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NUMERICAL SOLUTION FOR A NONLINEAR COUPLED SYSTEM OF LANGEVIN EQUATIONS VIA ABC FRACTIONAL OPERATORS

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Abstract. The aims of this work is to investigate the behavior of a coupled system of Langevin equations with modified Atangana-Baleanu-Caputo (MABC) fractional derivatives, which has broad applicability in understanding and simulating physical and biological processes characterized by random forces and fluctuations. We seek to analyze the effect of the MABC fractional operator on the properties of the solution and develop the conditions for the existence and uniqueness of solutions by employing some fixed-point theorems. Additionally, we investigate Ulam stability to assess solution stability and sensitivity to perturbations. Furthermore, a numerical scheme is provided to showcase the obtained results, allowing for practical simulations and a deeper understanding of the system's behavior. Overall, this study contributes valuable insights into the coupled Langevin equations with the modified Atangana-Baleanu-Caputo (MABC) fractional derivative, benefiting researchers in diverse scientific fields.

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1. Introduction and motivation

In recent decades, fractional calculus has emerged as a valuable tool in various branches of applied mathematics. Researchers have successfully employed fractional order differential equations to yield significant findings in diverse fields such as fluid mechanics, rheology, physiology, control theory, and electrodynamics [23, 27]. Numerous classical equations have been used to derive their fractional counterparts, and extensive investigations have been carried out to analyze the latter's solutions [7, 8].

In 1908, Paul Langevin conducted a groundbreaking study on the erratic (Brownian) motion of molecules, proposing an equation to describe the random movement of a single Brownian particle [13]. In recognition of his significant contribution, this equation was later named the Langevin equation [17]. The Langevin equation holds immense importance as a stochastic differential equation, finding extensive applications in diverse fields such as mathematics, physics, and biochemistry [20, 25]. One notable area where the Langevin equation has made a significant impact is in the study of multiatomic systems through direct molecular dynamics simulations. By employing Langevin dynamics-based approaches, researchers can gain insights into a wide range of phenomena. These simulations enable a detailed understanding of the complex dynamics and interactions within such systems [14, 29].

Moreover, the versatility and broad applicability of the Langevin equation have made it an invaluable tool for understanding and simulating various physical and biological processes characterized by random forces and fluctuations. In physics, the Langevin equation aids in studying Brownian motion, diffusion processes, and thermal fluctuations. In biochemistry, it has been employed to investigate the dynamics of biological macromolecules, such as proteins, nucleic acids, and lipid bilayers, shedding light on their folding pathways, conformational changes, and interactions with the surrounding environment [19, 24]. Overall, the Langevin equation stands as a cornerstone in the realm of stochastic modeling, providing a powerful framework for analyzing and simulating a wide range of phenomena in both physical and biological systems. Its continued utilization and development contribute to advancements in various scientific disciplines and pave the way for deeper insights into the intricate dynamics of the natural world [26].

Devi et al. [16] studied the stability, existence, and uniqueness of solutions of fractional Langevin equations FDEs. These equations involved Caputo Hadamard derivatives with independent orders and were subject to non-local integral and non-periodic boundary conditions. The researchers used the Krasnoselskii fixed point theorem and the Banach contraction mapping principle in their analysis. Ahmad et al. [4] utilized the Krasnoselskii fixed point theorem and the contraction mapping principle to establish the existence of solutions for the Langevin equation featuring two distinct Caputo fractional derivatives.

In [12], Baleanu et. al. studied some properties of the solution for the Mittag-Leffler-type fractional Langevin equation. Coupled system of fractional differential equations offers a more accurate basis for studying interrelated variables and their dynamics. It allows for the modeling of complex interactions, multiscale phenomena, memory effects, and generalizability to diverse applications. Many researchers have investigated properties of the solution for a coupled system of FDEs using different methodologies [5, 32].

Recently, Alrefai and Baleanu [10] investigated a new approach to fractional derivatives with a nonsingular kernel. This operator called the modified ABC fractional operator. By this operator, we can find new solutions which are not solvable with the ABC derivative [11]. For example, the homogenous FDEs in ABC derivative have only the trivial solution. While in the MABC-derivative it has a nonzero solution. In recent years, the MABC operator has garnered recognition for its versatility and applicability across various disciplines. It has proven effective in the analysis, control, and modeling of intricate phenomena. [3, 6, 22]. All studies above have made significant contributions, but none specifically focused on investigating a coupled system involving Langevin equations with MABC derivatives. This particular research area remains unexplored and offers intriguing possibilities for further investigation. When coupled with the MABC derivative, the coupled system of Langevin equations presents a unique and complex mathematical framework that warrants dedicated attention. Future research endeavors in this direction promise to unravel new insights and advance our understanding of the dynamics and behavior of such systems.

Motivated by [4, 12] with the significance as mentioned above of the Langevin equations [13] and MABC fractional operators [10], we analyze the system dynamics and gain insights into how the specific MABC fractional operator affects the properties of solution as well as the Ulam-Hyers stability of the following coupled system of fractional Langevin equation:

$$\begin{cases} {}^{MABC}\mathbb{D}^{\alpha_1}\left({}^{ABC}\mathbb{D}^{\sigma_1}+\mu_1\right)f_1(t)=g_1(t,f_1(t),f_2(t)), t\in(0,b), \\ {}^{MABC}\mathbb{D}^{\alpha_2}\left({}^{ABC}\mathbb{D}^{\sigma_2}+\mu_2\right)f_2(t)=g_2(t,f_1(t),f_2(t)), t\in(0,b) \end{cases} \tag{1.1}$$

with the initial conditions

$$\begin{cases} f_1(0)=A_1, {}^{ABC}\mathbb{D}^{\sigma_1}f_1(0)=B_1, \\ f_2(0)=A_2, {}^{ABC}\mathbb{D}^{\sigma_2}f_2(0)=B_2, \end{cases}$$

where ${}^{MABC}\mathbb{D}^{\alpha_i}$ is the MABC-FD of order $\alpha_i \in (0,1]$, $(i=1,2)$, ${}^{ABC}\mathbb{D}^{\sigma_i}$ is the ABC-FD of order $\sigma_i \in (0,1]$, $(i=1,2)$, $g_i : (0,b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions such that $g_i(0,0,0)| = 0$, $(i=1,2)$ and $\mu_i > 0$, and $A_{i,} B_i \in \mathbb{R}$, $(i=1,2)$.

In Langevin equations (1.1), $f_1(t)$ and $f_2(t)$ represent the positions of the particles, while the functions g_1 and g_2 denote the forces acting on the particles from the surrounding fluid molecules. The terms μ_1 and μ_2 correspond to the damping or viscosity coefficients, and A_i and B_i $(i=1,2)$ represent the initial positions of the particles.

Our study makes a valuable contribution to enhancing our understanding of the dynamics associated with coupled systems. The contributions of this work can be summarized as follows:

(1) Based on our current knowledge, no previous investigations have focused on analyzing this coupled system using a combination of the MABC operators. This work can be viewed as a generalization of [12] and [2].
(2) We establish necessary and sufficient conditions for the existence, uniqueness, and stability of a coupled system of Langevin equations using a new fractional operator by employing the Banach contraction principle and Leray-Schauder's alternative fixed-point theorem. These conditions provide a solid foundation for analyzing the dynamics of coupled systems and contribute to the understanding of their behavior.
(3) We propose a numerical scheme for solving the coupled system of Langevin equations using Lagrange's interpolation method. This numerical scheme not only extends the application of the MABC operator but also offers a practical approach to modeling problems in various fields. We demonstrate its application in areas such as Brownian motion, anomalous diffusion, and modeling the dynamics of population sizes. Our research findings emphasize the practical significance of this operator in the fields of physics and biology.

The paper is organized as follows. Section 2 provides a review of background definitions and lemmas from fractional calculus. Additionally, an important lemma is proven, which allows us to convert the coupled system described in equation (1.1) into an equivalent integral equation. In Section 3, the primary existence and uniqueness of solutions for the coupled system (1.1) are established. Section 4 focuses on establishing the Ulam-Hyers stability of the system. Section 5 presents a numerical example that serves to illustrate the aforementioned results. We provide two applications of coupled systems of Langevin equations in Section 6. Finally, the last Section concludes the paper, summarizing the findings and implications of the study.

2. Preliminary results and concepts

This section starts with a quick overview of MABC operator with a non-singular kernel. We introduce notations, definitions, and provide results that are needed later. Throughout this paper, we fix the notation to be as follows:

(1) $\Omega = [0, b] \subset \mathbb{R}$ where $b > 0$.
(2) $C(\Omega, \mathbb{R})$ be the Banach space of all continuous functions $f : \Omega \to \mathbb{R}$ equipped with the norm
$$\|f\| = \max\{|f(t)| : t \in \Omega\}.$$
(3) The product space $C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R})$ is a Banach space with the norm
$$\|(f_1, f_2)\| = \max\{\|f_1\|, \|f_2\|\}.$$
(4) $H^1(\Omega)$ denotes the Sobolev space $\{f \in L^2(\Omega) : f' \in L^2(\Omega)\}$.
(5) Π_δ denotes the closed ball in $C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R})$ with radius δ centered on $(0, 0)$ where 0 is the zero function. This closed ball is given by
$$\Pi_\delta = \{(f_1, f_2) \in C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R}) : \|(f_1, f_2)\| \leq \delta\}.$$
(6) For each $i = 1, 2$, let
$$\Delta_i = \left(1 + \frac{\mu_i (1 - \sigma_i)}{\mathfrak{B}(\sigma_i)}\right) \neq 0,$$
where $\alpha_i, \sigma_i, \mu_i$ as given in system (1.1), and $\mathfrak{B}(x)$ is a normalization function given in Definition 2.2 below.

Definition 2.1. ([27]) For $\alpha > 0$ and $f \in L_1(\Omega)$, the Riemann-Liouville fractional integral and the Riemann-Liouville fractional derivative of f with fractional order α are defined by the following formulas
$${}^{RL}\mathbb{I}_{0^+}^{\alpha} f(t) = \int_0^t \frac{(t - \theta)^{\alpha - 1}}{\Gamma(\alpha)} f(\theta) d\theta$$
and
$${}^{RL}\mathbb{D}_{0^+}^{\alpha} f(t) = \left(\frac{d}{dt}\right)^n \left({}^{RL}\mathbb{I}_{0^+}^{n-\alpha} f(t)\right),$$
respectively.

Another useful approach to fractional calculus is Atangana-Baleanu model, which has non-singular kernel [11].

Definition 2.2. ([11]) For $\alpha \in (0, 1]$ and $f \in H^1(\Omega)$. The ABC-FD of order α for the function f is defined by
$${}^{ABC}\mathbb{D}_{0^+}^{\alpha} f(t) = \frac{\mathfrak{B}(\alpha)}{1 - \alpha} \int_0^t f'(s) E_\alpha \left[-\Phi_\alpha (t - s)^\alpha\right] ds,$$

where, $\Phi_\alpha = \frac{\alpha}{1-\alpha}$ and E_α is the generalized Mittag-Leffler function, and the normalization function $\mathfrak{B}(\alpha)$ satisfies the conditions $\mathfrak{B}(0) = \mathfrak{B}(1) = 1$.

Definition 2.3. ([11]) For $\alpha \in (0,1]$ and $f \in H^1(\Omega)$, the following representation

$$^{AB}\mathbb{I}^{\alpha}_{0+} f(t) = \frac{1-\alpha}{\mathfrak{B}(\alpha)} f(t) + \frac{\alpha}{\mathfrak{B}(\alpha)}\ ^{RL}\mathbb{I}^{\alpha}_{0+} f(t)$$

is the AB fractional integral of ABC-FD of order α for the function f.

Lemma 2.4. ([28]) *For $\alpha \in (0,1]$ and $f \in H^1(\Omega)$. If ABC-FD exists, then*

$$^{AB}\mathbb{I}^{\alpha}_{0+}\ ^{ABC}\mathbb{D}^{\alpha}_{0+} f(t) = f(t) - f(0).$$

Lemma 2.5. ([1, 11]) *Let $f(t)$ be a function defined on Ω and $n < \alpha \leq n+1$. The following properties are hold*

(1) $\left(^{ABC}\mathbb{D}^{\alpha}_{0+} {}^{AB}\mathbb{I}^{\alpha}_{0+} f\right)(t) = f(t)$,

(2) $\left(^{AB}\mathbb{I}^{\alpha}_{0+} {}^{ABC}\mathbb{D}^{\alpha}_{0+} f\right)(t) = f(t) - \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} t^i$, *for some $n \in \mathbb{N}$.*

Definition 2.6. ([10]) For $\alpha \in (0,1]$ and $f \in H^1(\Omega)$. The MABC-FD of order α for the function f is defined by

$$\begin{aligned} ^{MABC}\mathbb{D}^{\alpha}_{0+} f(t) = \frac{\mathfrak{B}(\alpha)}{1-\alpha}\Big[& f(t) - E_\alpha\left(-\Phi_\alpha t^\alpha\right) f(0) \\ & - \Phi_\alpha \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\Phi_\alpha (t-s)^\alpha\right) f(s)\, ds\Big]. \end{aligned}$$

By [9], we have $^{MABC}\mathbb{D}^{\alpha}_{0+} C = 0$, where C is constant.

Definition 2.7. ([9, 10]) For $\alpha \in (0,1]$ and $f \in H^1(\Omega)$, the following representation

$$^{mAB}\mathbb{I}^{\alpha}_{0+} f(\iota) = \frac{1-\alpha}{\mathfrak{B}(\alpha)}\left[f(\iota) - f(0)\right] + \frac{\alpha}{\mathfrak{B}(\alpha)}\ ^{RL}\mathbb{I}^{\alpha}_{0+}\left[f(\iota) - f(0)\right]$$

or

$$^{mAB}\mathbb{I}^{\alpha}_{0+} f(\iota) = \frac{1-\alpha}{\mathfrak{B}(\alpha)}\left[f(\iota) + \Phi_\alpha {}^{RL}\mathbb{I}^{\alpha}_{0+} f(\iota) - f(0)\left(1 + \Phi_\alpha \frac{\iota^\alpha}{\Gamma(\alpha+1)}\right)\right]$$

is the mAB fractional integral of order α for the function f. By this definition, one can verify that $^{mAB}\mathbb{I}^{\alpha}_{0+} C = 0$, where C is constant.

Lemma 2.8. ([10]) *For $\alpha \in (0,1]$ and $\mathfrak{g} \in H^1(\Omega)$. If MABC-FD exists, then*

$$^{mAB}\mathbb{I}^{\alpha}_{0+}\ ^{MABC}\mathbb{D}^{\alpha}_{0+} f(\iota) = f(\iota) - f(0).$$

Lemma 2.9. ([10]) *Let $f(\iota)$ be a function defined on Ω and $n < \alpha \leq n+1$. The following properties are hold*

(1) $\left({}^{MABC}\mathbb{D}^{\alpha}_{0^+}\ {}^{mAB}\mathbb{I}^{\alpha}_{0^+} f\right)(\iota) = f(\iota)$,

(2) $\left({}^{mAB}\mathbb{I}^{\alpha}_{0^+}\ {}^{MABC}\mathbb{D}^{\alpha}_{0^+} f\right)(\iota) = f(\iota) - \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!}\iota^i$, *for some $n \in \mathbb{N}$.*

3. Main results

We start our study by investigating the corresponding linear problem and using the results to tackle the nonlinear fractional coupled system (1.1).

3.1. **Linear fractional coupled system.**

Lemma 3.1. *Let $\alpha_i, \sigma_i \in (0,1]$ and $y_i \in C(\Omega, \mathbb{R})$ with $y_i(0) = 0$. The pair of functions $(f_1, f_2) \in C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R})$ is a solution of the ABC-system*

$$ {}^{MABC}\mathbb{D}^{\alpha_i}_{0^+}\left({}^{ABC}\mathbb{D}^{\sigma_i}_{0^+} + \mu_i\right) f_i(t) = y_i(t) \quad i = 1, 2, \tag{3.1}$$

with conditions

$$ f_i(0) = A_i, \quad {}^{ABC}\mathbb{D}^{\sigma_i}_{0^+} f_i(0) = B_i, \quad i = 1, 2, \tag{3.2}$$

if and only (f_1, f_2) satisfies the following integral equations

$$\begin{aligned} f_i(t) = A_i + \frac{1}{\Delta_i}\Bigg\{ & \frac{(\mu_i A_i + B_i)t^{\sigma_i}}{\mathfrak{B}(\sigma_i)\Gamma(\sigma_i)} + \frac{(1-\sigma_i)(1-\alpha_i)}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)} y_i(t) \\ & + \frac{(1-\sigma_i)\alpha_i}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\ {}^{RL}\mathbb{I}^{\alpha_i}_{0^+} y_i(t) + \frac{\sigma_i(1-\alpha_i)}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\ {}^{RL}\mathbb{I}^{\sigma_i}_{0} y_i(t) \\ & + \frac{\sigma_i\alpha_i}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\ {}^{RL}\mathbb{I}^{\alpha_i+\sigma_i}_{0^+} y_i(t) - \frac{\mu_i\sigma_i}{\mathfrak{B}(\sigma_i)}\ {}^{RL}\mathbb{I}^{\sigma_i}_{0} f_i(t) \Bigg\} \end{aligned} \tag{3.3}$$

for each $i = 1, 2$.

Proof. Assume that (f_1, f_2) is a solution of Equations (3.1). Applying the operators ${}^{MAB}\mathbb{I}^{\alpha_1}_{0^+}$ and ${}^{MAB}\mathbb{I}^{\alpha_2}_{0^+}$ on both sides of the equations in (3.1) respectively, we have

$$ {}^{MAB}\mathbb{I}^{\alpha_i}_{0^+}{}^{MABC}\mathbb{D}^{\alpha_i}_{0^+}\left({}^{ABC}\mathbb{D}^{\sigma_i}_{0^+} + \mu_i\right) f_i(t) = {}^{MAB}\ \mathbb{I}^{\alpha_i}_{0^+} y_i(t), \quad i = 1, 2.$$

In view of Definition 2.7, and Lemma 2.8, we have

$$\begin{aligned} \left({}^{ABC}\mathbb{D}^{\sigma_i}_{0^+} + \mu_i\right) f_i(t) = & \left({}^{ABC}\mathbb{D}^{\sigma_i}_{0^+} + \mu_i\right) f_i(0) + \frac{1-\alpha_i}{\mathfrak{B}(\alpha_i)} y_i(t) \\ & + \frac{\alpha_i}{\mathfrak{B}(\alpha_i)}\ {}^{RL}\mathbb{I}^{\alpha_i}_{0^+} y_i(t) - y_i(0)\left(1 + \Phi_{\alpha_i}\frac{t^{\alpha_i}}{\Gamma(\alpha_i+1)}\right). \end{aligned} \tag{3.4}$$

By assumption $y_i(0) = 0$, and the conditions in (3.2), we obtain

$$ {}^{ABC}\mathbb{D}_{0^+}^{\sigma_i} f_i(t) = \frac{1-\alpha_i}{\mathfrak{B}(\alpha_i)} y_i(t) + \frac{\alpha_i}{\mathfrak{B}(\alpha_i)} {}^{RL}\mathbb{I}_{0^+}^{\alpha_i} y_i(t) + \mu_i A_i + B_i - \mu_i f_i(t), \quad i = 1, 2. \tag{3.5} $$

Applying the operators ${}^{AB}\mathbb{I}_{0^+}^{\sigma_1}$ and ${}^{AB}\mathbb{I}_{0^+}^{\sigma_2}$ to both sides of the two equations in (3.5) respectively, we have

$$ \begin{aligned} {}^{AB}\mathbb{I}_{0^+}^{\sigma_i}\,{}^{ABC}\mathbb{D}_{0^+}^{\sigma_i} f_i(t) = {}^{AB}\mathbb{I}_{0^+}^{\sigma_i} \Big[& \frac{1-\alpha_i}{\mathfrak{B}(\alpha_i)} y_i(t) + \frac{\alpha_i}{\mathfrak{B}(\alpha_i)} {}^{RL}\mathbb{I}_{0^+}^{\alpha_i} y_i(t) \\ & + \mu_i A_i + B_i - \mu_i f_i(t) \Big], \quad i = 1, 2. \end{aligned} $$

By Lemmas 2.4, and 2.5, we get

$$ \begin{aligned} f_i(t) = f_i(0) + {}^{AB}\mathbb{I}_{0^+}^{\sigma_i} \Big[& \frac{1-\alpha_i}{\mathfrak{B}(\alpha_i)} y_i(t) + \frac{\alpha_i}{\mathfrak{B}(\alpha_i)} {}^{RL}\mathbb{I}_{0^+}^{\alpha_i} y_i(t) \\ & + \mu_i A_i + B_i - \mu_i f_i(t) \Big], \quad i = 1, 2. \end{aligned} \tag{3.6} $$

By Definition 2.3, we have

$$ \begin{aligned} f_i(t) = & \frac{(1-\sigma_i)(1-\alpha_i)}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)} y_i(t) + \frac{(1-\sigma_i)\,\alpha_i}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)} {}^{RL}\mathbb{I}_{0^+}^{\alpha_i} y_i(t) \\ & + \frac{(1-\sigma_i)\,\mu_i A_i}{\mathfrak{B}(\sigma_i)} + \frac{(1-\sigma_i)}{\mathfrak{B}(\sigma_i)} B_i - \frac{(1-\sigma_i)}{\mathfrak{B}(\sigma_i)} \mu_i f_i(t) \\ & + \frac{\sigma_i(1-\alpha_i)}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)} {}^{RL}\mathbb{I}_{0^+}^{\sigma_i} y_i(t) + \frac{\sigma_i}{\mathfrak{B}(\sigma_i)} \frac{\alpha_i}{\mathfrak{B}(\alpha_i)} {}^{RL}\mathbb{I}_{0^+}^{\sigma_i+\alpha_i} y_i(t) \\ & + \frac{\sigma_i}{\mathfrak{B}(\sigma_i)} {}^{RL}\mathbb{I}_{0^+}^{\sigma_i} \mu_i A_i + \frac{\sigma_i}{\mathfrak{B}(\sigma_i)} {}^{RL}\mathbb{I}_{0^+}^{\sigma_i} B_i - \frac{\sigma_i}{\mathfrak{B}(\sigma_i)} {}^{RL}\mathbb{I}_{0^+}^{\sigma_i} \mu_i f_i(t) + c_i, \end{aligned} \tag{3.7} $$

where c_1 and c_2 are constants. Substituting $t = 0$ in the above equations and using the fact $y_i(0) = 0$, for $i = 1, 2$, $f_1(0) = A_1$, and $f_2(0) = A_2$, we get

$$ c_i = A_i - \frac{1-\sigma_i}{\mathfrak{B}(\sigma_i)} B_i, \quad i = 1, 2. $$

Substituting c_1 and c_2 in Equation (3.7), we get the equations in (3.3). So, (f_1, f_2) is a solution of (3.3).

Conversely, if (f_1, f_2) is a solution of (3.3), then by applying the operators ${}^{ABC}\mathbb{D}_{0^+}^{\sigma_1}$ and ${}^{ABC}\mathbb{D}_{0^+}^{\sigma_2}$ to (3.3), we get (3.4). Next, applying the operators ${}^{ABC}\mathbb{D}_{0^+}^{\alpha_1}$ and ${}^{ABC}\mathbb{D}_{0^+}^{\alpha_2}$ to (3.4), we can get initial value problem (3.1). In view of (3.3) and (3.4), the conditions (3.2) follows. □

3.2. **Nonlinear fractional coupled system.** This part considers a coupled nonlinear system (1.1). The following lemma follows directly from Lemma 3.1.

Lemma 3.2. *For $i=1,2$, let $\alpha_i,\sigma_i\in(0,1]$ and $\mathbb{G}^i_{f_1,f_2}=g_i(t,f_1(t),f_2(t)):\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous and differentiable function such that $\mathbb{G}^i_{f_1,f_2}(0)=0, i=1,2$. Then the pair of functions $(f_1,f_2)\in C(\Omega,\mathbb{R})\times C(\Omega,\mathbb{R})$ is a solution of the system (1.1) if and only if (f_1,f_2) satisfies the following integral equations*

$$
\begin{aligned}
f_i(t) = A_i + \frac{1}{\Delta_i}\Bigg\{ & \frac{(\mu_i A_i + B_i)\, t^{\sigma_i}}{\mathfrak{B}(\sigma_i)\Gamma(\sigma_i)} + \frac{(1-\sigma_i)(1-\alpha_i)}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\mathbb{G}^i_{f_1,f_2}(t) \\
& + \frac{(1-\sigma_i)\,\alpha_i}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\,{}^{RL}\mathbb{I}^{\alpha_i}_{0^+}\mathbb{G}^i_{f_1,f_2}(t) + \frac{\sigma_i(1-\alpha_i)}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\,{}^{RL}\mathbb{I}^{\sigma_i}_{0}\mathbb{G}^i_{f_1,f_2}(t) \\
& + \frac{\sigma_i\alpha_i}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\,{}^{RL}\mathbb{I}^{\alpha_i+\sigma_i}_{0^+}\mathbb{G}^i_{f_1,f_2}(t) - \frac{\mu_i\sigma_i}{\mathfrak{B}(\sigma_i)}\,{}^{RL}\mathbb{I}^{\sigma_i}_{0}f_i(t)\Bigg\}.
\end{aligned}
$$

To obtain our main results, the following conditions must be assumed.

H_1 : For $i=1,2$, $\mathbb{G}^i_{f_1,f_2}(t)=g_i(t,f_1(t),f_2(t))$ are continuous functions and there exists a constant numbers $\mathfrak{L}_i>0$ such that

$$\left|\mathbb{G}^i_{f_1,f_2}(t)-\mathbb{G}^i_{\widehat{f}_1,\widehat{f}_2}(t)\right| \le \mathfrak{L}_i\left(\left|f_1(t)-\widehat{f}_1(t)\right|+\left|f_2(t)-\widehat{f}_2(t)\right|\right)$$

for any $f_i,\widehat{f}_i\in C(\Omega,\mathbb{R})$.

H_2 : For $i=1,2$, the functions $\mathbb{G}^i_{f_1,f_2}(t):\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions such that for $(t,f_1,f_2)\in\Omega\times\mathbb{R}\times\mathbb{R}$ we have

$$\left|\mathbb{G}^i_{f_1,f_2}(t)\right| \le \eta_{g_i}(t)+\Upsilon_{g_i}(t)\left|f_1(t)\right|+\varpi_{g_i}(t)\left|f_2(t)\right|,$$

where $\eta_{g_i},\Upsilon_{g_i},\varpi_{g_i}\in C(\Omega,\mathbb{R})$ are nonnegative functions.

To simplify our discussion, the following notations are used. For $i=1,2$, let

$$\eta^*_{g_i}=\max_{t\in\Omega}\left|\eta_{g_i}(t)\right|,\ \Upsilon^*_{g_i}=\max_{t\in\Omega}\left|\Upsilon_{g_i}(t)\right|,\ \varpi^*_{g_i}=\max_{t\in\Omega}\left|\varpi_{g_i}(t)\right|$$

and

$$\psi_{g_i}=\max_{t\in\Omega}\left|\mathbb{G}^i_{0,0}(t)\right|<\infty.$$

We also use

$$
\begin{aligned}
\mathbb{Q}_i = & \frac{(1-\sigma_i)(1-\alpha_i)}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)} + \frac{(1-\sigma_i)\, b^{\alpha_i}}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)\Gamma(\alpha_i)} \\
& + \frac{(1-\alpha_i)\, b^{\sigma_i}}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)\Gamma(\sigma_i)} + \frac{\sigma_i\alpha_i b^{\sigma_i+\alpha_i}}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)\Gamma(\sigma_i+\alpha_i+1)},
\end{aligned}
\tag{3.8}
$$

$$\mathcal{Z}_i = \frac{2\mathfrak{L}_i}{\Delta_i}\left(\mathbb{Q}_i+\frac{\mu_i b^{\sigma_i}}{\mathfrak{B}(\sigma_i)\Gamma(\sigma_i)}\right), \tag{3.9}$$

and

$$\mathbb{K} = \sum_{i=1}^{2} |A_i| + \frac{(\mu_i A_i + B_i)\, b^{\sigma_i}}{\Delta_i \mathfrak{B}(\sigma_i)\Gamma(\sigma_i)} + \frac{\eta^*_{g_i}}{\Delta_i}\left[\mathbb{Q}_i + \frac{\mu_i b^{\sigma_i}}{\mathfrak{B}(\sigma_i)\Gamma(\sigma_i)}\right] + \frac{\Upsilon^*_{g_i}}{\Delta_i}\left[\mathbb{Q}_i + \frac{\mu_i b^{\sigma_i}}{\mathfrak{B}(\sigma_i)\Gamma(\sigma_i)}\right]\|f_1\| + \frac{\varpi^*_{g_i}}{\Delta_i}\left[\mathbb{Q}_i + \frac{\mu_i b^{\sigma_i}}{\mathfrak{B}(\sigma_i)\Gamma(\sigma_i)}\right]\|f_2\|.$$

Using hypotheses (H_1), we have

$$\begin{aligned}\left|\mathbb{G}^i_{f_1,f_2}(t)\right| &\le \left|\mathbb{G}^i_{f_1,f_2}(t) - \mathbb{G}^i_{0,0}(t)\right| + \left|\mathbb{G}^i_{0,0}(t)\right| \\ &\le \mathfrak{L}_i\left(\|f_1\| + \|f_2\|\right) + \psi_{g_i}. \end{aligned} \tag{3.10}$$

We end the subsection with the following definition which is needed later .

Definition 3.3. For $i = 1, 2$, let the operators $\mathcal{T}_1, \mathcal{T}_2 : C(\Omega,\mathbb{R}) \times C(\Omega,\mathbb{R}) \to C(\Omega,\mathbb{R}) \times C(\Omega,\mathbb{R})$ be given as

$$\begin{aligned}\mathcal{T}_i\left(f_1,f_2\right)(t) = A_i + \frac{1}{\Delta_i}\Bigg\{&\frac{(\mu_i A_i + B_i)\, t^{\sigma_i}}{\mathfrak{B}(\sigma_i)\Gamma(\sigma_i)} + \frac{(1-\sigma_i)(1-\alpha_i)}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\mathbb{G}^i_{f_1,f_2}(t) \\ &+ \frac{(1-\sigma_i)\,\alpha_i}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\ {}^{RL}\mathbb{I}^{\alpha_i}_{0^+}\mathbb{G}^i_{f_1,f_2}(t) + \frac{\sigma_i(1-\alpha_i)}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\ {}^{RL}\mathbb{I}^{\sigma_i}_{0}\mathbb{G}^i_{f_1,f_2}(t) \\ &+ \frac{\sigma_i\alpha_i}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\ {}^{RL}\mathbb{I}^{\alpha_i+\sigma_i}_{0^+}\mathbb{G}^i_{f_1,f_2}(t) - \frac{\mu_i\sigma_i}{\mathfrak{B}(\sigma_i)}\ {}^{RL}\mathbb{I}^{\sigma_i}_{0} f_i(t)\Bigg\}.\end{aligned}$$

Define the operator $\mathcal{T} : C(\Omega,\mathbb{R}) \times C(\Omega,\mathbb{R}) \to C(\Omega,\mathbb{R}) \times C(\Omega,\mathbb{R})$ by

$$\mathcal{T}\left(f_1,f_2\right)(t) = \left(\mathcal{T}_1\left(f_1,f_2\right)(t), \mathcal{T}_2\left(f_1,f_2\right)(t)\right). \tag{3.11}$$

3.3. **Uniqueness of the solution.** In this part, we state the Banach fixed point theorem which will be used in proving the uniqueness result of the MABC coupled system (1.1).

Theorem 3.4. ([15]) *Let $\mathbb{X}$ be a Banach space and $\mathbb{K} \subset \mathbb{X}$ be a closed subspace. If $\mathbb{G} : \mathbb{K} \to \mathbb{K}$ is a mapping such that*

$$\|\mathbb{G}(x) - \mathbb{G}(y)\| \le \mathbb{L}\,\|x - y\| \ \text{ for some } 0 < \mathbb{L} < 1 \text{ and all } x, y \in \mathbb{K},$$

then $\mathbb{G}$ has a unique fixed point in $\mathbb{K}$.

Theorem 3.5. *Assume that (H_1) holds and that $0 < \mathcal{Z}_1, \mathcal{Z}_2 < 1$. If we choose*

$$\delta \ge \max_{i\in\{1,2\}}\left\{\frac{|A_i| + \frac{1}{\Delta_i}\left(\frac{(\mu_i A_i + B_i) b^{\sigma_i}}{\mathfrak{B}(\sigma_i)\Gamma(\sigma_i)} + \mathbb{Q}_i\psi_{g_i}\right)}{1 - \mathcal{Z}_i}\right\},$$

then the MABC coupled system (1.1) has a unique solution $(f_1, f_2) \in \Pi_\delta$.

Proof. We first show that $\mathcal{T}(\Pi_\delta) \subseteq \Pi_\delta$. For all $(f_1, f_2) \in \Pi_\delta$ and $t \in \Omega$, we have

$$\begin{aligned}|\mathcal{T}_1(f_1,f_2)(t)| \leq |A_1| + \frac{1}{\Delta_1}\Bigg\{ & \frac{(\mu_1 A_1 + B_1)\, t^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)} + \frac{(1-\sigma_1)(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)} \left|\mathbb{G}^1_{f_1,f_2}(t)\right| \\ & + \frac{(1-\sigma_1)\,\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)} {}^{RL}\mathbb{I}^{\alpha_1}_{0^+} \left|\mathbb{G}^1_{f_1,f_2}(t)\right| + \frac{\sigma_1(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)} {}^{RL}\mathbb{I}^{\sigma_1}_{0} \left|\mathbb{G}^1_{f_1,f_2}(t)\right| \\ & + \frac{\sigma_1\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)} {}^{RL}\mathbb{I}^{\alpha_1+\sigma_1}_{0^+} \left|\mathbb{G}^1_{f_1,f_2}(t)\right| - \frac{\mu_1\sigma_1}{\mathfrak{B}(\sigma_1)} {}^{RL}\mathbb{I}^{\sigma_1}_{0} |f_1(t)| \Bigg\}. \end{aligned} \tag{3.12}$$

In view of definition of RL-fractional integral, we get

$$ {}^{RL}\mathbb{I}^{\sigma_1}_{0^+} \left|\mathbb{G}^1_{f_1,f_2}(t)\right| = \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} \left|\mathbb{G}^1_{f_1,f_2}(\theta)\right| d\theta.$$

By (3.10), we have

$$\left|\mathbb{G}^1_{f_1,f_2}(t)\right| \leq \mathfrak{L}_1(\|f_1\| + \|f_2\|) + \psi_{g_1}.$$

Thus, we get

$$\begin{aligned} {}^{RL}\mathbb{I}^{\sigma_1}_{0^+} \left|\mathbb{G}^1_{f_1,f_2}(t)\right| &\leq \left[\mathfrak{L}_1(\|f_1\| + \|f_2\|) + \psi_{g_1}\right] \int_0^t \frac{(t-\theta)^{\sigma_1-1}}{\Gamma(\sigma_1)} d\theta \\ &\leq \mathfrak{L}_1(\|f_1\| + \|f_2\|) \frac{t^{\sigma_1}}{\Gamma(\sigma_1+1)} + \psi_{g_1} \frac{t^{\sigma_1}}{\Gamma(\sigma_1+1)}. \end{aligned}$$

Similarly, we get

$$ {}^{RL}\mathbb{I}^{\alpha_1}_{0^+} \left|\mathbb{G}^1_{f_1,f_2}(t)\right| \leq \mathfrak{L}_1(\|f_1\| + \|f_2\|) \frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)} + \psi_{g_1} \frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}$$

and

$$\begin{aligned} {}^{RL}\mathbb{I}^{\sigma_1+\alpha_1}_{0^+} \left|\mathbb{G}^1_{f_1,f_2}(t)\right| \leq\ & \mathfrak{L}_1(\|f_1\| + \|f_2\|) \frac{t^{\sigma_1+\alpha_1}}{\Gamma(\sigma_1+\alpha_1+1)} \\ & + \psi_{g_1} \frac{t^{\sigma_1+\alpha_1}}{\Gamma(\sigma_1+\alpha_1+1)}. \end{aligned}$$

Thus by (3.12), we have

$$\begin{aligned}
&|\mathcal{T}_1(f_1,f_2)(t)| \\
&\le |A_1| + \frac{1}{\Delta_1}\Bigg\{ \frac{(\mu_1 A_1 + B_1)\, t^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)} + \frac{(1-\sigma_1)(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)} \left(\mathfrak{L}_1(\|f_1\| + \|f_2\|) + \psi_{g_1}\right) \\
&\quad + \frac{(1-\sigma_1)\,\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)} \left(\mathfrak{L}_1(\|f_1\| + \|f_2\|) \frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)} + \psi_{g_1} \frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)} \right) \\
&\quad + \frac{\sigma_1(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)} \left(\mathfrak{L}_1(\|f_1\| + \|f_2\|) \frac{t^{\sigma_1}}{\Gamma(\sigma_1+1)} + \psi_{g_1} \frac{t^{\sigma_1}}{\Gamma(\sigma_1+1)} \right) \\
&\quad + \frac{\sigma_1\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)} \left(\mathfrak{L}_1(\|f_1\| + \|f_2\|) \frac{t^{\sigma_1+\alpha_1}}{\Gamma(\sigma_1+\alpha_1+1)} + \psi_{g_1} \frac{t^{\sigma_1+\alpha_1}}{\Gamma(\sigma_1+\alpha_1+1)} \right) \\
&\quad + \frac{\mu_1\sigma_1 b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1+1)} \|f_1\| \Bigg\} \\
&\le |A_1| + \frac{1}{\Delta_1}\left(\frac{(\mu_1 A_1 + B_1)\, b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)} + \mathbb{Q}_1\psi_{g_1} \right) \\
&\quad + \frac{\mathfrak{L}_1(\|f_1\| + \| \quad f_2\|)}{\Delta_1}\mathbb{Q}_1 + \frac{1}{\Delta_1}\frac{\mu_1 b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)} \|f_1\|,
\end{aligned}$$

where $\mathbb{Q}_1$ is given by (3.8). Taking the maximum on both sides of the above inequality, we obtain

$$\begin{aligned}
\|\mathcal{T}_1(f_1,f_2)\| &\le |A_1| + \frac{1}{\Delta_1}\left(\frac{(\mu_1 A_1 + B_1)\, b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)} + \mathbb{Q}_1\psi_{g_1} \right) \\
&\quad + \frac{2\mathfrak{L}_1}{\Delta_1}\left(\mathbb{Q}_1 + \frac{\mu_1 b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)} \right) \|(f_1,f_2)\| \\
&\le \mathcal{Z}_1\delta + |A_1| + \frac{1}{\Delta_1}\left(\frac{(\mu_1 A_1 + B_1)\, b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)} + \mathbb{Q}_1\psi_{g_1} \right) \\
&\le \delta.
\end{aligned}$$

In similar manner, starting with the operator $\mathcal{T}_2$, we get

$$\begin{aligned}
\|\mathcal{T}_2(f_1,f_2)\| &\le \mathcal{Z}_2\delta + |A_2| + \frac{1}{\Delta_2}\left(\frac{(\mu_2 A_2 + B_2)\, b^{\sigma_2}}{\mathfrak{B}(\sigma_2)\Gamma(\sigma_2)} + \mathbb{Q}_2\psi_{g_2} \right) \\
&\le \delta.
\end{aligned}$$

Choose a real number $\delta > 0$ as in the theorem's statement. Since $0 < \mathcal{Z}_1, \mathcal{Z}_2 < 1$, we have $\mathcal{T}(\Pi_\delta) \subseteq \Pi_\delta$.

Next, we show that the operator $\mathcal{T}$ is a contraction mapping.

Let $(f_1, f_2), \left(\widehat{f}_1, \widehat{f}_2\right) \in \Pi_\delta$ and $t \in \Omega$. Then for the operator $\mathcal{T}_1$, we have

$$\begin{aligned}\left|\mathcal{T}_1(f_1,f_2)(t)-\mathcal{T}_1\left(\widehat{f}_1,\widehat{f}_2\right)(t)\right| \leq \frac{1}{\Delta_1}&\left\{\frac{(1-\sigma_1)(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\left|\mathbb{G}^1_{f_1,f_2}(t)-\mathbb{G}^1_{\widehat{f}_1,\widehat{f}_2}(t)\right|\right.\\
&+\frac{(1-\sigma_1)\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}{}^{RL}\mathbb{I}^{\alpha_1}_{0^+}\left|\mathbb{G}^1_{f_1,f_2}(t)-\mathbb{G}^1_{\widehat{f}_1,\widehat{f}_2}(t)\right|\\
&+\frac{\sigma_1(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}{}^{RL}\mathbb{I}^{\sigma_1}_{0}\left|\mathbb{G}^1_{f_1,f_2}(t)-\mathbb{G}^1_{\widehat{f}_1,\widehat{f}_2}(t)\right|\\
&+\frac{\sigma_1\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}{}^{RL}\mathbb{I}^{\alpha_1+\sigma_1}_{0^+}\left|\mathbb{G}^1_{f_1,f_2}(t)-\mathbb{G}^1_{\widehat{f}_1,\widehat{f}_2}(t)\right|\\
&\left.+\frac{\mu_1\sigma_1}{\mathfrak{B}(\sigma_1)}{}^{RL}\mathbb{I}^{\sigma_1}_{0}\left|f_1(t)-\widehat{f}_1(t)\right|\right\}.\end{aligned}$$

Using (H_1), and taking the maximum on both sides of the above inequality, we obtain

$$\begin{aligned}\left\|\mathcal{T}_1(f_1,f_2)-\mathcal{T}_1\left(\widehat{f}_1,\widehat{f}_2\right)\right\| &\leq \frac{2\mathfrak{L}_1}{\Delta_1}\left(\mathbb{Q}_1+\frac{\mu_1 b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)}\right)\left\|(f_1,f_2)-\left(\widehat{f}_1,\widehat{f}_2\right)\right\|\\
&\leq \mathcal{Z}_1\left(\left\|(f_1,f_2)-\left(\widehat{f}_1,\widehat{f}_2\right)\right\|\right).\end{aligned}$$

In a similar manner for the operator $\mathcal{T}_2$, we get

$$\left\|\mathcal{T}_2(f_1,f_2)-\mathcal{T}_2\left(\widehat{f}_1,\widehat{f}_2\right)\right\| \leq \mathcal{Z}_2\left(\left\|(f_1,f_2)-\left(\widehat{f}_1,\widehat{f}_2\right)\right\|\right).$$

So,

$$\left\|\mathcal{T}(f_1,f_2)-\mathcal{T}\left(\widehat{f}_1,\widehat{f}_2\right)\right\| \leq \max\{\mathcal{Z}_1,\mathcal{Z}_2\}\left\|(f_1,f_2)-\left(\widehat{f}_1,\widehat{f}_2\right)\right\|.$$

Thus, the operator $\mathcal{T}$ is a contraction operator since $\max\{\mathcal{Z}_1,\mathcal{Z}_2\} < 1$. By Theorem 3.4, we conclude that the operator $\mathcal{T}$ has a unique fixed point, and the MABC coupled system (1.1) has a unique solution. □

3.4. **Existence of solutions.** In this subsection, we study the existence of a solution for the MABC coupled system (1.1), using Leray-Schauder alternative fixed point theorem.

Lemma 3.6. ([18]) *If the operator $\mathcal{T}: C(\Omega,\mathbb{R}) \to C(\Omega,\mathbb{R})$ is completely continuous, then either the set*

$$\Phi(\mathcal{T}) = \{f \in C(\Omega,\mathbb{R}) : f = \xi\mathcal{T}(f) \text{ } for \text{ } some \text{ } \xi \in (0,1)\}$$

is unbounded, or $\mathcal{T}$ has at least one fixed point.

Theorem 3.7. *For $i = 1, 2$, let*

$$\Im_i = \frac{1}{\Delta_i}\left[\mathbb{Q}_i + \frac{\mu_i b^{\sigma_i}}{\mathfrak{B}(\sigma_i)\Gamma(\sigma_i)}\right]\left(\Upsilon^*_{g_i} + \varpi^*_{g_i}\right). \tag{3.13}$$

If (H_2) holds and $0 < \Im_1, \Im_2 < 1$, then the MABC coupled system (1.1) has at least one solution on Ω.

Proof. To enhance readability, we provide proof in the following steps.

Step 1: We show that the operator $\mathcal{T} : C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R}) \to C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R})$, defined by (3.11), is continuous and uniformly bounded. Note that the operator $\mathcal{T}$ is continuous since $\mathbb{G}^i_{f_1,f_2}(t), i = 1, 2$ are continuous. For $(f_1, f_2) \in C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R})$, $t \in \Omega$, we have

$$\begin{aligned}|\mathcal{T}_1(f_1, f_2)(t)| \le |A_1| + \frac{1}{\Delta_1}\Bigg\{&\frac{(\mu_1 A_1 + B_1) b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)} + \frac{(1-\sigma_1)(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\left|\mathbb{G}^1_{f_1,f_2}(t)\right| \\ &+ \frac{(1-\sigma_1)\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}{}^{RL}\mathbb{I}^{\alpha_1}_{0^+}\left|\mathbb{G}^1_{f_1,f_2}(t)\right| + \frac{\sigma_1(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}{}^{RL}\mathbb{I}^{\sigma_1}_{0}\left|\mathbb{G}^1_{f_1,f_2}(t)\right| \\ &+ \frac{\sigma_1\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}{}^{RL}\mathbb{I}^{\alpha_1+\sigma_1}_{0^+}\left|\mathbb{G}^1_{f_1,f_2}(t)\right| + \frac{\mu_1\sigma_1}{\mathfrak{B}(\sigma_1)}{}^{RL}\mathbb{I}^{\sigma_1}_{0}|f_1(t)|\Bigg\}.\end{aligned}$$

By (H_2), we get

$$\begin{aligned}|\mathcal{T}_1(f_1, f_2)(t)| \le |A_1| + \frac{1}{\Delta_1}\Bigg\{&\frac{(\mu_1 A_1 + B_1) b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)} \\ &+ \frac{(1-\sigma_1)(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\left[\eta_{g_1}(t) + \Upsilon_{g_1}(t)|f_1(t)| + \varpi_{g_1}(t)|f_2(t)|\right] \\ &+ \frac{(1-\sigma_1)\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}{}^{RL}\mathbb{I}^{\alpha_1}_{0^+}\left[\eta_{g_1}(t) + \Upsilon_{g_1}(t)|f_1(t)| + \varpi_{g_1}(t)|f_2(t)|\right] \\ &+ \frac{\sigma_1(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}{}^{RL}\mathbb{I}^{\sigma_1}_{0}\left[\eta_{g_1}(t) + \Upsilon_{g_1}(t)|f_1(t)| + \varpi_{g_1}(t)|f_2(t)|\right] \\ &+ \frac{\sigma_1\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}{}^{RL}\mathbb{I}^{\alpha_1+\sigma_1}_{0^+}\left[\eta_{g_1}(t) + \Upsilon_{g_1}(t)|f_1(t)| + \varpi_{g_1}(t)|f_2(t)|\right] \\ &+ \frac{\mu_1\sigma_1}{\mathfrak{B}(\sigma_1)}{}^{RL}\mathbb{I}^{\sigma_1}_{0}|f_1(t)|\Bigg\}.\end{aligned}$$

By taking the maximum on both sides of the above inequality, we obtain

$$\begin{aligned}\|\mathcal{T}_1(f_1, f_2)\| \le |A_1| &+ \frac{(\mu_1 A_1 + B_1) b^{\sigma_1}}{\Delta_1\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)} \\ &+ \frac{1}{\Delta_1}\left[\mathbb{Q}_1 + \frac{\mu_1 b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)}\right]\left(\eta^*_{g_1} + \Upsilon^*_{g_1}\|f_1\| + \varpi^*_{g_1}\|f_2\|\right).\end{aligned}$$

Similarly, we find that

$$\|\mathcal{T}_2(f_1,f_2)\| \leq |A_2| + \frac{(\mu_2 A_2 + B_2)\, b^{\sigma_2}}{\Delta_2 \mathfrak{B}(\sigma_2)\Gamma(\sigma_2)} + \frac{1}{\Delta_2}\left[\mathbb{Q}_2 + \frac{\mu_2 b^{\sigma_2}}{\mathfrak{B}(\sigma_2)\Gamma(\sigma_2)}\right]\left(\eta_{g_2}^* + \Upsilon_{g_2}^* \|f_1\| + \varpi_{g_2}^* \|f_2\|\right).$$

Consequently, we get

$$\|\mathcal{T}(f_1,f_2)\| \leq \mathbb{K}.$$

Therefore, $\mathcal{T}$ is uniformly bounded.

Step 2: We show that the operator $\mathcal{T}$ is equicontinuous. If $t_1, t_2 \in \Omega$ with $t_1 < t_2$, then

$$\begin{aligned}
&|\mathcal{T}_1(f_1,f_2)(t_2) - \mathcal{T}_1(f_1,f_2)(t_1)| \\
&\leq \frac{1}{\Delta_1}\left\{ \frac{(1-\sigma_1)(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)} \left|\mathbb{G}^1_{f_1,f_2}(t_2) - \mathbb{G}^1_{f_1,f_2}(t_1)\right| \right.\\
&\quad + \frac{(1-\sigma_1)\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)} \int_0^{t_1} \left[\frac{(t_2-\theta)^{\alpha_1-1}}{\Gamma(\alpha_1)} - \frac{(t_1-\theta)^{\alpha_1-1}}{\Gamma(\alpha_1)}\right] \left|\mathbb{G}^1_{f_1,f_2}(\theta)\right| d\theta \\
&\quad + \frac{\sigma_1(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)} \int_0^{t_1} \left[\frac{(t_2-\theta)^{\sigma_1-1}}{\Gamma(\sigma_1)} - \frac{(t_1-\theta)^{\sigma_1-1}}{\Gamma(\sigma_1)}\right] \left|\mathbb{G}^1_{f_1,f_2}(\theta)\right| d\theta \\
&\quad + \frac{\sigma_1\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)} \int_0^{t_1} \left[\frac{(t_2-\theta)^{\alpha_1+\sigma_1-1}}{\Gamma(\alpha_1+\sigma_1)} - \frac{(t_1-\theta)^{\alpha_1+\sigma_1-1}}{\Gamma(\alpha_1+\sigma_1)}\right] \left|\mathbb{G}^1_{f_1,f_2}(\theta)\right| d\theta \\
&\quad + \frac{\mu_1\sigma_1}{\mathfrak{B}(\sigma_1)} \int_0^{t_1} \left[\frac{(t_2-\theta)^{\sigma_1-1}}{\Gamma(\sigma_1)} - \frac{(t_1-\theta)^{\sigma_1-1}}{\Gamma(\sigma_1)}\right] |f_1(\theta)|\, d\theta \\
&\quad + \frac{(1-\sigma_1)\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)} \int_{t_1}^{t_2} \frac{(t_2-\theta)^{\alpha_1-1}}{\Gamma(\alpha_1)} \left|\mathbb{G}^1_{f_1,f_2}(\theta)\right| d\theta \\
&\quad + \frac{\sigma_1(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)} \int_{t_1}^{t_2} \frac{(t_2-\theta)^{\sigma_1-1}}{\Gamma(\sigma_1)} \left|\mathbb{G}^1_{f_1,f_2}(\theta)\right| d\theta \\
&\quad + \frac{\sigma_1\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)} \int_{t_1}^{t_2} \frac{(t_2-\theta)^{\alpha_1+\sigma_1-1}}{\Gamma(\alpha_1+\sigma_1)} \left|\mathbb{G}^1_{f_1,f_2}(\theta)\right| d\theta \\
&\quad \left. + \frac{\mu_1\sigma_1}{\mathfrak{B}(\sigma_1)} \int_{t_1}^{t_2} \frac{(t_2-\theta)^{\sigma_1-1}}{\Gamma(\sigma_1)} \left|\mathbb{G}^1_{f_1,f_2}(\theta)\right| d\theta \right\}.
\end{aligned}$$

By using (H_2), we get

$$\begin{aligned}
&|\mathcal{T}_1(f_1,f_2)(t_2)-\mathcal{T}_1(f_1,f_2)(t_1)| \\
&\leq \frac{1}{\Delta_1}\Bigg\{\frac{(1-\sigma_1)(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\left|\mathbb{G}^1_{f_1,f_2}(t_2)-\mathbb{G}^1_{f_1,f_2}(t_1)\right| \\
&\quad+\frac{(1-\sigma_1)\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\left[\frac{t_2^{\alpha_1}}{\Gamma(\alpha_1+1)}-\frac{t_1^{\alpha_1}}{\Gamma(\alpha_1+1)}\right]\left(\eta^*_{g_1}+\Upsilon^*_{g_1}\|f_1\|+\varpi^*_{g_1}\|f_2\|\right) \\
&\quad+\frac{\sigma_1(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\left[\frac{t_2^{\sigma_1}}{\Gamma(\sigma_1+1)}-\frac{t_1^{\sigma_1}}{\Gamma(\sigma_1+1)}\right]\left(\eta^*_{g_1}+\Upsilon^*_{g_1}\|f_1\|+\varpi^*_{g_1}\|f_2\|\right) \\
&\quad+\frac{\sigma_1\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\left[\frac{t_2^{\alpha_1+\sigma_1}}{\Gamma(\alpha_1+\sigma_1+1)}-\frac{t_1^{\alpha_1+\sigma_1}}{\Gamma(\alpha_1+\sigma_1+1)}\right]\left(\eta^*_{g_1}+\Upsilon^*_{g_1}\|f_1\|+\varpi^*_{g_1}\|f_2\|\right) \\
&\quad+\frac{\mu_1\sigma_1}{\mathfrak{B}(\sigma_1)}\left[\frac{t_2^{\sigma_1}}{\Gamma(\sigma_1+1)}-\frac{t_1^{\sigma_1}}{\Gamma(\sigma_1+1)}\right]\|f_1\|\Bigg\} \\
&\longrightarrow 0 \text{ as } t_2\to t_1.
\end{aligned}$$

Analogously, we can obtain

$$|\mathcal{T}_2(f_1,f_2)(t_2)-\mathcal{T}_2(f_1,f_2)(t_1)|\to 0 \text{ as } t_2\to t_1.$$

Thus, the operator $\mathcal{T}(f_1,f_2)$ is equicontinuous. In view of the above arguments, and by Arzelà-Ascoli theorem, the operator $\mathcal{T}(f_1,f_2)$ is completely continuous.

Step 3: We show that the set

$$\varphi=\{(f_1,f_2)\in C(\Omega,\mathbb{R})\times C(\Omega,\mathbb{R}):(f_1,f_2)=\xi\mathcal{T}(f_1,f_2),\ \text{for some } 0<\xi<1\}$$

is bounded. Let $(f_1,f_2)\in\varphi$ with $(f_1,f_2)(t)=\xi\mathcal{T}(f_1,f_2)(t)$ such that

$$f_i(t)=\xi\mathcal{T}_i(f_1,f_2)(t),\ i=1,2.$$

For any $t\in\Omega$, we have

$$f_i(t)=\xi\mathcal{T}_i(f_1,f_2)(t)\leq\mathcal{T}_i(f_1,f_2)(t),\ i=1,2.$$

In view of condition (H_2) and taking the maximum on both sides of the above inequality, we obtain

$$\begin{aligned}
\|f_1\| \leq |A_1| &+ \frac{(\mu_1 A_1 + B_1)\, b^{\sigma_1}}{\Delta_1 \mathfrak{B}(\sigma_1)\Gamma(\sigma_1)} \\
&+ \frac{1}{\Delta_1}\left[\mathbb{Q}_1 + \frac{\mu_1 b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)}\right]\left(\eta^*_{g_1} + \Upsilon^*_{g_1}\|f_1\| + \varpi^*_{g_1}\|f_2\|\right) \\
\leq |A_1| &+ \frac{(\mu_1 A_1 + B_1)\, b^{\sigma_1}}{\Delta_1 \mathfrak{B}(\sigma_1)\Gamma(\sigma_1)} + \frac{\eta^*_{g_1}}{\Delta_1}\left[\mathbb{Q}_1 + \frac{\mu_1 b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)}\right] \\
&+ \frac{\Upsilon^*_{g_1}}{\Delta_1}\left[\mathbb{Q}_1 + \frac{\mu_1 b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)}\right]\|f_1\| + \frac{\varpi^*_{g_1}}{\Delta_1}\left[\mathbb{Q}_1 + \frac{\mu_1 b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)}\right]\|f_2\| \\
\leq |A_1| &+ \frac{(\mu_1 A_1 + B_1)\, b^{\sigma_1}}{\Delta_1 \mathfrak{B}(\sigma_1)\Gamma(\sigma_1)} + \frac{\eta^*_{g_1}}{\Delta_1}\left[\mathbb{Q}_1 + \frac{\mu_1 b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)}\right] \\
&+ \frac{1}{\Delta_1}\left[\mathbb{Q}_1 + \frac{\mu_1 b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)}\right]\left(\Upsilon^*_{g_1} + \varpi^*_{g_1}\right)\max\left(\|f_1\|, \|f_2\|\right).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\|f_2\| \leq |A_2| &+ \frac{(\mu_2 A_2 + B_2)\, b^{\sigma_2}}{\Delta_2 \mathfrak{B}(\sigma_2)\Gamma(\sigma_2)} + \frac{\eta^*_{g_2}}{\Delta_2}\left[\mathbb{Q}_2 + \frac{\mu_2 b^{\sigma_2}}{\mathfrak{B}(\sigma_2)\Gamma(\sigma_2)}\right] \\
&+ \frac{1}{\Delta_2}\left[\mathbb{Q}_2 + \frac{\mu_2 b^{\sigma_2}}{\mathfrak{B}(\sigma_2)\Gamma(\sigma_2)}\right]\left(\Upsilon^*_{g_2} + \varpi^*_{g_2}\right)\max\left(\|f_1\|, \|f_2\|\right).
\end{aligned}$$

Choose a real number $\Psi > 0$ with

$$\Psi \geq \max_{i\in\{1,2\}}\left\{\frac{|A_1| + \frac{(\mu_1 A_1 + B_1)b^{\sigma_1}}{\Delta_1 \mathfrak{B}(\sigma_1)\Gamma(\sigma_1)} + \frac{\eta^*_{g_1}}{\Delta_1}\left[\mathbb{Q}_1 + \frac{\mu_1 b^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)}\right]}{1 - \Im_i}\right\},$$

where $0 < \Im_1 < 1$ and $0 < \Im_2 < 1$, then

$$\|(f_1, f_2)\| = \max\left(\|f_1\|, \|f_2\|\right) \leq \Psi,$$

which means that the set φ is bounded by Ψ. By Lemma 3.6, the operator $\mathcal{T}$ has at least one solution. Thus, there exists a solution of the MABC coupled system (1.1) on $[0, 1]$. □

3.5. **Ulam-Hyers stability.** In this subsection, we state the definitions of Ulam-Hyers stability and prove the stability results of MABC coupled system (1.1). For more information about the stability analysis see [21, 30]. Using the results in [31], we state the following definition.

Definition 3.8. Let $\mathcal{T}_1, \mathcal{T}_2 : C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R}) \to C(\Omega, \mathbb{R})$ be two operators. The system

$$\left\{\begin{array}{l} f_1(t) = \mathcal{T}_1\left(f_1, f_2\right)(t), \\ f_2(t) = \mathcal{T}_2\left(f_1, f_2\right)(t) \end{array}\right. \tag{3.14}$$

is Ulam-Hyers stable if there is $\ell >, \varepsilon_1, \varepsilon_2 > 0$ such that for all $\left(\widehat{f}_1, \widehat{f}_2\right) \in C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R})$ satisfying

$$\begin{cases} \left\|\widehat{f}_1 - \mathcal{T}_1(\widehat{f}_1, \widehat{f}_2)\right\| \leq \varepsilon_1, \\ \\ \left\|\widehat{f}_2 - \mathcal{T}_2\left(\widehat{f}_1, \widehat{f}_2\right)\right\| \leq \varepsilon_2, \end{cases} \tag{3.15}$$

there is a unique solution $(f_1, f_2) \in C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R})$ of the system (3.14) with

$$\left\|\left(\widehat{f}_1, \widehat{f}_2\right) - (f_1, f_2)\right\| \leq \ell\varepsilon.$$

Remark 3.9. A function $\left(\widehat{f}_1, \widehat{f}_2\right) \in C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R})$ satisfies the inequality (3.15) if and only if there is a function $z_i(t) \in C(\Omega, \mathbb{R}),\ i = 1, 2$ such that

$$|z_i(t)| \leq \varepsilon_i \text{ for all } t \in (0, b), (z_1 \text{ depends on } \ f),\ i = 1, 2.$$

Remark 3.10. We have

$$^{MABC}\mathbb{D}^{\alpha_i}\left(^{ABC}\mathbb{D}^{\sigma_i} + \mu_i\right)\widehat{f}_i(t) = g_i(t, \widehat{f}_1(t), \widehat{f}_2(t)) + z_i(t), t \in (0, b),\ i = 1, 2.$$

Lemma 3.11. *If* $(\widehat{f}_1, \widehat{f}_2) \in C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R})$ *satisfies* (3.15)*, then* $(\widehat{f}_1, \widehat{f}_2)$ *is a solution of the inequalities*

$$\left|\widehat{f}_i(\iota) - \Sigma_{\widehat{f}_i}\right| \leq \frac{\mathbb{Q}_i}{\Delta_i}\varepsilon_i, \quad i = 1, 2,$$

where

$$\begin{aligned} \Sigma_{\widehat{f}_i} = A_i + \frac{1}{\Delta_i}\Bigg\{ & \frac{(\mu_i A_i + B_i)\,\sigma_i t^{\sigma_i}}{\mathfrak{B}(\sigma_i + 1)} + \frac{(1-\sigma_1)(1-\alpha_i)}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\mathbb{G}^i_{\widehat{f}_1, \widehat{f}_2}(t)) \\ & + \frac{(1-\sigma_i)\,\alpha_i}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}{}^{RL}\mathbb{I}^{\alpha_i}_{0^+}\mathbb{G}^i_{\widehat{f}_1, \widehat{f}_2}(t) + \frac{\sigma_i(1-\alpha_i)}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i\ \ )}{}^{RL}\mathbb{I}^{\sigma_i}_0\mathbb{G}^i_{\widehat{f}_1, \widehat{f}_2}(t) \\ & + \frac{\sigma_i\alpha_i}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}{}^{RL}\mathbb{I}^{\alpha_i+\sigma_i}_{0^+}\mathbb{G}^i_{\widehat{f}_1, \widehat{f}_2}(t) - \frac{\mu_i\sigma_i}{\mathfrak{B}(\sigma_i)}{}^{RL}\mathbb{I}^{\sigma_i}_0\widehat{f}_i(t)\Bigg\},\ i = 1, 2. \end{aligned}$$

Proof. By Lemma 3.2 and Remark 3.9, $(\widehat{f}_1, \widehat{f}_2)$ is a solution of (1.1) with the conditions

$$\widehat{f}_i(0) = A_i, {}^{ABC}\mathbb{D}^{\sigma_i}\widehat{f}_i(0) = B_i, \quad i = 1, 2,$$

if and only if

$$\widehat{f}_i(t) = A_i + \frac{1}{\Delta_i}\left\{\frac{(\mu_i A_i + B_i)\,\sigma_i t^{\sigma_i}}{\mathfrak{B}(\sigma_i+1)} + \frac{(1-\sigma_i)(1-\alpha_i)}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\left[\mathbb{G}^i_{\widehat{f}_1,\widehat{f}_2}(t) + z_i(t)\right]\right.$$
$$+ \frac{(1-\sigma_i)\,\alpha_i}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\,{}^{RL}\mathbb{I}^{\alpha_i}_{0^+}\left[\mathbb{G}^i_{\widehat{f}_1,\widehat{f}_2}(t) + z_i(t)\right]$$
$$+ \frac{\sigma_i(1-\alpha_i)}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\,{}^{RL}\mathbb{I}^{\sigma_i}_{0}\left[\mathbb{G}^i_{\widehat{f}_1,\widehat{f}_2}(t) + z_i(t)\right]$$
$$\left.+\frac{\sigma_i\alpha_i}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\,{}^{RL}\mathbb{I}^{\alpha_i+\sigma_i}_{0^+}\left[\mathbb{G}^i_{\widehat{f}_1,\widehat{f}_2}(t) + z_i(t)\right] - \frac{\mu_i\sigma_i}{\mathfrak{B}(\sigma_i)}\,{}^{RL}\mathbb{I}^{\sigma_i}_{0}\widehat{f}_i(t)\right\}, i = 1, 2.$$

Hence, we have

$$\left|\widehat{f}_i(\iota) - \Sigma_{\widehat{f}_i}\right| \le \frac{\mathbb{Q}_i}{\Delta_i}\varepsilon_i, \quad i = 1, 2.$$

□

Theorem 3.12. *Under the hypothesis* (H_1)*, we have*

$$\begin{cases} {}^{MABC}\mathbb{D}^{\alpha_1}\left({}^{ABC}\mathbb{D}^{\sigma_1} + \mu_1\right) f_1(t) = g_1(t, f_1(t), f_2(t)), t \in (0, b), \\ {}^{MABC}\mathbb{D}^{\alpha_2}\left({}^{ABC}\mathbb{D}^{\sigma_2} + \mu_2\right) f_2(t) = g_2(t, f_1(t), f_2(t)), t \in (0, b), \\ f_1(0) = A_1, {}^{ABC}\mathbb{D}^{\sigma_1} f_1(0) = B_1, \\ f_2(0) = A_2, {}^{ABC}\mathbb{D}^{\sigma_2} f_2(0) = B_2, \end{cases} \tag{3.16}$$

is Ulam-Hyers (UH) stable, provided that $0 < \frac{\mathbb{Q}_i}{\Delta_i - 2\mathfrak{L}_i\mathbb{Q}_i} < 1, \ i = 1, 2.$

Proof. Let $(\widehat{f}_1, \widehat{f}_2) \in C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R})$ satisfies the inequality (3.15) and let $(f_1, f_2) \in C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R})$ be the unique solution of the system (3.16). By virtue of Lemma 3.2, we obtain

$$f_i(t) = A_i + \frac{1}{\Delta_i}\left\{\frac{(\mu_i A_i + B_i)\,t^{\sigma_i}}{\mathfrak{B}(\sigma_i)\Gamma(\sigma_i)} + \frac{(1-\sigma_i)(1-\alpha_i)}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\mathbb{G}^i_{f_1,f_2}(t)\right.$$
$$+ \frac{(1-\sigma_i)\,\alpha_i}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\,{}^{RL}\mathbb{I}^{\alpha_i}_{0^+}\mathbb{G}^i_{f_1,f_2}(t) + \frac{\sigma_i(1-\alpha_i)}{\mathfrak{B}(\sigma_i\ )\mathfrak{B}(\alpha_i)}\,{}^{RL}\mathbb{I}^{\sigma_i}_{0}\mathbb{G}^i_{f_1,f_2}(t)$$
$$\left.+ \frac{\sigma_i\alpha_i}{\mathfrak{B}(\sigma_i)\mathfrak{B}(\alpha_i)}\,{}^{RL}\mathbb{I}^{\alpha_i+\sigma_i}_{0^+}\mathbb{G}^i_{f_1,f_2}(t) - \frac{\mu_i\sigma_i}{\mathfrak{B}(\sigma_i)}\,{}^{RL}\mathbb{I}^{\sigma_i}_{0} f_i(t)\right\}, \quad i = 1, 2.$$

Therefore, we get

$$\begin{aligned}\left|\widehat{f_1}-f_1\right| &= \Bigg|\widehat{f_1}-A_1-\frac{1}{\Delta_1}\Bigg\{\frac{(\mu_1 A_1+B_1)\,t^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)}+\frac{(1-\sigma_1)(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\mathbb{G}^1_{f_1,f_2}(t)\\
&\quad+\frac{(1-\sigma_1)\,\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\,{}^{RL}\mathbb{I}^{\alpha_1}_{0^+}\mathbb{G}^1_{f_1,f_2}(t)+\frac{\sigma_1(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\,{}^{RL}\mathbb{I}^{\sigma_1}_{0}\mathbb{G}^1_{f_1,f_2}(t)\\
&\quad+\frac{\sigma_1\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\,{}^{RL}\mathbb{I}^{\alpha_1+\sigma_1}_{0^+}\mathbb{G}^1_{f_1,f_2}(t)-\frac{\mu_1\sigma_1}{\mathfrak{B}(\sigma_1)}\,{}^{RL}\mathbb{I}^{\sigma_1}_{0}f_1(t)\Bigg\}\Bigg|\\
&\le \Bigg|\widehat{f_1}-A_1-\frac{1}{\Delta_1}\Bigg\{\frac{(\mu_1 A_1+B_1)\,t^{\sigma_1}}{\mathfrak{B}(\sigma_1)\Gamma(\sigma_1)}+\frac{(1-\sigma_1)(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\mathbb{G}^1_{\widehat{f_1},\widehat{f_2}}(t)\\
&\quad+\frac{(1-\sigma_1)\,\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\,{}^{RL}\mathbb{I}^{\alpha_1}_{0^+}\mathbb{G}^1_{\widehat{f_1},\widehat{f_2}}(t)+\frac{\sigma_1(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\,{}^{RL}\mathbb{I}^{\sigma_1}_{0}\mathbb{G}^1_{\widehat{f_1},\widehat{f_2}}(t)\\
&\quad+\frac{\sigma_1\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\,{}^{RL}\mathbb{I}^{\alpha_1+\sigma_1}_{0^+}\mathbb{G}^1_{\widehat{f_1},\widehat{f_2}}(t)-\frac{\mu_1\sigma_1}{\mathfrak{B}(\sigma_1)}\,{}^{RL}\mathbb{I}^{\sigma_1}_{0}\widehat{f_1}(t)\Bigg\}\Bigg|\\
&\quad+\frac{1}{\Delta_1}\Bigg\{\frac{(1-\sigma_1)(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\left|\mathbb{G}^1_{\widehat{f_1},\widehat{f_2}}(t)-\mathbb{G}^1_{f_1,f_2}(t)\right|\\
&\quad+\frac{(1-\sigma_1)\,\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\,{}^{RL}\mathbb{I}^{\alpha_1}_{0^+}\left|\mathbb{G}^1_{\widehat{f_1},\widehat{f_2}}(t)-\mathbb{G}^1_{f_1,f_2}(t)\right|\\
&\quad+\frac{\sigma_1(1-\alpha_1)}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\,{}^{RL}\mathbb{I}^{\sigma_1}_{0}\left|\mathbb{G}^1_{\widehat{f_1},\widehat{f_2}}(t)-\mathbb{G}^1_{f_1,f_2}(t)\right|\\
&\quad+\frac{\sigma_1\alpha_1}{\mathfrak{B}(\sigma_1)\mathfrak{B}(\alpha_1)}\,{}^{RL}\mathbb{I}^{\alpha_1+\sigma_1}_{0^+}\left|\mathbb{G}^1_{\widehat{f_1},\widehat{f_2}}(t)-\mathbb{G}^1_{f_1,f_2}(t)\right|\Bigg\}.\end{aligned}$$

Thus, by Lemma 3.11 and (H_1), we get

$$\left|\widehat{f_1}-f_1\right|\le\frac{\mathbb{Q}_1}{\Delta_1}\varepsilon_1+\frac{\mathbb{Q}_1\mathfrak{L}_1}{\Delta_1}\left(\left|f_1-\widehat{f_1}\right|+\left|f_2-\widehat{f_2}\right|\right).$$

Thus,

$$\left\|\widehat{f_i}-f_i\right\|\le\varepsilon_i\frac{\mathbb{Q}_i}{\Delta_i-2\mathfrak{L}_i\mathbb{Q}_i},\ \ i=1,2,$$

and

$$\left\|\left(\widehat{f_1},\widehat{f_2}\right)-(f_1,f_2)\right\|\le\varepsilon\ell, \tag{3.17}$$

where

$$\ell=\max_{i\in\{1,2\}}\left\{\frac{\mathbb{Q}_i}{\Delta_i-2\mathfrak{L}_i\mathbb{Q}_i}\right\}>0.$$

By the inequality (3.17) and Definition 3.8, the solution of the MABC coupled system (1.1) is Ulam-Hyers stable. □

4. Numerical solution for nonlinear MABC coupled system (1.1)

Consider the MABC coupled system (1.1)

$$\begin{cases} {}^{MABC}\mathbb{D}^{\alpha_1}\left({}^{ABC}\mathbb{D}^{\sigma_1}+\mu_1\right)f_1(t)=\mathbb{G}^1_{f_1,f_2}(t), \\ {}^{MABC}\mathbb{D}^{\alpha_2}\left({}^{ABC}\mathbb{D}^{\sigma_2}+\mu_2\right)f_2(t)=\mathbb{G}^2_{f_1,f_2}(t), \end{cases}$$

with fixed point $f_i, i=1,2$, given as

$$\begin{aligned} f_i(t)=f_i(0)+{}^{AB}\,\mathbb{I}^{\sigma_i}_{0^+}\Big[&\frac{1-\alpha_i}{\mathfrak{B}(\alpha_i)}\mathbb{G}^i_{f_1,f_2}(t)+\frac{\alpha_i}{\mathfrak{B}(\alpha_i)}\;{}^{RL}\mathbb{I}^{\alpha_i}_{0^+}\mathbb{G}^i_{f_1,f_2}(t) \\ &+\mu_iA_i+B_i-\mu_if_i(t)\Big]. \end{aligned} \tag{4.1}$$

Define the nonlinear function

$$\mathbb{H}^i_{f_1,f_2}(t)=\left[\frac{1-\alpha_i}{\mathfrak{B}(\alpha_i)}\mathbb{G}^i_{f_1,f_2}(t)+\frac{\alpha_i}{\mathfrak{B}(\alpha_i)}\;{}^{RL}\mathbb{I}^{\alpha_i}_{0^+}\mathbb{G}^i_{f_1,f_2}(t)+\mu_iA_i+B_i-\mu_if_i(t)\right]. \tag{4.2}$$

Thus, we get

$$f_i(t)=f_i(0)+\frac{1-\sigma_i}{\mathfrak{B}(\sigma_i)}\mathbb{H}^i_{f_1,f_2}(t)+\frac{\sigma_i}{\mathfrak{B}(\sigma_i)\Gamma(\sigma_i)}\int_0^t(t-s)^{\sigma_i-1}\,\mathbb{H}^i_{f_1,f_2}(s)\,ds. \tag{4.3}$$

By discretizing the function $\mathbb{H}_i$ in equations (4.3) at $t=t_{n+1}$, we obtain the following discrete equations

$$\begin{aligned} f_i(t_{n+1})=f_i(0)&+\frac{1-\sigma_i}{\mathfrak{B}(\sigma_i)}\mathbb{H}^i_{f_1,f_2}(t_n) \\ &+\frac{\sigma_i}{\mathfrak{B}(\sigma_i)\Gamma(\sigma_i)}\int_0^{t_{n+1}}(t_{n+1}-s)^{\sigma_i-1}\mathbb{H}^i_{f_1,f_2}(s)\,ds. \end{aligned} \tag{4.4}$$

Let $\mathbb{H}_i$ in the interval $[t_k,t_{k+1}]$, using two points Lagrange interpolation polynomial, we have

$$\begin{aligned} \mathbb{H}^i_{f_1,f_2}(t)&=\frac{\mathbb{H}^i_{f_1,f_2}(t_k)}{t_k-\imath_{k-1}}(t-t_{k-1})-\frac{\mathbb{H}^i_{f_1,f_2}(t_{k-1})}{t_k-\imath_{k-1}}(t-t_k) \\ &\simeq\frac{\mathbb{H}^i_{f_1,f_2}(t_k)}{h}(t-t_{k-1})-\frac{\mathbb{H}^i_{f_1,f_2}(t_{k-1})}{h}(t-t_k). \end{aligned} \tag{4.5}$$

By the help of (4.4) and (4.5), we have

$$
\begin{aligned}
f_i(t_{n+1}) = f_i(0) &+ \frac{1-\sigma_i}{\mathfrak{B}(\sigma_i)}\mathbb{H}^i_{f_1,f_2}(t_n)\\
&+ \frac{\sigma_i}{\mathfrak{B}(\sigma_i)\Gamma(\sigma_i)}\left(\sum_{k=0}^{n}\frac{\mathbb{H}^i_{f_1,f_2}(t_k)}{h}\int_{t_k}^{t_{k+1}}(s-t_{k-1})(t_{k+1}-s)^{\sigma_i-1}ds\right.\\
&\left. -\frac{\mathbb{H}^i_{f_1,f_2}(t_{k-1})}{h}\int_{t_k}^{t_{k+1}}(s-t_k)(t_{k+1}-s)^{\sigma_i-1}ds\right).
\end{aligned}
$$

Now, after computing the above two integrals, the numerical scheme for MABC coupled system (1.1) is given

$$
\begin{aligned}
f_i(t_{n+1}) \;=\; & f_i(0) + \frac{1-\sigma_i}{\mathfrak{B}(\sigma_i)}\mathbb{H}^i_{f_1,f_2}(t_n)\\
& + \frac{\sigma_i}{\mathfrak{B}(\sigma_i)}\sum_{k=0}^{n}\left[\frac{h^{\sigma_i}\mathbb{H}^i_{f_1,f_2}(t_k)}{\Gamma(\sigma_i+2)}\left[(n-k+1)^{\sigma_i}(n-k+2+\sigma_i)\right.\right.\\
& \left. -(n-k)^{\sigma_i}(n-k+2+2\sigma_i)\right]\\
& \left. -\frac{h^{\sigma_i}\mathbb{H}^i_{f_1,f_2}(t_{k-1})}{\Gamma(\sigma_i+2)}\left[(n-k+1)^{\sigma_i+1}-(n-k)^{\sigma_i}(n-k+1+\sigma_i)\right]\right].
\end{aligned}
$$

To illustrate the validity of our main findings, we consider

$$
\mathbb{G}^1_{f_1,f_2}(t) = \mathbb{G}^2_{f_1,f_2}(t) = \frac{t}{50}\left(\frac{|f_1(t)|}{1+|f_1(t)|} + \frac{|f_2(t)|}{1+|f_2(t)|}\right).
$$

Clearly, both $\mathbb{G}^1_{f_1,f_2}(t)$ and $\mathbb{G}^2_{f_1,f_2}(t)$ are continuous, and $\mathbb{G}^1_{f_1,f_2}(0) = \mathbb{G}^2_{f_1,f_2}(0) = 0$. For $t \in (0,1]$ and $f_1, \widehat{f}_1, f_2, \widehat{f}_2 \in C(\Omega,\mathbb{R})$, we have

$$
\begin{aligned}
\left|\mathbb{G}^1_{f_1,f_2}(t) - \mathbb{G}^1_{\widehat{f}_1,\widehat{f}_2}(t)\right| &= \left|\mathbb{G}^2_{f_1,f_2}(t) - \mathbb{G}^2_{\widehat{f}_1,\widehat{f}_2}(t)\right|\\
&\le \frac{1}{50}\left[\left|f_1-\widehat{f}_1\right| + \left|f_2-\widehat{f}_2\right|\right].
\end{aligned}
$$

Therefore, (H_1) holds with $\mathfrak{L}_i = \frac{1}{50}$. Also, for $\alpha_i = \sigma_i = \frac{1}{2} \in (0,1], i = 1,2, b = 1, A_1 = A_2 = 1, B_1 = B_2 = 4$, we have $\Delta_1 = \Delta_2 = 4$, where $\mathfrak{B}(\sigma_i) = \frac{\sigma_i}{2-\sigma_i}$. With some calculations, we get $\mathbb{Q}_1 = \mathbb{Q}_2 \simeq 11.82$. Hence $\mathcal{Z}_1 = \mathcal{Z}_2 \simeq 0.152$. Thus, all conditions in Theorem 3.5 are satisfied. Consequently, the ABC-system (1.1) has a unique solution. Moreover, for each $\varepsilon = \max\{\varepsilon_1,\varepsilon_2\} > 0$ and every $\left(\widehat{f}_1,\widehat{f}_2\right) \in C(\Omega,\mathbb{R}) \times C(\Omega,\mathbb{R})$ satisfies the inequalities

$$
\left\|\widehat{f}_1 - \mathcal{T}_1\left(\widehat{f}_1,\widehat{f}_2\right)\right\| \le \varepsilon_1 \text{ and } \left\|\widehat{f}_2 - \mathcal{T}_2\left(\widehat{f}_1,\widehat{f}_2\right)\right\| \le \varepsilon_2,
$$

and (f_1, f_2) is the unique solution of ABC problem (3.14) with

$$\left\| \left(\widehat{f}_1, \widehat{f}_2\right) - (f_1, f_2) \right\| \leq \ell\varepsilon,$$

where $\ell = \max_{i\in\{1,2\}} \left\{ \frac{\mathbb{Q}_i}{\Delta_i - 2\mathfrak{L}_i\mathbb{Q}_i} \right\} = 3.35 > 0$. Thus, the MABC-system (1.1) is UH stable.

5. Application of the coupled system of Langevin equations

Langevin fractional equations find applications in various areas of physics and biology. In Physics: Brownian motion, anomalous diffusion: In certain systems, such as porous media or complex fluids, the diffusion of particles may deviate from the standard Gaussian behavior. The Langevin fractional equation can describe anomalous diffusion processes by incorporating fractional derivatives that account for the non-local and memory effects. In Biology: Modeling biological processes: The Langevin fractional equation can be applied to model various biological processes, such as gene regulation, enzyme kinetics, or population dynamics. Fractional derivatives account for these systems' memory effects and long-range interactions, allowing for more accurate descriptions of their behavior.

Here, we provide two applications for the Langevin equation using modified ABC fractional operators.

(1) The coupled system of Langevin equations for the oscillators with anomalous diffusion and memory effects using modified ABC fractional operators can be written as:

$$\begin{aligned} {}^{MABC}\mathbb{D}^{\alpha_1}\left(m_1\ {}^{ABC}\mathbb{D}^{\sigma_1} + \gamma_1\right)x_1(t) &= C_{11}x_1(t) + C_{12}x_2(t) + F_1(t), \\ {}^{MABC}\mathbb{D}^{\alpha_2}\left(m_2\ {}^{ABC}\mathbb{D}^{\sigma_2} + \gamma_2\right)x_2(t) &= C_{21}x_1(t) + C_{22}x_2(t) + F_2(t), \end{aligned}$$

where, m_1 and m_2 are the masses of the first and second oscillators, respectively. $x_1(t)$ and $x_2(t)$ represent the positions of the first and second oscillators as functions of time, respectively. γ_1 and γ_2 are the friction coefficients for the first and second oscillators, respectively. C_{11}, C_{12}, C_{21} and C_{22} represent the coupling strengths between the oscillators. $F_1(t)$ and $F_2(t)$ are stochastic force terms acting on the first and second oscillators, respectively. The friction coefficients γ_1 and γ_2 can be chosen based on the damping properties of the oscillators. This example demonstrates how the model can be customized by setting specific values for the masses, fractional orders, friction coefficients, coupling strengths, and stochastic forces to capture the desired dynamics and phenomena in the system. In practice, the choices for these parameters would depend on the specific application and the desired behavior or characteristics being modeled.

(2) Consider a population dynamics problem involving two interacting species, where fractional order derivatives influence the growth rates of the species. We denote the population sizes of the two species as $N_1(t)$ and $N_2(t)$ at time t. The dynamics of the population sizes can be modeled using the following equations:

$$\begin{cases} {}^{MABC}\mathbb{D}^{\alpha_1}\left({}^{ABC}\mathbb{D}^{\sigma_1}+\mu_1\right)N_1(t)=r_1N_1(t)\left[1-\frac{N_1(t)+a_1N_2(t)}{K_1}\right], \\ {}^{MABC}\mathbb{D}^{\alpha_2}\left({}^{ABC}\mathbb{D}^{\sigma_2}+\mu_2\right)N_2(t)=r_2N_2(t)\left[1-\frac{b_2N_1(t)+N_2(t)}{K_2}\right], \end{cases}$$

where, μ_1 and μ_2 are constant parameters representing additional effects or factors. $N_1(t)$ and $N_2(t)$ are the population sizes of species 1 and 2, respectively. r_1 and r_2 are the intrinsic growth rates of species 1 and 2, respectively. a_1 and b_2 are the interaction coefficients representing the influence of each species on the other. K_1 and K_2 are the carrying capacities of species 1 and 2, respectively. In this model, the equations describe the growth of two interacting populations, where the fractional derivatives capture memory effects and anomalous diffusion in the population dynamics. The terms inside the square brackets represent the logistic growth model with interaction terms. By this model, one can study the populations' long-term behavior, stability, and coexistence. The values of the parameters $(r_1,\ r_2, a_1, b_2, K_1, K_2)$ would depend on the specific ecological system or problem you are modeling. This example demonstrates how the coupled fractional differential equations with modified ABC fractional operators can be applied to population dynamics problems, capturing memory effects and fractional order dynamics in ecological systems.

6. Conclusion

The Langevin equation is indeed fundamental in mathematical physics, particularly in the context of fluctuating environments such as Brownian motion. In our study, we focused on a specific aspect of the Langevin equation, namely the initial value problem of a coupled system of Langevin equations, where we incorporated modified Atangana-Baleanu fractional derivatives. Our investigation primarily revolved around the analysis of the existence, uniqueness, and stability of solutions to this coupled system. To achieve this, we employed fixed point theorems and applied Ulam's method, a technique used to assess the stability of functional equations. By utilizing various fixed-point theorems and discussing Ulam stability within the framework of MABC fractional derivatives, we were able to gain insights into the behavior of the system. We believe that our work makes a valuable contribution to the existing literature by providing a comprehensive understanding of dynamic processes governed by coupled Langevin equations with MABC fractional derivatives. By investigating the existence, uniqueness, and stability of solutions, we enhance our understanding of the underlying mathematical properties and shed light on the behavior of such systems in fluctuating environments. In our future endeavors, we are dedicated to further exploring combined structures that integrate both physical and mathematical models. This interdisciplinary approach allows us to capture the complex dynamics of real-world phenomena more accurately. To improve the precision of our numerical results, we employ piecewise non-singular fractional operators, which offer enhanced computational efficiency and accuracy in solving fractional differential equations.

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