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FIXED POINT THEOREMS OF CONDENSED KANNAN-TYPE CONTRACTION IN $\mathcal{G}$-METRIC SPACES

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Abstract. In this paper, we introduce the notion of condensed Kannan-type contraction in $\mathcal{G}$-metric spaces. We establish and prove some fixed point theorems of operators satisfying the condensed Kannan-type map. To evaluate the efficacy of this condition, we define a new criterion, called the $\alpha-$measure, in $\mathcal{G}$-metric spaces to seamlessly assess the effectiveness and dynamicity of the condensed map. We consider some practical examples to validate and demonstrate the dominance of the condensed map over some existing Kannan-type maps in $\mathcal{G}$-metric spaces. The results obtained ensure suitability for solving unique and non-unique fixed points and suggest a framework for selecting an appropriate real constant $\alpha \in (0,1)$ for studying nonlinear operators.

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1. Introduction

In 2006, Mustafa and Sims [15] introduced a generalization of the conventional metric spaces called the $\mathcal{G}$-metric spaces. Instead of a two-point distance, the $\mathcal{G}$-metric spaces measure the distance between three points simultaneously, allowing for more flexibility in spaces that may not fit the two-point distance. The $\mathcal{G}$-metric spaces continue to evolve as a focus in fixed point theory and nonlinear analysis exploration. For instance, the notion has been used for the extensions of classical theorems, exploring new types of contractive conditions, and applying the framework to study various branches of nonlinear analysis, see [1, 2, 3, 4, 8, 14, 17]. In what follows, we present some useful definitions and terminologies of $\mathcal{G}$-metric spaces.

Definition 1.1. ([15]) Let $\mathcal{M}$ be a non-empty set and let $\mathcal{G} : \mathcal{M}\times\mathcal{M}\times\mathcal{M} \longrightarrow \mathbb{R}^+$ be a function satisfying:

(G_1) $\mathcal{G}(u, v, w) = 0$ if $u = v = w$;
(G_2) $0 < \mathcal{G}(u, u, v)$ for all $u, v \in \mathcal{M}$ with $u \not\equiv v$;
(G_3) $\mathcal{G}(u, u, v) \leq \mathcal{G}(u, v, w)$ for all $u, v, w \in \mathcal{M}$ with $v \not\equiv w$;
(G_4) $\mathcal{G}(u, v, w) = \mathcal{G}(u, w, v) = \mathcal{G}(v, u, w) = \ldots$;
(G_5) $\mathcal{G}(u, v, w) \leq \mathcal{G}(u, a, a) + \mathcal{G}(a, u, w)$ for all $u, v, w, a \in \mathcal{M}$.

Then, the function $\mathcal{G}$ is called a generalized metric or simply a $\mathcal{G}$-metric on $\mathcal{M}$, and the pair $(\mathcal{M}, \mathcal{G})$ is called a $\mathcal{G}$-metric space.

Example 1.2. Let $(\mathcal{M}, d)$ be a metric space. Then the pairs $(\mathcal{M}, \mathcal{G}_e)$ and $(\mathcal{M}, \mathcal{G}_f)$ given by

$$\mathcal{G}_e(u, v, w) = d(u, v) + d(u, w) + d(v, w),$$
$$\mathcal{G}_f(u, v, w) = \max\{d(u, v), d(u, w), (v, w)\}$$

for all $u, v, w \in \mathcal{M}$, are $\mathcal{G}$-metric spaces.

Definition 1.3. ([15]) Let $(\mathcal{M}, \mathcal{G})$ be a $\mathcal{G}$-metric space and let $\{u_n\}$ be a sequence in $\mathcal{M}$. Then $\{u_n\}$ is $\mathcal{G}$-convergent to u if $\lim_{n,m\longrightarrow\infty} \mathcal{G}(u, u_n, u_m) = 0$, that is, for any $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that $\mathcal{G}(u, u_n, u_m) < \epsilon$ for all $n, m > n_0$.

Proposion 1.4. ([15]) *Let $(\mathcal{M}, \mathcal{G})$ be a $\mathcal{G}$-metric space. Then the following are equivalent:*

(1) *$\{u_n\}$ is $\mathcal{G}$-convergent to u.*
(2) *$\mathcal{G}(u, u_n, u_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$.*
(3) *$\mathcal{G}(u_n, u, u) \longrightarrow 0$ as $n \longrightarrow \infty$.*
(4) *$\mathcal{G}(u_n, u_n, u) \longrightarrow 0$ as $n \longrightarrow \infty$.*

Definition 1.5. ([15]) Let $(\mathcal{M}, \mathcal{G})$ be a $\mathcal{G}$-metric space. A sequence $\{u_n\}$ is $\mathcal{G}$-Cauchy if given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{G}(u_n, u_m, u_l) < \epsilon$ for all $n, m, l \geq n_0$, that is, $\mathcal{G}(u_n, u_m, u_l) \longrightarrow 0$ as $n, m, l \longrightarrow \infty$.

Proposion 1.6. ([15]) *In a $\mathcal{G}$-metric space $(\mathcal{M}, \mathcal{G})$, the following are equivalent:*

(1) *The sequence u_n is $\mathcal{G}$-Cauchy.*
(2) *For every $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that $\mathcal{G}(u_n, u_m, u_m) < \epsilon$ for all $n, m \geq n_0$.*

Definition 1.7. ([15]) A $\mathcal{G}$-metric space $(\mathcal{M}, \mathcal{G})$ is said to be $\mathcal{G}$-complete if every $\mathcal{G}$-Cauchy sequence in $(\mathcal{M}, \mathcal{G})$ is $\mathcal{G}$-convergent in $(\mathcal{M}, \mathcal{G})$.

Proposion 1.8. ([15]) *A $\mathcal{G}$-metric space $(\mathcal{M}, \mathcal{G})$ is G-complete if and only if $(\mathcal{M}, d_{\mathcal{G}})$ is a complete metric space.*

Note that $d_{\mathcal{G}}$ is a two-point metric associated with the $\mathcal{G}$-metric. The following theorems are useful results in $\mathcal{G}$-metric spaces.

Theorem 1.9. ([14]) *Let $(\mathcal{M}, \mathcal{G})$ be a $\mathcal{G}$-metric space and let $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$ be a $\mathcal{G}$-continuous mapping satisfying the following conditions:*

$$\mathcal{G}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}w) \leq \lambda \left[\mathcal{G}(u, \mathcal{T}u, \mathcal{T}u) + \mathcal{G}(v, \mathcal{T}v, \mathcal{T}v) + \mathcal{G}(w, \mathcal{T}w, \mathcal{T}w)\right] \tag{1.1}$$

for all $u, v, w \in M$, where M is an everywhere dense subset of $\mathcal{M}$ (with respect to the topology of $\mathcal{T}$-metric convergence) and $0 < \lambda < \frac{1}{6}$. If there is $u \in \mathcal{M}$ such that $\mathcal{T}^n(u) \longrightarrow u$, then is a unique fixed point.

Theorem 1.10. ([14]) *Let $(\mathcal{M}, \mathcal{G})$ be $\mathcal{G}-$metric space and let $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$ be a $\mathcal{G}$-continuous mapping satisfying*

$$\mathcal{G}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}v) \leq \lambda \left[\mathcal{G}(u, \mathcal{T}u, \mathcal{T}u) + \mathcal{G}(v, \mathcal{T}v, \mathcal{T}v)\right] \tag{1.2}$$

for all $u, v \in M$, where M is an everywhere dense subset of $\mathcal{M}$ (with respect to the topology of $\mathcal{G}$-metric convergence and $0 < \lambda < \frac{1}{6}$). If there is $u \in \mathcal{M}$ such that $\mathcal{T}^n u \longrightarrow u$, then u is a unique fixed point.

In 2018, Karapinar [10] obtained the following result in a standard metric space.

Definition 1.11. ([10]) Let $(\mathcal{M}, d)$ be a metric space. A self-mapping $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$ is called an interpolative Kannan-type contraction if there exist $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$d(\mathcal{T}u, \mathcal{T}v) \leq \mu d(u, \mathcal{T}u)^{\alpha} d(v, \mathcal{T}v)^{1-\alpha} \tag{1.3}$$

for all $u, v \in \mathcal{M} \backslash Fix(T)$, where $Fix(\mathcal{T}) = \{u \in \mathcal{M} : \mathcal{T}u = u\}$.

Theorem 1.12. ([10]) *Let* $(\mathcal{M}, d)$ *be a complete metric space and let* $T : \mathcal{M} \longrightarrow \mathcal{M}$ *be an interpolative Kannan-type contraction. Then* T *has a unique fixed point in* $\mathcal{M}$.

The above result has two advantages over the Kannan contraction [9], namely, (a) it is suitable to study the operators with non-unique fixed points, and (b) the non-integer power α contributes to the contractiveness of the map. For a few robust versions and extensions of the condition (1.3), see [5, 6, 11, 12, 13, 16, 18]. In [7], Jiddah et al. introduced an independent version of the interpolative contraction (1.3) in $\mathcal{G}$-metric spaces as follows:

Definition 1.13. ([7]) Let $(\mathcal{M}, \mathcal{G})$ be a $\mathcal{G}$-metric space. A self-mapping $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is called an interpolative Kannan-type $\mathcal{G}_{\mu,\alpha}$-contraction if there exist $\mu \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$\mathcal{G}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}^2v) \leq \mu\mathcal{G}(u, \mathcal{T}u, \mathcal{T}^2u)^{\alpha}\mathcal{G}(v, \mathcal{T}v, \mathcal{T}^2v)^{1-\alpha} \tag{1.4}$$

for all $u, v \in \mathcal{M}\backslash Fix(\mathcal{T})$.

Brief comments on the poseness of (1.4): It is worthy of noting that condition (1.4) improves the adaptability to solve complex systems, but then it was ill-posed due to the constant μ defined between 0 and 1, whereas this should only valid in the standard metric space. Since each $\mathcal{G}$-metric has a three-metrical term, the choice of μ controls each of the metrical terms. That is, for any $\mathcal{G}$-metric $\mathcal{G}(u, v, w)$ and μ a real non-negative number, $\mu\mathcal{G}(u, v, w) = \mathcal{G}(\mu u, \mu v, \mu w)$.

Furthermore, if $\mathcal{G}(u_1, v_1, w_1)$ and $\mathcal{G}(u_2, v_2, w_2)$ are any two $\mathcal{G}$-metrical terms, then

$$\begin{aligned} \mu\mathcal{G}(u_1, v_1, w_1)\mathcal{G}(u_2, v_2, w_2) &= \mathcal{G}(\mu u_1, \mu v_1, \mu w_1)\mathcal{G}(u_2, v_2, w_2) \\ &= \mathcal{G}(u_1, v_1, w_1)\mathcal{G}(\mu u_2, \mu v_2, \mu w_2). \end{aligned} \tag{1.5}$$

Equality (1.5) shows that the constant μ controls only one $\mathcal{G}$-metric apart in the product of two $\mathcal{G}$-metrical terms.

Motivated by the result of Mustapha et al. [14] and (1.5), we redefine the condition (1.4) as follows:

Definition 1.14. Let $(\mathcal{M}, \mathcal{G})$ be a $\mathcal{G}$-metric space. A self-mapping $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is called an interpolative Kannan-type $\mathcal{G}_{\mu,\alpha}$-contraction if there exist $\alpha \in (0, 1)$ and $\mu \in [0, \frac{1}{3})$ such that

$$\mathcal{G}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}^2v) \leq \mu\mathcal{G}(u, \mathcal{T}u, \mathcal{T}^2u)^{\alpha}\mathcal{G}(v, \mathcal{T}v, \mathcal{T}^2v)^{1-\alpha} \tag{1.6}$$

for all $u, v \in \mathcal{M}\backslash Fix(\mathcal{T})$.

Remark 1.15. By replacing $\mu \in [0, \frac{1}{3})$ with the choice of $\mu \in [0, 1)$ in [7, Theorem 2], the proof therein is a routine.

In view of Definition 1.14, we define a Kannan-type $\mathcal{G}$-contraction of the form (1.2) by replacing the third entry operator $\mathcal{T}$ with $\mathcal{T}^2$.

Definition 1.16. Let $(\mathcal{M}, \mathcal{G})$ be a $\mathcal{G}$-metric space. A self-mapping $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is called a Kannan-type $\mathcal{G}_\lambda$-contraction if there exist $\lambda \in [0, \frac{1}{6})$ such that

$$\mathcal{G}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}^2 v) \leq \lambda \left[\mathcal{G}(u, \mathcal{T}u, \mathcal{T}^2 u) + \mathcal{G}(v, \mathcal{T}v, \mathcal{T}^2 v)\right] \tag{1.7}$$

for all $u, v \in \mathcal{M}$.

In the sequel, we shall define a more elaborate Kannan-type map in $\mathcal{G}$-metric spaces that harmonizes and improves the results of Jiddah et al. [7], Karapinar [10], and Mustapha et al. [15].

2. Main results

The goal of this section is to define and prove the existence properties of a new Kannan-type $\mathcal{G}$-contraction that can handle some operators that may not meet the hypotheses of previous Kannan-type conditions.

Definition 2.1. Let $(\mathcal{M}, \mathcal{G})$ be a $\mathcal{G}$-metric space. A self-map $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ will be called a condensed Kannan-type $\mathcal{G}_{\mu,\lambda,\alpha}$-contraction if there exist $\alpha \in (0, 1)$, $\lambda \in [0, \frac{1}{6})$, and $\mu \in [0, \frac{1}{3})$ such that

$$\mathcal{G}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}^2 v) \begin{cases} \leq \lambda \left[\mathcal{G}(u, \mathcal{T}u, \mathcal{T}^2 u)^{2\alpha} + \mathcal{G}(v, \mathcal{T}v, \mathcal{T}^2 v)^{2(1-\alpha)}\right] & \text{(CK1)} \\ \geq \mu \mathcal{G}(u, \mathcal{T}u, \mathcal{T}^2 u)^{\alpha} \mathcal{G}(v, \mathcal{T}v, \mathcal{T}^2 v)^{1-\alpha} & \text{(CK2)} \end{cases} \tag{2.1}$$

for all $u, v \in \mathcal{M} \backslash Fix(\mathcal{T})$ and contrariwise.

We note that the Definition 1.13 is suitable for approximating non-unique fixed points. However, if $\mathcal{M} \backslash Fix(\mathcal{T})$ is changed to $\mathcal{M}$, the map $\mathcal{T}$ is suitable for both unique and non-unique fixed points. In this respect, we redefine condition (2.1) as follows:

Definition 2.2. Let $(\mathcal{M}, \mathcal{G})$ be a $\mathcal{G}$-metric space. A self-map $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ will be called a refined Kannan-type $\mathcal{G}_{\lambda,\alpha}$-contraction if there exist $\alpha \in (0, 1)$, $\lambda \in [0, \frac{1}{6})$ such that

$$\mathcal{G}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}^2 v) \leq \lambda \left[\mathcal{G}(u, \mathcal{T}u, \mathcal{T}^2 u)^{2\alpha} + \mathcal{G}(v, \mathcal{T}v, \mathcal{T}^2 v)^{2(1-\alpha)}\right] \tag{2.2}$$

for all $u, v \in \mathcal{M}$.

By interchanging the fractional powers 2α and $2(1-\alpha)$, inequality (2.2) is still valid. Obviously, every Kannan-type $\mathcal{G}_\lambda$-contraction is a refined Kannan-type $\mathcal{G}_{\lambda,\alpha}$-contraction with $\alpha = \frac{1}{2}$, but the converse is false if otherwise. We justify this exclusion in the next example.

Example 2.3. Let $\mathcal{T} : [-1,1] \to [-1,1]$ be a self-map defined by $\mathcal{T}u = \frac{u}{6}$ if $u \in \{-1,1\}$ and $\mathcal{T}u = \frac{1}{6}$, if $u \in (-1,1)$ with the $\mathcal{G}$-metric $\mathcal{G}(u,v,w) = \max\{d(u,v), d(u,w), d(v,w)\}$, for all $u,v,w \in [-1,1]$. Then, $\mathcal{T}$ forms a refined $\mathcal{G}_{\lambda,\alpha}$-Kannan-type contraction for $\alpha > \frac{1}{2}$ but $\mathcal{T}$ does not satisfy the $\mathcal{G}_\lambda$-Kannan contraction.

To see this, we select $u = -1$, $v = 1$, $\lambda = 0.166$, and $\alpha = 0.6$ in (2.2) to get

$$\begin{aligned}\mathcal{G}(\mathcal{T}(-1), \mathcal{T}(1), \mathcal{T}^2(1)) \leq 0.166[&\mathcal{G}(-1, \mathcal{T}(-1), \mathcal{T}^2(-1))^{1.2} \\ &+ \mathcal{G}(1, \mathcal{T}(1), \mathcal{T}^2(1))^{0.8}].\end{aligned}$$

This implies that $0.3333 < 0.166(1.2032 + 0.8643) \approx 0.3432$. On the other hand, if $u = -1$, $v = 1$, $\lambda = 0.166$, and $\alpha = 0.5$, we obtain $0.3333 < 0.166(1.1667 + 0.8333) \approx 0.3320$ which is a contradiction.

The following Theorem is one of the main results.

Theorem 2.4. *Let $(\mathcal{M}, \mathcal{G})$ be a complete $\mathcal{G}$-metric space and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a condensed Kannan-type $\mathcal{G}_{\mu,\lambda,\alpha}$-contraction for $\alpha \in (0,1)$, $\lambda \in [0, \frac{1}{2})$, and $\mu \in [0,1)$ with $\mathcal{M}\backslash Fix(\mathcal{T})$. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{M}$.*

Proof. Let $u_0 \in \mathcal{M}$ be a random point and define a sequence $u_n \in \mathcal{M}$ by $u_n = \mathcal{T}^n u_0$. If there exists some $m \in \mathbb{N}$ such that $\mathcal{T}u_m = u_{m+1} = u_m$, then u_m is a fixed point of $\mathcal{T}$, and so the proof is complete.

On the other hand, assume that $u_n \neq u_{n+1}$ for any $n \in \mathbb{N}$. Using (CK1) in (2.1), we have that

$$\begin{aligned}\mathcal{G}(u_n, u_{n+1}, u_{n+2}) &= \mathcal{G}(\mathcal{T}u_{n-1}, \mathcal{T}u_n, \mathcal{T}^2 u_n) \\ &\leq \lambda \left[\mathcal{G}(u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}^2 u_{n-1})^{2\alpha} + \lambda \mathcal{G}(u_n, \mathcal{T}u_n, \mathcal{T}^2 u_n)^{2(1-\alpha)}\right] \\ &= \lambda \left[\mathcal{G}(u_{n-1}, u_n, u_{n+1})^{2\alpha} + \mathcal{G}(u_n, u_{n+1}, u_{n+2})^{2(1-\alpha)}\right]. \end{aligned} \tag{2.3}$$

Also, using (CK2) in (2.1), we get

$$\begin{aligned}\mathcal{G}(u_n, u_{n+1}, u_{n+2}) &= \mathcal{G}(\mathcal{T}u_{n-1}, \mathcal{T}u_n, \mathcal{T}^2 u_n) \\ &\geq \mu \mathcal{G}(u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}^2 u_{n-1})^{\alpha} \mathcal{G}(u_n, \mathcal{T}u_n, \mathcal{T}^2 u_n)^{1-\alpha} \\ &= \mu \mathcal{G}(u_{n-1}, u_n, u_{n+1})^{\alpha} \mathcal{G}(u_n, u_{n+1}, u_{n+2})^{1-\alpha}. \end{aligned} \tag{2.4}$$

Combination of both (2.3) and (2.4) gives

$$\left[\mathcal{G}(u_{n-1}, u_n, u_{n+1})^{\alpha} - \mathcal{G}(u_n, u_{n+1}, u_{n+2})^{1-\alpha}\right]^2 \geq 0, \tag{2.5}$$

where $\mu \equiv 2\lambda$. Two cases arise from the inequality (2.5), that is,
Case (a) $\mathcal{G}(u_{n-1}, u_n, u_{n+1})^{\alpha} \leq \mathcal{G}(u_n, u_{n+1}, u_{n+2})^{1-\alpha}$.
Case (b) $\mathcal{G}(u_n, u_{n+1}, u_{n+2})^{1-\alpha} \leq \mathcal{G}(u_{n-1}, u_n, u_{n+1})^{\alpha}$.

Case (a): If $\mathcal{G}(u_{n-1}, u_n, u_{n+1})^{\alpha} \leq \mathcal{G}(u_n, u_{n+1}, u_{n+2})^{1-\alpha}$, then inequality (2.3) becomes

$$\mathcal{G}(u_n, u_{n+1}, u_{n+2})^{2(1-\alpha)} \leq \mu < \frac{1}{3},$$

which is a contradiction for $\alpha > 0$. Thus, Case (a) fails.

Case (b): If $\mathcal{G}(u_n, u_{n+1}, u_{n+2})^{1-\alpha} \leq \mathcal{G}(u_{n-1}, u_n, u_{n+1})^{\alpha}$, then inequality (2.4) gives

$$\mathcal{G}(u_n, u_{n+1}, u_{n+2}) \geq \mu \mathcal{G}(u_n, u_{n+1}, u_{n+2})^{2(1-\alpha)}. \tag{2.6}$$

By combining inequalities (2.3) and (2.6), we get

$$\mu \mathcal{G}(u_n, u_{n+1}, u_{n+2})^{2(1-\alpha)} \leq \lambda \left[\mathcal{G}(u_{n-1}, u_n, u_{n+1})^{2\alpha} + \mathcal{G}(u_n, u_{n+1}, u_{n+2})^{2(1-\alpha)} \right]. \tag{2.7}$$

This further gives the recursive relation

$$\begin{aligned} \mathcal{G}(u_n, u_{n+1}, u_{n+2}) &\leq \left(\frac{\lambda}{\mu - \lambda} \right)^{\frac{1}{2(1-\alpha)}} \mathcal{G}(u_{n-1}, u_n, u_{n+1})^{\frac{\alpha}{1-\alpha}} \\ &\equiv \delta \mathcal{G}(u_{n-1}, u_n, u_{n+1})^{\rho}, \end{aligned} \tag{2.8}$$

where $\delta \equiv \left(\frac{\lambda}{\mu - \lambda} \right)^{\frac{1}{2(1-\alpha)}}$ and $\rho \equiv \frac{\alpha}{1-\alpha}$. Obviously, $\delta < 1$ and $\rho > 0$ for any $\alpha \in [0, 1]$, $\mu \in [0, \frac{1}{3})$, and $\lambda \in [0, \frac{1}{6})$. By induction,

$$\mathcal{G}(u_n, u_{n+1}, u_{n+2}) \leq \delta^{\sum_{k=0}^{n-1} \rho^k} \mathcal{G}(u_0, u_1, u_2)^{\rho^n}. \tag{2.9}$$

If $\alpha < \frac{1}{2}$, then the proof is completed. On other hand, let $\alpha \geq \frac{1}{2}$ and taking limit as $n \to \infty$ over the last inequality, we have

$$\lim_{n \to \infty} \mathcal{G}(u_n, u_{n+1}, u_{n+2}) = 0.$$

For any $n, m, l \in \mathbb{N}$ with $n < m < l$ and by G_5, we obtain

$$\mathcal{G}(u_n, u_m, u_l) \to 0 \quad \text{as} \quad n, m, l \to \infty.$$

Hence, $\{u_n\}$ is a $\mathcal{G}$-Cauchy sequence in $(\mathcal{M}, \mathcal{G})$ and so by the completeness of $(\mathcal{M}, \mathcal{G})$, there exists a point $p \in \mathcal{M}$ such that $\{u_n\}$ is $\mathcal{G}$-convergent to p, that is,

$$\lim_{n \to \infty} \mathcal{G}(u_n, u_n, p) = 0.$$

Next is to show that the map $\mathcal{T}$ is fixed. Let $p \in \mathcal{M}$ so that $p \neq \mathcal{T}p \neq \mathcal{T}^2 p$, by using (G_5):

$$\mathcal{G}(p, \mathcal{T}p, \mathcal{T}^2 p) \leq \mathcal{G}(p, \mathcal{T}u_n, \mathcal{T}u_n) + \mathcal{G}(\mathcal{T}u_n, \mathcal{T}p, \mathcal{T}^2 p). \tag{2.10}$$

But,

$$\begin{aligned}\mathcal{G}(\mathcal{T}u_n, \mathcal{T}p, \mathcal{T}^2p) &\leq \lambda \left[\mathcal{G}(u_n, \mathcal{T}u_n, \mathcal{T}^2u_n)^{2\alpha} + \mathcal{G}(p, \mathcal{T}p, \mathcal{T}^2p)^{2(1-\alpha)}\right] \\ &= \lambda \left[\mathcal{G}(u_n, u_{n+1}, u_{n+2})^{2\alpha} + \mathcal{G}(p, \mathcal{T}p, \mathcal{T}^2p)^{2(1-\alpha)}\right]. \qquad (2.11)\end{aligned}$$

Taking limit as $n \to \infty$ across (2.11) to get

$$\lim_{n\to\infty} \mathcal{G}(u_{n+1}, \mathcal{T}p, \mathcal{T}^2p) \leq \lambda\mathcal{G}(p, \mathcal{T}p, \mathcal{T}^2p)^{2(1-\alpha)}.$$

Therefore, inequality (2.10) becomes

$$\mathcal{G}(p, \mathcal{T}p, \mathcal{T}^2p) \leq \lim_{n\to\infty} \mathcal{G}(p, u_{n+1}, u_{n+1}) + \lambda\mathcal{G}(p, \mathcal{T}p, \mathcal{T}^2p)^{2(1-\alpha)},$$

which further reduces to

$$\mathcal{G}(p, \mathcal{T}p, \mathcal{T}^2p)^{2\alpha-1} \leq \lambda < \frac{1}{6}.$$

This leads to a contradiction for some $\alpha \in (0, 1)$. Thus, $\mathcal{G}(p, \mathcal{T}p, \mathcal{T}^2p) = 0$ only if $p = \mathcal{T}p = \mathcal{T}^2p$.

Finally, suppose that p and q are fixed points of $\mathcal{T}$ for which $p \neq q$, $p = \mathcal{T}p = \mathcal{T}^2p$, and $q = \mathcal{T}q = \mathcal{T}^2q$. By hypothesis,

$$\begin{aligned}\mathcal{G}(p, q, \mathcal{T}q) = \mathcal{G}(\mathcal{T}p, \mathcal{T}q, \mathcal{T}^2q) &\leq \lambda \left[\mathcal{G}(p, \mathcal{T}p, \mathcal{T}^2p)^{2\alpha} + \mathcal{G}(q, \mathcal{T}q, \mathcal{T}^2q)^{2(1-\alpha)}\right] \\ &= \lambda \left[\mathcal{G}(p, p, p)^{2\alpha} + \mathcal{G}(q, q, q)^{2(1-\alpha)}\right] \\ &= 0. \qquad (2.12)\end{aligned}$$

Also,

$$\begin{aligned}\mathcal{G}(p, q, \mathcal{T}q) = \mathcal{G}(\mathcal{T}p, \mathcal{T}q, \mathcal{T}^2q) &\geq \mu\mathcal{G}(p, \mathcal{T}p, \mathcal{T}^2p)^{\alpha}\mathcal{G}(q, \mathcal{T}q, \mathcal{T}^2q)^{1-\alpha} \\ &= \mu\mathcal{G}(p, p, p)^{\alpha}\mathcal{G}(q, q, q)^{1-\alpha} \qquad (2.13) \\ &= 0.\end{aligned}$$

From both (2.12) and (2.13), we have that $\mathcal{G}(p, q, \mathcal{T}q) = 0$, which is a contradiction. Hence, the fixed point of $\mathcal{T}$ is unique. □

Theorem 2.5. *Let* $(\mathcal{M}, \mathcal{G})$ *be a complete* $\mathcal{G}$*-metric space and* $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ *be a refined Kannan-type* $\mathcal{G}_{\lambda,\alpha}$*-contraction for* $\alpha \in (0, 1)$ *and* $\lambda \in [0, \frac{1}{6})$*. Then* $\mathcal{T}$ *has a unique fixed point in* $\mathcal{M}$*.*

Proof. By adopting only the (CK1) in Theorem 2.4, the proof is a routine. □

Corollary 2.6. *Let $(\mathcal{M}, \mathcal{G})$ be a complete $\mathcal{G}$-metric space and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a condensed Kannan-type $\mathcal{G}_{\mu,\lambda,\frac{1}{2}}$-contraction satisfying*

$$\mathcal{G}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}^2 v) \begin{cases} \leq \lambda \left[\mathcal{G}(u, \mathcal{T}u, \mathcal{T}^2 u) + \mathcal{G}(v, \mathcal{T}v, \mathcal{T}^2 v)\right] \\ \\ \geq \mu \mathcal{G}(u, \mathcal{T}u, \mathcal{T}^2 u)^{\frac{1}{2}} \mathcal{G}(v, \mathcal{T}v, \mathcal{T}^2 v)^{\frac{1}{2}} \end{cases} \tag{2.14}$$

for all $u, v \in \mathcal{M} \backslash Fix(\mathcal{T})$, $\lambda \in [0, \frac{1}{6})$ and $\mu \in [0, \frac{1}{3})$. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{M}$.

Proof. The proof is immediate from Theorem 2.4 with $\alpha = \frac{1}{2}$. □

Corollary 2.7. *Let $(\mathcal{M}, \mathcal{G})$ be a complete $\mathcal{G}$-metric space and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a Kannan-type $\mathcal{G}_{\lambda,\frac{1}{2}}$-contraction satisfying*

$$\mathcal{G}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}^2 v) \leq \lambda \left[\mathcal{G}(u, \mathcal{T}u, \mathcal{T}^2 u) + \mathcal{G}(v, \mathcal{T}v, \mathcal{T}^2 v)\right] \tag{2.15}$$

for all $u, v \in \mathcal{M}$, $\lambda \in [0, \frac{1}{6})$. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{M}$.

Proof. If $Fix(\mathcal{T}) \subset \mathcal{M}$ in Corollary 2.6, then the proof follows immediately. □

Example 2.8. Let $\mathcal{M} = [-1, 1]$ and let $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a self-map on $\mathcal{M}$ defined by

$$\mathcal{T}u = \begin{cases} \frac{u}{5}, & \text{if } u \in \{-1, 1\}; \\ \\ \frac{1}{5}, & \text{if } u \in (-1, 1) \text{ for all } u \in \mathcal{M}. \end{cases}$$

Define $\mathcal{G} : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathbb{R}_+$ by $\mathcal{G}(u, v, w) = |u - v| + |u - w| + |v - w|$ for all $u, v, w \in \mathcal{M}$ with $Fix(\mathcal{T}) = \{\frac{1}{5}\}$. Then, the map $\mathcal{T}$

(1) does not satisfy the Kannan $\mathcal{G}_\lambda$-continuous (1.2).

(2) does not satisfy the Kannan $\mathcal{G}_\lambda$-contraction (1.7).

(3) does not satisfy the interpolative Kannan-type $\mathcal{G}_\lambda$-contraction (1.6).

(4) satisfies the condensed Kannan-type $\mathcal{G}_{\mu,\lambda,\alpha}$-contraction (2.1).

(5) satisfies the refined Kannan-type $\mathcal{G}_{\lambda,\alpha}$-contraction (2.2).

Firstly, if $u, v \in (0, 1)$, then all of the above-mentioned conditions are satisfied. On the other hand, suppose that $u \neq v$ for $u, v \in [-1, 1] \backslash Fix(\mathcal{T})$, or specifically $u = -1$ and $v = 1$, then the left side $\mathcal{G}(-\frac{1}{5}, \frac{1}{5}, \frac{1}{5}) = 0.8$. To know the condition with the dominant right side, we let $\alpha = \frac{5}{6}$, $\mu = \frac{8}{25}$, and $\lambda = \frac{4}{25}$ to compare some $\mathcal{G}$-Kannan-type maps.

Table 1. Comparison of various $\mathcal{G}$-Kannan-type maps

No	Conditions	Right sides	Remark
1	Kannan $\mathcal{G}_{\frac{4}{25}}$-continuous (1.2)	0.5120	Not satisfy
2	Kannan $\mathcal{G}_{\frac{4}{25}}$-contraction (1.7)	0.6400	Not satisfy
3	Interpolative Kannan $\mathcal{G}_{\frac{8}{25},\frac{5}{6}}$-contraction (1.6)	0.7178	Not satisfy
4	Refined Kannan $\mathcal{G}_{\frac{4}{25},\frac{5}{6}}$-contraction (2.2)	0.8755	Satisfy

As presented in Table 1, only the refined $\mathcal{G}_{\lambda,\alpha}$-contraction is dominant. Thus, Theorem 2.5 is applicable for studying Example 2.8. Also, it is observed that the value of $\mathcal{G}(-\frac{1}{5},\frac{1}{5},\frac{1}{5})$ lies between the condensed conditions (CK1) and (CK2), that is, $0.8 \in (0.7178, 0.8755)$. Therefore, Example 2.8 satisfies all hypotheses of Theorem 2.4.

3. Discrepancy measures

In this section, we study another variant for measuring the discrepancy between the condensed conditions (CK1) and (CK2) associated with the positive real constant α. This measure suggests all suitable values of $\alpha \in (0,1)$ that are reliable for approximating any nonlinear map $\mathcal{T}$ satisfying (2.1). We define a discrepancy measure for the condensed Kannan-type $\mathcal{G}_{\mu,\lambda,\alpha}$ map as follows:

Definition 3.1. Let $\Upsilon : \mathbb{R}^+ \times \mathbb{R}^+ \times (0,1) \to \mathbb{R}^+$ be a multivalued function and α-measure of the condensed Kannan-type map be given by a non-negative real number

$$\Delta_\alpha = |\Upsilon(\upsilon,\nu,\alpha)|,$$

where $\Upsilon(\upsilon,\nu,\alpha) = \upsilon^\alpha - \nu^{1-\alpha}$, $\upsilon,\nu \in \mathbb{R}^+$ are $\mathcal{G}$-metric terms with $\upsilon \neq \nu$ and $\alpha \in (0,1)$.

(1) If $\Delta_\alpha = 0$, then the condensed Kannan-type map is highly reliable.

(2) If $0 < \Delta_\alpha < 1$, then the condensed Kannan-type map is reliable.

(3) If $\Delta_\alpha \geq 1$, then the condensed Kannan-type map is not reliable.

Theorem 3.2. *Let $(\mathcal{M},\mathcal{G})$ be a complete $\mathcal{G}$-metric space and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a self-map. If $\mathcal{T}$ is a condensed Kannan-type $\mathcal{G}_{\mu,\lambda,\alpha}$-contraction, then there exists an $\alpha \in (0,1)$ for which $\Delta_\alpha = 0$.*

Proof. Since $\mathcal{T}$ is a condensed Kannan-type $\mathcal{G}_{\mu,\lambda,\alpha}$-contraction, then for $u_n \in \mathcal{M}$ such that $u_n \neq u_{n+1} \neq u_{n+2}$ for each $n \in \mathbb{N}$, we have from (CK1) that

$$\begin{aligned}\mathcal{G}(u_n, u_{n+1}, u_{n+2}) &= \mathcal{G}(\mathcal{T}u_{n-1}, \mathcal{T}u_n, \mathcal{T}^2 u_n) \qquad (3.1)\\ &\leq \lambda\left[\mathcal{G}(u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}^2 u_{n-1})^{2\alpha} + \lambda\mathcal{G}(u_n, \mathcal{T}u_n, \mathcal{T}^2 u_n)^{2(1-\alpha)}\right]\\ &= \lambda\left[\mathcal{G}(u_{n-1}, u_n, u_{n+1})^{2\alpha} + \mathcal{G}(u_n, u_{n+1}, u_{n+2})^{2(1-\alpha)}\right].\end{aligned}$$

Also from (CK2), we get

$$\begin{aligned}\mathcal{G}(u_n, u_{n+1}, u_{n+2}) &= \mathcal{G}(\mathcal{T}u_{n-1}, \mathcal{T}u_n, \mathcal{T}^2 u_n)\\ &\geq \mu\mathcal{G}(u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}^2 u_{n-1})^{\alpha}\mathcal{G}(u_n, \mathcal{T}u_n, \mathcal{T}^2 u_n)^{1-\alpha}\\ &= \mu\mathcal{G}(u_{n-1}, u_n, u_{n+1})^{\alpha}\mathcal{G}(u_n, u_{n+1}, u_{n+2})^{1-\alpha}. \qquad (3.2)\end{aligned}$$

By letting $\upsilon_n = \mathcal{G}(u_{n-1}, u_n, u_{n+1})$, $\nu_n = \mathcal{G}(u_n, u_{n+1}, u_{n+2})$, and combining both (3.1) and (3.2), these give

$$\Delta_\alpha^2 \equiv \left(\upsilon_n^\alpha - \nu_n^{1-\alpha}\right)^2 \geq 0, \qquad (3.3)$$

where $\Delta_\alpha \equiv |\upsilon_n^\alpha - \nu_n^{1-\alpha}|$. Here, two possibilities arise from inequality (3.3), that is, $\upsilon^\alpha \leq \nu_n^{1-\alpha}$ and $\upsilon^\alpha \geq \nu_n^{1-\alpha}$.

By inserting $\upsilon^\alpha \leq \nu_n^{1-\alpha}$ into (3.2), we have

$$\nu_n \geq \mu\nu_n^{2(1-\alpha)}. \qquad (3.4)$$

Solving (3.1) and (3.4), we obtain a recursive relation

$$\nu_n^\alpha \leq \left(\frac{\lambda}{\mu-\lambda}\right)^{\frac{1}{2}} \upsilon_n^{1-\alpha}. \qquad (3.5)$$

Similarly, by inserting $\upsilon^\alpha \geq \nu_n^{1-\alpha}$ into (3.2), and then solve using (3.4), we obtain another recursive relation

$$\upsilon_n^{1-\alpha} \leq \left(\frac{\lambda}{\mu-\lambda}\right)^{\frac{1}{2}} \nu_n^\alpha. \qquad (3.6)$$

By subtracting (3.6) from (3.5), we obtain

$$(\nu_n^\alpha - \upsilon_n^{1-\alpha})^2 \leq \frac{\lambda}{\mu-\lambda}(\upsilon_n^{1-\alpha} - \nu_n^\alpha)^2$$

or simply,

$$\Delta_\alpha^2 \leq \frac{\lambda}{\mu-\lambda}\Delta_\alpha^2. \qquad (3.7)$$

Since $\frac{\lambda}{\mu-\lambda} < 1$ for any $\lambda \in (0, \frac{1}{6})$ and $\mu \in (0, \frac{1}{3})$, then inequality (3.7) gives a contradiction, that is, $\Delta_\alpha \leq \left(\frac{\lambda}{\mu-\lambda}\right)^{\frac{1}{2}} \Delta_\alpha < \Delta_\alpha$. Therefore, $\Delta_\alpha = 0$. □

Remark 3.3. (1) The converse of Theorem 3.2 is true and the proof is obvious.

(2) If there exists an $\alpha^* \in (0,1)$ such that $\Delta_{\alpha^*} = 0$, then the point $(\alpha^*, \Delta_{\alpha^*})$ is stationary.

As an application of Theorem 3.2, we demonstrate the reliability of the map $\mathcal{T}$ in Example 2.8. This provides a clear framework for choosing suitable real constants α in the interval $(0,1)$.

Example 3.4. Let $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a self-map as presented in Example 2.8. Then, $\mathcal{T}$ forms the condensed Kannan-type $\mathcal{G}_{\mu,\lambda,\alpha}$-contraction for all $\alpha \in (0,1)$ at $u = v = 1$ with stationary value $(\alpha^*, \Delta_{\alpha^*}) = (0.5, 0)$.

To see this, we select the pair of points

$$(u,v) = \{(-1,-1),(1,1),(-1,1),(1,-1)\}$$

in the set $\{-1,1\} \subset \mathcal{M}$ to plot four graphs of Δ_α against α.

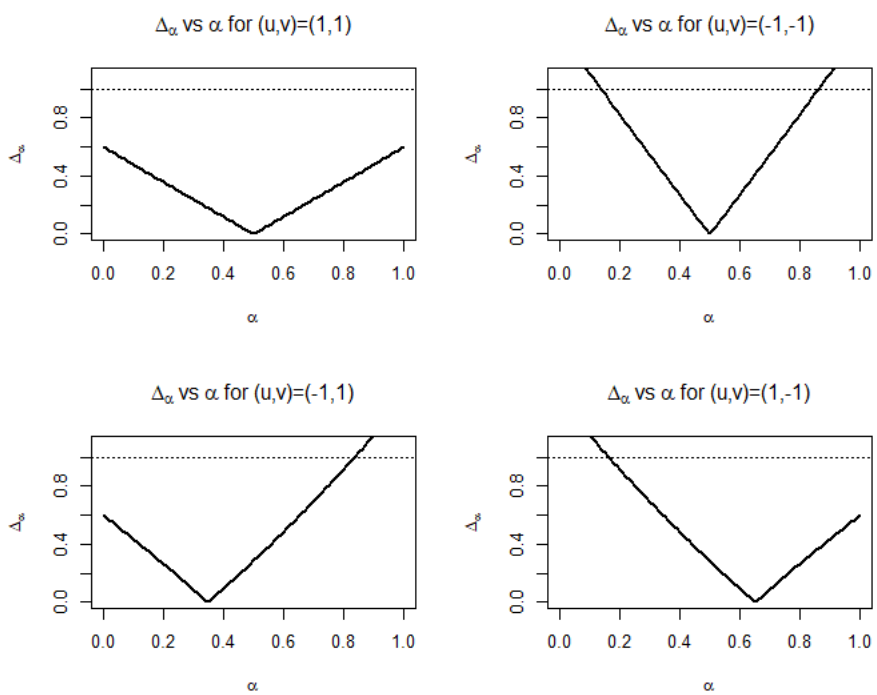

FIGURE 1. Graphs of Δ_α against α for any pair of $\{-1,1\}$

Table 2. Reliability of the condensed Kannan-type map

Reliability of $\mathcal{T}$ in Example 2.8			
Point (u, v)	Range of Δ_α	Range of α	$(\alpha^*, \Delta_{\alpha^*})$
$(1, 1)$	$(0, 0.6)$	$(0, 1)$	$(0.5, 0)$
$(-1, -1)$	$(0, 1)$	$(0.139, 0.861)$	$(0.5, 0)$
$(-1, 1)$	$(0, 1)$	$(0, 0.861)$	$(0.3493, 0)$
$(1, -1)$	$(0, 1)$	$(0.139, 1)$	$(0.6507, 0)$

As shown in Figure 1, the map $\mathcal{T}$ forms a condensed Kannan-type $\mathcal{G}_{\mu,\lambda,\alpha}$-contraction and is reliable at the point $(u, v) = (1, 1)$ for all $\alpha \in (0, 1)$ with stationary point $(0.5, 0)$. The map $\mathcal{T}$ is also reliable at the points $(-1, -1)$, $(-1, 1)$, and $(1, -1)$, respectively, in the intervals $(0.139, 0.861)$, $(0, 0.861)$, and $(0.139, 1)$.

4. Conclusion

The study introduced the notion of condensed Kannan-type $\mathcal{G}_{\mu,\lambda,\alpha}$-contraction to prove the fixed point theorems of nonlinear operators in $\mathcal{G}$-metric spaces. The study also proposed a framework for selecting appropriate real constants α of the condensed map for examining nonlinear operators. We investigated the condensed map using Example 2.8 to validate and show its generality for any $u, v \in \mathcal{M}$. The practical experiments indicated that the condensed Kannan-type $\mathcal{G}_{\mu,\lambda,\alpha}$-contraction is strictly a larger class than some of the existing Kannan-type maps; it is robust and efficient for solving unique and non-unique fixed points in $\mathcal{G}$-metric spaces. Furthermore, we showed that the reliability of the condensed map varies and highly depends on $\mathcal{M}$ as presented in Figure 1. Lastly, the study enhanced the flexibility of existing Kannan-type mappings to study more complex systems in various fields where pairwise interactions may not be suitable.

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