, let $(s_j, t_j) \in V \times V$ denote the source and sink vertices, respectively, from which the player selects paths for routing clients.

Assume that each vertex $i \in V$ is an $M/M/1$ queue, with service rate μ_i. The arrival rate λ_i at vertex i is determined by the paths chosen by the players.

Each players objective is to minimize the mean expected time of their clients in the system by appropriately distributing them over available paths. Let $R^{(j)}$ denote the set of available paths for player $j \in K$. The arrival process for clients of players 1 and 2 is assumed to follow a Poisson process with rates $\lambda^{(1)}$ and $\lambda^{(2)}$, respectively, where $\lambda^{(1)} \neq \lambda^{(2)}$.

A strategy of player j is represented by a vector $S^{(j)} = (x_r^{(j)})_{r \in R^{(j)}}$, where $x_r^{(j)}$ is the fraction of clients routed through path r. The arrival rate at a vertex $i \in V$ under the full strategy profile $S = (S^{(1)}, S^{(2)})$ is then

$$\lambda_i = \sum_{\substack{r \in R^{(1)} \\ i \in r}} x_r^{(1)} \lambda^{(1)} + \sum_{\substack{r \in R^{(2)} \\ i \in r}} x_r^{(2)} \lambda^{(2)}. \tag{3.1}$$

A strategy profile $S = (S^{(1)}, S^{(2)})$ is said to be feasible if $\lambda_i < \mu_i$ for all $i \in V$. The set of all feasible strategies is denoted by

$$\Omega = \left\{ S \;\middle|\; \sum_{r \in R^{(j)}} x_r^{(j)} = 1,\, x_r^{(j)} \geq 0,\, \lambda_i < \mu_i \text{ for all } i \in V,\, j = 1, 2 \right\}. \tag{3.2}$$

For any $S \in \Omega$, the mean expected time spent at vertex i is given by

$$\frac{1}{\mu_i - \lambda_i}.$$

Thus, the mean expected time for the clients of player j in the system is

$$\pi^{(j)}(S) = \sum_{i=1}^{n} \frac{\sum_{\substack{r \in R^{(j)} \\ i \in r}} x_r^{(j)}}{\mu_i - \lambda_i}, \quad j = 1, 2. \tag{3.3}$$

The total mean expected time for the system is given by

$$\pi(S) = \sum_{j=1}^{2} \pi^{(j)}(S). \tag{3.4}$$

Since both players aim to minimize the mean expected time of their clients, Equation (3.3) defines the individual objective function (payoff) for player j. Therefore, the 2-player non-cooperative game played on the network $D = (V, E)$ is defined by the tuple:

$$G = \left(K, V, S^{(j)}, \pi^{(j)} \right),$$

where

- K is the set of players,
- V is the set of vertices in the network,
- $S^{(j)}$ is the strategy space of player j, and
- $\pi^{(j)}$ is the payoff function of player j as defined in (3.3).

4. Equilibrium analysis

This section studies the existence of equilibrium solutions in the game G. We begin by introducing definitions related to dominant strategies, pure-strategy Nash equilibria (PNE), the concept of the Price of Anarchy (PoA), and the matrix form of payoffs.

Definition 4.1. (Dominant and Best Response Strategies) A strategy $s^{(j)}$ is said to be a *dominant strategy* for player j if it yields the lowest mean expected time in the system for any possible strategy s_{-j} of the opponent:

$$\pi^{(j)}(s^{(j)}, s_{-j}) \leq \pi^{(j)}(\bar{s}^{(j)}, s_{-j}) \quad \text{for all} \;\; \bar{s}^{(j)} \in S^{(j)}.$$

The *best response* of player j to s_{-j} is the strategy that minimizes their cost, defined by:

$$\mathrm{BR}^{(j)}(s_{-j}) = \left\{ \hat{s}^{(j)} \in S^{(j)} \;\middle|\; \pi^{(j)}(\hat{s}^{(j)}, s_{-j}) \leq \pi^{(j)}(\bar{s}^{(j)}, s_{-j}), \; \forall \; \bar{s}^{(j)} \in S^{(j)} \right\}.$$

Definition 4.2. (Pure-Strategy Nash Equilibrium (PNE)) A strategy profile $S = (s^{(1)}, s^{(2)})$ is a *pure-strategy Nash equilibrium* if no player can reduce their own expected time by unilaterally deviating from their strategy. Formally, for each $j \in \{1, 2\}$:

$$s^{(j)} \in \mathrm{BR}^{(j)}(s_{-j}).$$

Definition 4.3. ([20, Finite Improvement Property]) Let S be a strategy profile. An *improvement step* for player j is a change to a strategy $\bar{s}^{(j)}$ such that:

$$\pi^{(j)}(\bar{s}^{(j)}, s_{-j}) < \pi^{(j)}(s^{(j)}, s_{-j}).$$

An *improvement path* is a sequence of profiles $\gamma = (S^1, S^2, \dots)$, where each S^{r+1} results from an improvement step by a single player from S^r. Let p_r denote the player improving at step r.

A game G is said to have the *finite improvement property* (FIP) if all such improvement paths are finite. If a game has FIP, then any sequence of better responses leads to a pure-strategy Nash equilibrium.

Price of Anarchy (PoA). The concept of the *Price of Anarchy* is used to measure the inefficiency of equilibrium outcomes due to selfish behavior. It compares the worst equilibrium outcome to the optimal (socially best) solution.

Although introduced formally by Koutsoupias and Papadimitriou, the idea has roots in earlier work in game theory and economics [2, 3, 10, 19].

Consider the game $G = ([K], V, S^{(j)}, \pi^{(j)})$ with $K = \{1, 2\}$. This game always admits a mixed-strategy Nash equilibrium, but may or may not admit a pure-strategy Nash equilibrium (PNE), depending on the structure of the game.

Some games may have multiple PNEs with different performance. To evaluate the quality of equilibria, we define the PoA as follows:

Definition 4.4. ([2, 3, Price of Anarchy]) Let $S_{\mathrm{Eq}} \subseteq S$ be the set of all Nash equilibria. Then the PoA is defined as:

$$\mathrm{PoA} = \frac{\max_{S \in S_{\mathrm{Eq}}} \pi(S)}{\min_{S \in S} \pi(S)}. \tag{4.1}$$

A PoA close to 1 implies that equilibrium outcomes are nearly socially optimal.

Definition 4.5. ([2, 3, Price of Anarchy]) Let $S_{\mathrm{Eq}} \subseteq S$ be the set of strategy profiles that are in Nash equilibrium. The Price of Anarchy (PoA) is the ratio between the worst equilibrium outcome and the optimal (socially best) outcome:

$$\mathrm{PoA} = \frac{\max_{S \in S_{\mathrm{Eq}}} \pi(S)}{\min_{S \in S} \pi(S)}. \tag{4.2}$$

Any equilibrium (that is, NE) is considered socially acceptable if $\mathrm{PoA} \approx 1$.

Our game, which involves two players with finitely many strategies, can be described by a bimatrix form:

- Let $A = (a_{s_1,s_2})_{s_1 \in S_1, s_2 \in S_2}$ be the payoff matrix for Player I (row player),
- and $B = (b_{s_1,s_2})_{s_1 \in S_1, s_2 \in S_2}$ be the payoff matrix for Player II (column player).

If Player I chooses row s_1 and Player II chooses column s_2, the payoffs are:

$$\pi_1(s) = a_{s_1,s_2}, \quad \pi_2(s) = b_{s_1,s_2}.$$

Example 4.6. Assume a network with three vertices and two players. The sets of available paths are

$$R^{(1)} = \{\{1\},\{2\}\}, \quad R^{(2)} = \{\{2\},\{3\}\}.$$

Each player selects one path to route all their clients. Assume:

$$\mu_1 = \mu_3 = 3, \quad \mu_2 = 4, \quad \lambda^{(1)} = 1, \quad \lambda^{(2)} = 2.$$

Using the payoff function from Equation (3.3), the mean expected time matrix for both players is

	$\{1\}$	$\{3\}$
$\{1\}$	$(0.5, 1)$	$(0.5, 1)$
$\{2\}$	$(1, 1)$	$(1, 0.5)$

Based on this model, the game is non-cooperative and each player aims to minimize their clients' expected time in the system independently from the global system optimum.

We now distinguish two types of strategies:

(1) **Pure strategies:** each player j selects one path $r \in R^{(j)}$ to send all their clients. Players are fully informed about the choices of their opponents.

(2) **Mixed strategies:** player j selects a probability distribution over the paths $r \in R^{(j)}$, denoted by $x_r^{(j)}$, where $x_r^{(j)} \in [0,1]$. Each player is informed of the opponents distribution.

If a player chooses a single path, then the strategy space is discrete and finite:

$$x_r^{(j)} \in \{0,1\}, \quad \forall\, r \in R^{(j)},\, j = 1,2.$$

If a player is allowed to split clients over multiple paths, the strategy space becomes continuous:

$$x_r^{(j)} \in [0,1], \quad \forall\, r \in R^{(j)},\, j = 1,2.$$

We now explore the existence of Nash equilibrium and the associated PoA under both discrete and continuous strategy assumptions.

4.1. **Discrete Strategy Space.** We consider the game G where each player is allowed to choose a single path. Given the strategy s_{-j} of the opponent, player j solves the following non-linear program:

$$\min_{S^{(j)}} \quad \pi^{(j)}(s) \tag{4.3}$$

$$\text{subject to:} \quad \lambda_i = \sum_{\substack{r\in R^{(1)}\\ i\in r}} x_r^{(1)}\lambda^{(1)} + \sum_{\substack{r\in R^{(2)}\\ i\in r}} x_r^{(2)}\lambda^{(2)}, \quad \forall\, i \in V, \tag{4.4}$$

$$\lambda_i \le \mu_i, \quad \forall\, i \in V, \tag{4.5}$$

$$\sum_{r\in R^{(j)}} x_r^{(j)} = 1, \tag{4.6}$$

$$x_r^{(j)} \in \{0,1\}, \quad \forall\, r \in R^{(j)}. \tag{4.7}$$

Solving this program for both players simultaneously yields potential pure Nash equilibria (PNE). However, as we will show in the following examples, the existence of a PNE is not guaranteed even in the discrete case.

Example 4.7. Consider a game of 2-player played on a network with three vertices. The sets of available paths (strategies) for both players are: $S^{(1)} = \{s_1^{(1)} = \{1\}, s_2^{(1)} = \{2\}\}$ and $S^{(2)} = \{s_1^{(2)} = \{2\}, s_2^{(2)} = \{3\}\}$. Each player has to select just one path for all its clients. For $\mu_1 = \mu_3 = 5$, $\mu_2 = 6$, $\lambda^{(1)} = 3$, $\lambda^{(2)} = 2$, where $\lambda^{(1)} + \lambda^{(2)} \le \mu_2$ is the condition of feasibility.

Using formula (3.3), the payoff functions for player 1 and player 2 take the form:

$$\pi^{(1)}(s) = \sum_{r\in S^{(1)}} \frac{1}{\mu_i - \lambda^{(1)}} + \sum_{r\in S^{(1)}\cap S^{(2)}} \frac{1}{\mu_i - \lambda^{(1)} - \lambda^{(2)}},$$

$$\pi^{(2)}(s) = \sum_{r\in S^{(2)}} \frac{1}{\mu_i - \lambda^{(2)}} + \sum_{r\in S^{(1)}\cap S^{(2)}} \frac{1}{\mu_i - \lambda^{(1)} - \lambda^{(2)}}.$$

We get:

$$\pi^{(1)}(s_1^{(1)}, s_1^{(2)}) = \frac{1}{\mu_1 - \lambda^{(1)}} = \frac{1}{2}, \quad \pi^{(2)}(s_1^{(1)}, s_1^{(2)}) = \frac{1}{\mu_2 - \lambda^{(2)}} = \frac{1}{4}.$$

Using this approach, the payoff matrix with the players mean expected time of their clients in the system for this game is:

	$s_1^{(2)}$	$s_2^{(2)}$
$s_1^{(1)}$	$(0.5, 0.25)$	$(0.5, 0.5)$
$s_2^{(1)}$	$(1, 1)$	$(0.33, 0.33)$

If player 1 selects path $\{1\}$ and player 2 selects path $\{2\}$, the entry $(0.5, 0.25)$ indicates that the mean expected time for player 1s clients is 0.5 and for player 2s clients is 0.25.

Depending on the definition of PNE, we see that the strategy profiles $(s_1^{(1)}, s_1^{(2)})$ and $(s_2^{(1)}, s_2^{(2)})$ are the PNE for this game. The PoA of this game is $\frac{0.75}{0.66} = 1.14$.

For $\mu_1 = \mu_3 = 3$, $\mu_2 = 4$, and equal arrival rates $\lambda^{(1)} = \lambda^{(2)} = 1$, the payoff matrix for the players is

$$\begin{pmatrix} (0.5, 0.33) & (0.5, 0.5) \\ (0.5, 0.5) & (0.33, 0.5) \end{pmatrix}.$$

Obviously, this game has multiple PNEs: $(s_1^{(1)}, s_1^{(2)})$, $(s_1^{(1)}, s_2^{(2)})$, and $(s_2^{(1)}, s_2^{(2)})$. The PoA in this case is $\frac{1}{0.83} = 1.20$. This result shows that if the arrival rates of both players are equal, the game has multiple PNEs and the PoA is close to one.

Example 4.8. Consider a game as in Example (4.6) with the following sets of available paths for each player

$$S^{(1)} = \{s_1^{(1)} = \{1, 2, 3\}, s_2^{(1)} = \{4, 5, 6\}\},$$
$$S^{(2)} = \{s_1^{(2)} = \{1, 2, 4\}, s_2^{(2)} = \{6, 5, 3\}\}.$$

Each player must select just one path for its clients. We discuss the existence of PNE in three cases:

Case 1: For $\lambda^{(1)} = \lambda^{(2)} = 1$, $\mu_i = 3$ for all $i \in \{1, 2, 3, 4, 5, 6\}$. A sufficient condition for feasibility is $\lambda^{(1)} + \lambda^{(2)} \leq \mu_i$ for all nodes $i \in S^{(1)} \cap S^{(2)}$. The payoff matrix is:

$$\begin{pmatrix} (2.5, 2.5) & (2, 2) \\ (2, 2) & (2.5, 2.5) \end{pmatrix}.$$

There are two pure Nash equilibria: $(s_1^{(1)}, s_2^{(2)})$ and $(s_2^{(1)}, s_1^{(2)})$. The PoA is 1.

Case 2: For $\lambda^{(1)} = 1$, $\lambda^{(2)} = 2$, $\mu_i = 6$ for $i \in \{1, 2, 5, 6\}$, and $\mu_i = 5$ for $i \in \{3, 4\}$. The payoff matrix is:

$$\begin{pmatrix} (0.917, 1) & (0.90, 1) \\ (0.90, 1) & (0.917, 1) \end{pmatrix}.$$

The strategy profiles $(s_1^{(1)}, s_2^{(2)})$ and $(s_2^{(1)}, s_1^{(2)})$ are pure Nash equilibria. The PoA is 1.

Case 3: For $\lambda^{(1)} = 1$, $\lambda^{(2)} = 2$, $\mu_i = 6$ for $i \in \{1, 2, 5, 6\}$, and $\mu_i = 4.95$ for $i \in \{3, 4\}$. The payoff matrix is:

$$\begin{pmatrix} (0.920, 1.006) & (0.913, 1.013) \\ (0.913, 1.013) & (0.920, 1.006) \end{pmatrix}.$$

The improvement cycle for this case is shown in:

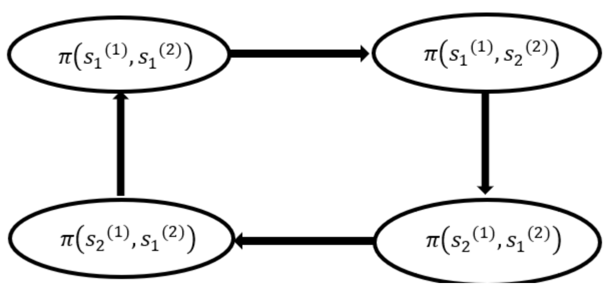

FIGURE 2. Improvement cycle

Since there are no strategy profiles outside this cycle, we conclude that this game has no PNE. Therefore, the PoA cannot be computed.

From Example (4.6) and Example (4.7), we conclude that a PNE in discrete space need not exist.

4.2. **Continuous strategy space.** If the player is permitted to split his clients over multiple paths, this leads to continuous strategy spaces for the players and therefore $x_r^{(j)} \in [0,1], \ \forall\ r \in R^{(j)},\ j = 1,2$. In this case, we show that there exists a unique PNE and find this strategy using a method of Lagrange multipliers with KKT conditions [1, 11].

Since the players goal is to decrease the mean expected time in the system of its clients, we can find the optimal strategy for any player j by finding the optimal solution of the non-linear mathematical model (4.7)–(4.11):

$$\min_{S^{(j)}} \pi^{(j)}(s). \tag{4.8}$$

Subject to:

$$\lambda_i = \sum_{\substack{i\in r\\ r\in R^{(1)}}} x_r^{(1)}\lambda^{(1)} + \sum_{\substack{i\in r\\ r\in R^{(2)}}} x_r^{(2)}\lambda^{(2)}, \quad \forall\ i \in V, \tag{4.9}$$

$$\lambda_i \le \mu_i, \quad \forall\ i \in V, \tag{4.10}$$

$$\sum_{r\in R^{(j)}} x_r^{(j)} = 1, \tag{4.11}$$

$$x_r^{(j)} \in [0,1], \quad \forall\ r \in R^{(j)}. \tag{4.12}$$

This mathematical program has a unique PNE. We will prove the existence and uniqueness of this strategy first by proving the convexity of the payoff function $\pi^{(j)}(s)$ defined by (3.3). After that, we show the existence of the unique PNE of this game.

To prove that $\pi^{(j)}(s)$ is convex, we introduce the following lemma:

Lemma 4.9. *A twice-differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex if and only if for every $x \in \mathbb{R}^n$, the Hessian matrix $\nabla^2 f(x)$ is positive definite (i.e., $f''(x) > 0,\ \forall x \in \mathbb{R}^n$).*

Proof. See [11]. □

Theorem 4.10. *The payoff function $\pi^{(j)}(s)$ of player j is strictly convex in $S^{(j)}$, $j = \{1,2\}$, for all $s \in \Omega$.*

Proof. Recall formulas (3.2) and (3.3). If we define $\tilde{x}_i^{(j)} = \sum_{\substack{r\in R^{(j)}\\ i\in r}} x_r^{(j)}$, then the payoff function can be written as:

$$\pi^{(j)}(s) = \sum_{i=1}^{n} \frac{\tilde{x}_i^{(j)}}{\mu_i - \tilde{x}_i^{(j)}\lambda_i}, \quad j = 1,2 \tag{4.13}$$

The first and second partial derivatives of (4.7) with respect to $\tilde{x}_i^{(j)}$ $(\forall i \in V)$ are given by:

$$\frac{\partial \pi^{(j)}(s)}{\partial \tilde{x}_i^{(j)}} = \sum_{i=1}^{n} \frac{\mu_i}{(\mu_i - \tilde{x}_i^{(j)}\lambda_i)^2}, \quad \frac{\partial^2 \pi^{(j)}(s)}{\partial (\tilde{x}_i^{(j)})^2} = \sum_{i=1}^{n} \frac{2\mu_i\lambda_i}{(\mu_i - \tilde{x}_i^{(j)}\lambda_i)^3}.$$

Moreover, for all nodes $i, k \in V$

$$\frac{\partial^2 \pi^{(j)}(s)}{\partial \tilde{x}_i^{(j)} \partial \tilde{x}_k^{(j)}} = 0.$$

The Hessian matrix $\nabla^2 \pi^{(j)}(s)$ is given by

$$\nabla^2 \pi^{(j)}(s) = \begin{bmatrix} \frac{\partial^2 \pi^{(j)}(s)}{\partial (\tilde{x}_1^{(j)})^2} & 0 & 0 & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & & & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \frac{\partial^2 \pi^{(j)}(s)}{\partial (\tilde{x}_n^{(j)})^2} \end{bmatrix}.$$

It is clear that the non-zero entries of this matrix take the form $\frac{\partial^2 \pi^{(j)}(s)}{\partial (\tilde{x}_i^{(j)})^2}$, which are all positive. Therefore, the Hessian matrix is positive definite, which means that $\pi^{(j)}(s)$ is strictly convex in $S^{(j)}$. □

The following theorem shows the conditions for a unique PNE to exist for the game G.

Theorem 4.11. *If the set of all feasible strategy profiles Ω is non-empty, the game $\mathcal{G}$ has a unique PNE.*

Proof. Since $x_r^{(j)} \in [0, 1]$ and $\sum_{r \in R^{(j)}} x_r^{(j)} = 1$, the set Ω is bounded. Moreover, since the inequalities and equations that construct Ω are linear, this set is convex. We already proved that the payoff function $\pi^{(j)}(s)$ is convex. For each p, the amount $\mu_i - p\lambda_i > 0$ implies that $\frac{p}{\mu_i - p\lambda_i}$ is continuous for all i, and thus $\pi^{(j)}(s)$ is continuous on Ω. Therefore, according to Theorem 1 in [1], the game $\mathcal{G}$ has a unique PNE. □

Now we introduce an example for a game $\mathcal{G}$ with continuous strategy space.

Example 4.12. Consider a game which is represented in Figure (3). The available paths for players are:

$$S^{(1)} = \{s_1^{(1)} = \{1\}, s_2^{(1)} = \{3\}\}, \quad S^{(2)} = \{s_1^{(2)} = \{2\}, s_2^{(2)} = \{3\}\}.$$

Assume the following feasibility conditions:

$$\lambda^{(1)} \leq \mu_1, \quad \lambda^{(2)} \leq \mu_2, \quad \lambda^{(1)} + \lambda^{(2)} \leq \mu_3.$$

The payoff functions for both players are:

$$\pi^{(1)}(s) = \frac{x_1^{(1)}}{\mu_1 - x_1^{(1)}\lambda^{(1)}} + \frac{x_3^{(1)}}{\mu_3 - (x_3^{(1)}\lambda^{(1)} + x_3^{(2)}\lambda^{(2)})},$$

$$\pi^{(2)}(s) = \frac{x_2^{(2)}}{\mu_2 - x_2^{(2)}\lambda^{(2)}} + \frac{x_3^{(2)}}{\mu_3 - (x_3^{(1)}\lambda^{(1)} + x_3^{(2)}\lambda^{(2)})}.$$

Since $\sum_{r \in R^{(j)}} x_r^{(j)} = 1$ for $j = 1, 2$, the payoff functions become:

$$\pi^{(1)}(s) = \frac{x_1^{(1)}}{\mu_1 - x_1^{(1)}\lambda^{(1)}} + \frac{(1 - x_1^{(1)})}{\mu_3 - [(1 - x_1^{(1)})\lambda^{(1)} + (1 - x_2^{(2)})\lambda^{(2)}]},$$

$$\pi^{(2)}(s) = \frac{x_2^{(2)}}{\mu_2 - x_2^{(2)}\lambda^{(2)}} + \frac{(1 - x_2^{(2)})}{\mu_3 - [(1 - x_1^{(1)})\lambda^{(1)} + (1 - x_2^{(2)})\lambda^{(2)}]}.$$

Solving this mathematical program simultaneously for player 1 and player 2 produces the optimal strategy profile S, which is the NE of the game $\mathcal{G}$.

We use the Lagrange multipliers method and KKT conditions as follows:

The Lagrangians for both players are:

$$\begin{aligned} L_1(S^{(1)}, \alpha_1, \alpha_2) &= \pi^{(1)}(s) + \alpha_1 x_1^{(1)} + \alpha_2(1 - x_1^{(1)}) \\ &= \frac{x_1^{(1)}}{\mu_1 - x_1^{(1)}\lambda^{(1)}} + \frac{(1 - x_1^{(1)})}{\mu_3 - [(1 - x_1^{(1)})\lambda^{(1)} + (1 - x_2^{(2)})\lambda^{(2)}]} \\ &\quad + \alpha_1 x_1^{(1)} + \alpha_2(1 - x_1^{(1)}), \end{aligned}$$

$$\begin{aligned} L_2(S^{(2)}, \beta_1, \beta_2) &= \pi^{(2)}(s) + \beta_1 x_2^{(2)} + \beta_2(1 - x_2^{(2)}) \\ &= \frac{x_2^{(2)}}{\mu_2 - x_2^{(2)}\lambda^{(2)}} + \frac{(1 - x_2^{(2)})}{\mu_3 - [(1 - x_1^{(1)})\lambda^{(1)} + (1 - x_2^{(2)})\lambda^{(2)}]} \\ &\quad + \beta_1 x_2^{(2)} + \beta_2(1 - x_2^{(2)}), \end{aligned}$$

$$\left.\begin{array}{c}
\frac{\partial L_1}{\partial {x_1}^{(1)}} = \frac{\mu_1}{\left(\mu_1 - {x_1}^{(1)}\lambda^{(1)}\right)^2} - \frac{\mu_3 - \left(1 - {x_2}^{(2)}\right)\lambda^{(2)}}{\left(\mu_3 - (1 - {x_1}^{(1)})\lambda^{(1)} - (1 - {x_2}^{(2)})\lambda^{(2)}\right)^2} + \alpha_1 - \alpha_2 = 0 \\
\frac{\partial L_2}{\partial {x_2}^{(2)}} = \frac{\mu_2}{\left(\mu_2 - {x_2}^{(2)}\lambda^{(2)}\right)^2} - \frac{\mu_3 - \left(1 - {x_1}^{(1)}\right)\lambda^{(1)}}{\left(\mu_3 - (1 - {x_1}^{(1)})\lambda^{(1)} - (1 - {x_2}^{(2)})\lambda^{(2)}\right)^2} + \beta_1 - \beta_2 = 0 \\
\alpha_1 {x_1}^{(1)} = 0 \\
\alpha_2\left(1 - {x_1}^{(1)}\right) = 0 \\
\beta_1 {x_2}^{(2)} = 0 \\
\beta_2\left(1 - {x_2}^{(2)}\right) = 0 \\
0 \le {x_1}^{(1)}, {x_2}^{(2)} \le 1
\end{array}\right\} \quad (*)$$

Now in system $(*)$, we consider $\lambda^{(1)} \neq \lambda^{(2)}$, for instance let $\lambda^{(1)} = 1$, $\lambda^{(2)} = 2$, and let $\mu_3 = 4$, $\mu_1 = \mu_2 = 3$. We get

$$3(x_1^{(1)} + 2x_2^{(2)} + 1)^2 = 2(3 - x_1^{(1)})^2(1 + x_2^{(2)}), \tag{4.14}$$

$$3(x_1^{(1)} + 2x_2^{(2)} + 1)^2 = (3 - 2x_2^{(2)})^2(3 + x_1^{(1)}). \tag{4.15}$$

Assuming symmetry $x_1^{(1)} = x_2^{(2)}$, we find $x_1^{(1)} = x_2^{(2)} = 0.703$, and the expected sojourn time for player 1 and 2 is 0.40 and 0.54 respectively.

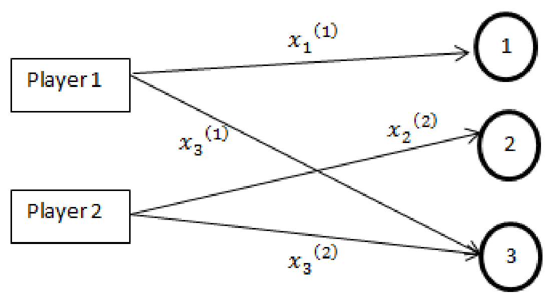

FIGURE 3. Queueing game with three nodes.

5. Queueing games as congestion game

An n-player game is considered as a congestion game if each player's strategy consists of a certain set of resources, and the cost of the approach is solely dependent on how many players are utilizing each resource, i.e., the cost takes the form $\sum d_i f(i)$, where $f(i)$ is the number of players using resource i, and d_i is a non-negative increasing function.

The game of network congestion on a directed graph is a common example. In this game, each player must choose a path from a source to a destination, and each edge has a "delay" function that grows as more players use the edge [9, 25]. Such congestion games always admit a PNE, according to Rosenthal [25], who demonstrated that these games have an exact potential function. Thus, for our game G to have a PNE, it is sufficient to show that this game is a congestion game.

The strategy set for each player in the congestion game may differ because each player chooses a different set of finite subsets of elements (in this case, vertices) as their strategy. Each element's delay function should be positive and monotonic in the number of players using this element (vertex). We demonstrate these characteristics of the game G in the following cases:

Case 1: The case of equal arrival rates ($\lambda^{(1)} = \lambda^{(2)} = \lambda$):
If we define $a_i^{(j)}(s)$ for any strategy profile S as:

$$a_i^{(j)}(s) = \begin{cases} 1, & \text{if } i \in R^{(j)}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the number of players using node i is:

$$y_i(S) = \sum_j a_i^{(j)}(S), \quad j = \{1, 2\}.$$

The payoff function becomes:

$$\pi^{(j)}(s) = \sum_{i=1}^{n} \frac{a_i^{(j)}(s)}{\mu_i - \lambda_i}. \tag{5.1}$$

Since $\lambda^{(1)} = \lambda^{(2)} = \lambda$, then $\lambda_i = y_i(s)\lambda$.

Now, we define the delay function d_i as:

$$d_i = \begin{cases} \frac{1}{\mu_i - y_i\lambda}, & y_i < \frac{\mu_i}{\lambda}, \\ \infty, & y_i \geq \frac{\mu_i}{\lambda}. \end{cases}$$

Therefore, the payoff function becomes:

$$\pi^{(j)}(s) = \sum_{i=1}^{n} a_i^{(j)}(s) \cdot d_i(y_i(s)). \tag{5.2}$$

The function d_i is positive and monotone increasing whenever $y_i < \frac{\mu_i}{\lambda}$. Thus, our game G with equal arrival rates is a congestion game and therefore has a PNE.

Case 2: The case of unequal arrival rates ($\lambda^{(1)} \neq \lambda^{(2)}$):

In this case, the delay function d_i becomes:

$$d_i = \begin{cases} \frac{1}{\mu_i - y_i \lambda}, & y_i < \frac{\mu_i}{\lambda_i}, \\ \infty, & y_i \geq \frac{\mu_i}{\lambda_i}, \end{cases}$$

where $\lambda_i = \sum_j a_i^{(j)}(s)\lambda^{(j)}, \quad j = 1, 2$. Therefore, the game G has a PNE provided that $y_i < \frac{\mu_i}{\lambda_i}$ holds for the strategy profile S to be feasible.

6. Conclusion

This article studied the equilibrium solution in a 2-player non-cooperative queueing game played on a network of $M/M/1$ queues with equal and unequal arrival rates, where the players have two choices:

(1) First, each player can select just one path for all its clients.

(2) Second, each player can split its clients over multiple paths and share some servers with the other player.

We analyzed and discussed the existence of PNE for both strategy types. In the case where each player selects one path, the game G has a Nash equilibrium in mixed strategies, but a PNE is not guaranteed. In contrast, if players can split their clients over multiple paths, we have shown that a unique PNE exists and we calculated it using the Lagrange multipliers method with KKT conditions.

Finally, we defined and described our game G as a congestion game. In this case, we derived the condition for the game G to have a PNE, which is:

$$y_i < \frac{\mu_i}{\lambda_i} \quad \text{where} \quad \lambda_i = \sum_j a_i^{(j)}(s)\lambda^{(j)}, \quad j = 1, 2.$$

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