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EQUILIBRIUM SOLUTION IN QUEUING GAMES PLAYED ON 3×3-GRID

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Abstract. In this paper, we study the existence of equilibrium play in a specific class of queuing game played on a 3×3-gird as non-cooperative 2-players game. In this game, both players compete to select, from a set of paths available for each player, just one path to use for their customers and exit from the grid after getting the required service as soon as possible. Every node in this grid is a $M/M/1$ queue with different service rates. We discuss the existence of equilibrium strategies in two cases: first case, both players can share the same entering and ending nodes. Second case, both players cannot share the same entering and ending nodes. We compute the pay-off matrices for both players in two cases. Using a numerical calculation, we find the best reply and the equilibrium strategies for both players if they have equal or unequal arrival rates. Also, the equilibrium play studied in which there is a relation between these rates in this queuing game.

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1. Introduction

In the long history of operations research, the queueing theory has been applied. Studying the dynamics of different service systems with constrained resources, like computer systems, communication networks, industrial systems, and transportation systems, can be greatly aided by it. The phenomenon of customer competition for scarce service resources is common in queueing systems. As a result, the idea of game theory offers a promising line of inquiry for queueing theory. A finite number of independent players, each optimizing a different objective function, often determine the state of a system. Non-cooperative games provide an appropriate framework for investigating such decentralized systems. Determining the conditions under which a Nash equilibrium in pure strategies can occur is one of the main objectives of game theory. For finite non-cooperative games, it is generally known that a Nash equilibrium in mixed strategies exists; however, this does not always necessary to a Nash equilibrium in pure strategies [1, 2, 17, 18, 20].

While game theory is used in communication networks to create the best routing algorithms, queueing theory is used in security to describe the stochastic elements of interdiction games. [7, 10, 14]. For several decades, queueing systems in terms of player equilibrium behavior have attracted increasing attention due to their applicability in many aspects. Naor [16] in 1969, pioneered the study of equilibrium and social optimal methods for the joining-balking problem in an M/M/1 queue with a basic linear reward-cost structure. Kelly [12] examines a queueing game in which Braess' dilemma occurs on a network of queue. Several publications in the literature incorporate models from queueing theory and game theory. The game theoretic analysis of queueing systems, which began with Naor's groundbreaking work, has attracted a lot of interest in the literature [3, 4, 8, 9, 11, 23]. Many researchers were worked on equilibrium solutions in queuing games [6, 5, 21, 22], the existence of this solution in which the arrival rates for players are equal was discussed in [13].

In this paper, we study the equilibrium play in a specific class of queuing game played on a 3×3-gird as a non-cooperative 2-players game. The players in this game compete to select, from a set of paths available (strategies) for each player, just one path to use for their customers and exit from the grid after getting the required service as soon as possible. Every node in this grid is a $M/M/1$ queue with different service rates and with equals or different arrival rate. We discuss the existence of equilibrium strategies in case that both players can share the same entering and ending nodes and can't share the entering and ending nodes. We drive the sufficient conditions for the existence of the equilibrium strategies for both players using the best respond algorithm and compute the pay-off matrices for both players. Using numerical calculations, we find the dominant strategies, best reply and the equilibrium strategies for both players if they have equal and un-equal arrival rates.

The remainder of this paper is organized as follows. In Section 2 we introduce an overview on traditional concept in game theory which is non-cooperative games on networks. In Section 3 we list the possible strategies of players and compute the payoff matrices in two cases. First, both players can share the entering and ending nodes with either equal service rates or different service rates of every node in this grid. Second case, both players have different entering and ending nodes, also the player must exit from the opposite side of the entering node and with either equal service rates or different service rates of every node in this grid. The sufficient conditions for the equilibrium solution in these cases of the given game discussed in section 4. In section 5, the equilibrium play studied in which there is a relation between the service rates of all nodes in this grid, and also there is a relation between the arrival rates of both players.

2. Model description

We consider a game of two players where the players are competing on a grid of form 3×3. Each player has finite paths and he must select one path to use for his customers to get the required service as soon as possible. Assuming that the arrival process of customers for player 1 and player 2 is a Poisson process with different arrival rates $\lambda^{(1)}, \lambda^{(2)}$ respectively. The strategies of both players are a set of different paths along the grid starting at a fixed entering node and ending on the opposite side of the grid at a fixed node. Each path must contain a minimum number of nodes which required to move from the entering node to the ending node. Each node in this grid is $M/M/1$ queue with a different service rate and FCFO queue discipline. At node$i$ of the grid, the mean sojourn time is given by $\frac{1}{\mu_i - \lambda_i}$ where λ_i is the rate of arrivals to this node. The goal of each player is to minimize the mean sojourn time of its customers using a suitable path. Since each player must choose one path to route his customer, the strategy space will be discrete, which means that the player will choose a specific path with probability 1 and the other paths with probability 0. In this situation, a strategy (available paths for player) of both players can be represented by a vector

$$S^{(j)} = (s_1^{(j)}, s_2^{(j)}, s_3^{(j)}, , , s_k^{(j)}),$$

where $j = 1, 2$ and k the number of paths available for players. If we denote by $S = (S^{(1)}, S^{(2)})$ to the strategy profile for both players, depending on the previous assumptions, the pay-off function for both players is given by:

$$\pi^{(j)} = \sum_{i \in S^{(j)}} \frac{1}{\mu_i - \lambda^{(j)}} + \sum_{i \in (S^{(1)} \cap S^{(2)})} \frac{1}{\mu_i - (\lambda^{(1)} + \lambda^{(2)})}, j = 1, 2. \tag{2.1}$$

And for at each node in this grid, the inequality $\lambda^{(1)} + \lambda^{(2)} < min(\mu_i)$ must hold.

3. Strategies of Players in 3×3-grid

The following two scenarios allow for the analysis of the players' strategies in this game:

Case 1: We suppose that the players will share the entering and ending node Figure (1). Also, players should select just the shortest path from entering node to ending node, in this case each path will contain just five nodes, therefore player 1 and player 2 will have the same six different paths from node a to node k. these paths represent the strategies of both players and given by:

$$S_1^{(1)} = S_1^{(2)} = \{a, b, c, f, k\}, S_2^{(1)} = S_2^{(2)} = \{a, b, e, f, k\},$$
$$S_3^{(1)} = S_3^{(2)} = \{a, b, e, h, k\}, S_4^{(1)} = S_4^{(2)} = \{a, d, e, h, k\},$$
$$S_5^{(1)} = S_5^{(2)} = \{a, d, e, f, k\}, S_6^{(1)} = S_6^{(2)} = \{a, d, g, h, k\}.$$

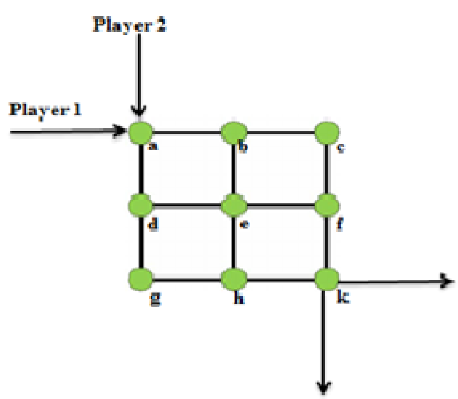

Figure 1. Grid of Case 1

If we set $\lambda = \lambda^{(1)} + \lambda^{(2)}$, the payoff matrix of player j when he plays with strategy $S_k^{(j)}$ against player 2 when he plays with strategy $S_k^{(j)}$, where $j = 1, 2, k = 1, 2, 3, 4, 5, 6$, given by table 1 and table 2 respectively.

	$S_1^{(2)}$	$S_2^{(2)}$	$S_3^{(2)}$	$S_4^{(2)}$	$S_5^{(2)}$	$S_6^{(2)}$
$S_1^{(1)}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_c-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_c-\lambda^{(1)}}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_c-\lambda^{(1)}}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_c-\lambda^{(1)}}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_c-\lambda^{(1)}}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_c-\lambda^{(1)}}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$
$S_2^{(1)}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda^{(1)}}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$
$S_3^{(1)}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda^{(1)}}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda^{(1)}}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$
$S_4^{(1)}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda^{(1)}}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda^{(1)}}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$
$S_5^{(1)}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda^{(1)}}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda^{(1)}}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$
$S_6^{(1)}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_g-\lambda^{(1)}}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_g-\lambda^{(1)}}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_g-\lambda^{(1)}}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_g-\lambda^{(1)}}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_g-\lambda^{(1)}}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_g-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$

Table 1 : Payoff matrix of player 1 for Case 1.

	$S_1^{(1)}$	$S_2^{(1)}$	$S_3^{(1)}$	$S_4^{(1)}$	$S_5^{(1)}$	$S_6^{(1)}$
$S_1^{(2)}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_c-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(2)}}+\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(2)}}+\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(2)}}+\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$
$S_2^{(2)}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$
$S_3^{(2)}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda^{(2)}}+\frac{1}{\mu_h-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda^{(2)}}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$
$S_4^{(2)}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda^{(2)}}+\frac{1}{\mu_h-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda^{(2)}}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$
$S_5^{(2)}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$
$S_6^{(2)}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}+\frac{1}{\mu_h-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}+\frac{1}{\mu_h-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_g-\lambda^{(2)}}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_g-\lambda^{(2)}}+\frac{1}{\mu_h-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_g-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$

Table 2 : Payoff matrix of player 2 for Case 1.

Example 3.1. (Payoff matrix with identical service rates) If we suppose that each node in this network have the same service rate ($\mu_i = 4$) for all node i in the grid, and the arrival rate for both players given by $\lambda^{(1)} = 2, \lambda^{(2)} = 1$ such that $\lambda < 4$. The payoff matrix for both players in this case is given by:

	$S_1^{(2)}$	$S_2^{(2)}$	$S_3^{(2)}$	$S_4^{(2)}$	$S_5^{(2)}$	$S_6^{(2)}$
$S_1^{(1)}$	(5,5)	(4.5,4.3)	(4,3.67)	(3.5,3)	(4,3.67)	(3.5,3)
$S_2^{(1)}$	(4.5,4.3)	(5,5)	(4.5,4.3)	(4,3.67)	(4.5,4.3)	(3.5,3)
$S_3^{(1)}$	(4,3.67)	(4.5,4.3)	(5,5)	(4.5,4.3)	(4,3.67)	(4,3.67)
$S_4^{(1)}$	(3.5,3)	(4,3.67)	(4.5,4.3)	(5,5)	(4.5,4.3)	(4.5,4.3)
$S_5^{(1)}$	(4,3.67)	(4.5,4.3)	(4,3.67)	(4.5,4.3)	(5,5)	(4,3.67)
$S_6^{(1)}$	(3.5,3)	(3.5,3)	(4,3.67)	(4.5,4.3)	(4,3.67)	(5,5)

Table 3: Payoff matrix with identical service rates of Case 1

Example 3.2. (Payoff matrix with different service rates) If we suppose that each node in this network have the following service rate ($\mu_a = \mu_b = \mu_e = \mu_k = 5, \mu_c = \mu_g = \mu_d = \mu_f = \mu_h = 4$), $\lambda^{(1)} = 2, \lambda^{(2)} = 1$, where $\lambda < max(\mu_i)$ for all node i in the grid. The payoff matrix for both players in this case is given by :

	$S_1^{(2)}$	$S_2^{(2)}$	$S_3^{(2)}$	$S_4^{(2)}$	$S_5^{(2)}$	$S_6^{(2)}$
$S_1^{(1)}$	(3.5,3)	(3,2.3)	(2.5,2.08)	(2.3,1.83)	(2.8,2.08)	(2.3,1.83)
$S_2^{(1)}$	(2.8,2.25)	(3,2.5)	(2.5,2.25)	(2.3,2)	(2.8,2.25)	(2.17,1.75)
$S_3^{(1)}$	(2.3,2.08)	(2.5,2.3)	(3,3)	(2.8,2.75)	(2.3,2.08)	(2.67,2.5)
$S_4^{(1)}$	(2.3,1.91)	(2.5,2.17)	(3,2.83)	(3.5,3.5)	(3,2.83)	(3.3,3.25)
$S_5^{(1)}$	(2.8,2.08)	(3,2.3)	(2.5,2.08)	(3,2.75)	(3.5,3)	(2.8,2.5)
$S_6^{(1)}$	(2.5,2)	(2.5,2)	(3,2.67)	(3.5,3.3)	(3,2.67)	(4,4)

Table 4 : Payoff matrix with different service rates of Case 1.

Case 2: In this case, the players have different entering and ending node Figure 2. Also, players should select just the shortest path from entering node to the ending node, in this case each path will contain just five nodes. If the entering node of player 1 is node a and the ending node is node k, then this player will have 6 different paths which represent the strategies of player 1 given by:

$$S_1^{(1)} = \{a, b, c, f, k\}, S_2^{(1)} = \{a, b, e, f, k\}, S_3^{(1)} = \{a, b, e, h, k\},$$

$$S_4^{(1)} = \{a, d, e, h, k\}, S_5^{(1)} = \{a, d, e, f, k\}, S_6^{(1)} = \{a, d, g, h, k\}.$$

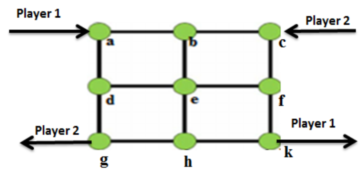

FIGURE 2. Grid of Case 2

If we assume that player 2's starting node is node c and that player 2's ending node is node g, then this player will also have six different paths that correspond to the following player 2 strategies:

$$S_1^{(2)} = \{a, b, c, f, k\}, S_2^{(2)} = \{a, b, e, f, k\}, S_3^{(2)} = \{a, b, e, h, k\},$$
$$S_4^{(2)} = \{a, d, e, h, k\}, S_5^{(2)} = \{a, d, e, f, k\}, S_6^{(2)} = \{a, d, g, h, k\}.$$

Using the same assumption of case 1 which is $\lambda = \lambda^{(1)} + \lambda^{(2)}$, the payoff matrix of player 1 and player 2 given by Table 5 and table 6 respectively.

	$S_1^{(2)}$	$S_2^{(2)}$	$S_3^{(2)}$	$S_4^{(2)}$	$S_5^{(2)}$	$S_6^{(2)}$
$S_1^{(1)}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_c-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_c-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_c-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_c-\lambda}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_c-\lambda}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_c-\lambda}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$
$S_2^{(1)}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda^{(1)}}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda^{(1)}}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$
$S_3^{(1)}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda^{(1)}}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_b-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda^{(1)}}+\frac{1}{\mu_h-\lambda^{\lambda(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$
$S_4^{(1)}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda^{(1)}}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda^{(1)}}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$
$S_5^{(1)}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda^{(1)}}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_e-\lambda^{(1)}}+\frac{1}{\mu_f-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$
$S_6^{(1)}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_g-\lambda^{(1)}}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_g-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_g-\lambda}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_d-\lambda^{(1)}}+\frac{1}{\mu_g-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda^{(1)}}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_g-\lambda}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$	$\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_g-\lambda}+\frac{1}{\mu_h-\lambda^{(1)}}+\frac{1}{\mu_k-\lambda^{(1)}}$

Table 5 : Payoff matrix of player 1 for Case 2.

	$S_1^{(1)}$	$S_2^{(1)}$	$S_3^{(1)}$	$S_4^{(1)}$	$S_5^{(1)}$	$S_6^{(1)}$
$S_1^{(2)}$	$\frac{1}{\mu_c-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}+\frac{1}{\mu_h-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}+\frac{1}{\mu_h-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_k-\lambda}+\frac{1}{\mu_k-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_g-\lambda}$
$S_2^{(2)}$	$\frac{1}{\mu_c-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_e-\lambda^{(2)}}+\frac{1}{\mu_h-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_k-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda^{(2)}}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_g-\lambda}$
$S_3^{(2)}$	$\frac{1}{\mu_c-\lambda}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_e-\lambda^{(2)}}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_k-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_f-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda^{(2)}}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_g-\lambda}$
$S_4^{(2)}$	$\frac{1}{\mu_c-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda^{(2)}}+\frac{1}{\mu_h-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_b-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_k-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_b-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda^{(2)}}+\frac{1}{\mu_h-\lambda}+\frac{1}{\mu_g-\lambda}$
$S_5^{(2)}$	$\frac{1}{\mu_c-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda^{(2)}}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_b-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_b-\lambda^{(2)}}+\frac{1}{\mu_e-\lambda^{(2)}}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_g-\lambda}$
$S_6^{(2)}$	$\frac{1}{\mu_c-\lambda}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_b-\lambda}+\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_b-\lambda^{(2)}}+\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_k-\lambda}+\frac{1}{\mu_e-\lambda}+\frac{1}{\mu_e-\lambda^{(2)}}+\frac{1}{\mu_g-\lambda^{(2)}}$	$\frac{1}{\mu_c-\lambda^{(2)}}+\frac{1}{\mu_b-\lambda^{(2)}}+\frac{1}{\mu_a-\lambda}+\frac{1}{\mu_d-\lambda}+\frac{1}{\mu_g-\lambda}$

Table 6 : Payoff matrix of player 2 for Case 2.

Example 3.3. (Payoff matrix with identical service rates) If we suppose that each node in this network have the same service rate ($\mu_i = 4$)for all node i in the grid. Also, the arrival rate for both players given by $\lambda^{(1)} = 2, \lambda^{(2)} = 1$ such that $\lambda < 4$. The payoff matrix for both players in this case is given by:

	$S_1^{(2)}$	$S_2^{(2)}$	$S_3^{(2)}$	$S_4^{(2)}$	$S_5^{(2)}$	$S_6^{(2)}$
$S_1^{(1)}$	(4,3.67)	(3.5,3)	(3.5,3)	(3.5,3)	(3.5,3)	(3.5,3.67)
$S_2^{(1)}$	(3.5,3)	(3.5,3)	(3.5,3)	(3.5,3)	(3.5,3)	(3.5,3)
$S_3^{(1)}$	(3.5,3)	(3.5,3)	(3,2.33)	(4,2.33)	(3.5,3)	(3.5,3)
$S_4^{(1)}$	(3.5,3)	(3.5,3)	(3.5,3.67)	(3.5,2.33)	(3.5,2.33)	(3.5,3)
$S_5^{(1)}$	(3.5,3)	(3.5,3)	(4,3)	(3,2.33)	(3.5,3)	(3.5,3)
$S_6^{(1)}$	(3.5,3.67)	(3.5,3)	(3.5,3)	(3.5,3)	(3.5,3)	(4,3.67)

Table 7 : Payoff matrix with identical service rates of Case 2.

Example 3.4. (Payoff matrix with different service rates) If we suppose that each node in this network have the following service rate ($\mu_a = \mu_b = \mu_e = \mu_k = 5, \mu_c = \mu_g = \mu_d = \mu_f = \mu_h = 4$), $\lambda^{(1)} = 2, \lambda^{(2)} = 1$ such that $\lambda < max(\mu_i)$ for all node i. The payoff matrix for both players in this case is given by:

	$S_1^{(2)}$	$S_2^{(2)}$	$S_3^{(2)}$	$S_4^{(2)}$	$S_5^{(2)}$	$S_6^{(2)}$
$S_1^{(1)}$	(3.17,3.17)	(3,2.5)	(3,2.5)	(2.67,2.5)	(2.67,2.5)	(2.67,3.17)
$S_2^{(1)}$	(2.5,2.92)	(2.5,2.5)	(2.5,2.5)	(2.17,2.5)	(2.17,2.5)	(2.17,2.92)
$S_3^{(1)}$	(2.5,2.92)	(2.5,2.5)	(2,1.83)	(2.67,1.83)	(2.17,2.5)	(2.17,2.92)
$S_4^{(1)}$	(2.67,2.42)	(2.67,2.42)	(2.67,2.67)	(2.67,2.17)	(2.67,1.75)	(2.67,2.83)
$S_5^{(1)}$	(2.67,2.42)	(2.67,2)	(3.17,2)	(2.17,1.75)	(2.67,2.42)	(2.67,2.83)
$S_6^{(1)}$	(2.5,2.67)	(3.17,2)	(3.17,2)	(3.17,2.42)	(3.17,2)	(3.33,3.08)

Table 8 : Payoff matrix with different service rates of Case 2.

4. Existence of equilibrium strategies

In this section we study the existence of equilibrium of our model. First, we introduce some basic definitions related to existence of Nash equilibrium concept [15, 19, 22].

Definition 4.1. A strategy $S^{(j)}$ for player j which hold the inequality

$$\Pi^{(j)}(s^j, s_{-j}) \leq \Pi^{(j)}(\bar{s})$$

for all s_{-j}, $\bar{s}$ is called dominant strategy. The payoff of a dominant strategy for player j is the lowest average sojourn time no matter the strategies selected by the other player.

Definition 4.2. The best-response strategy of player j against the strategies s_{-j} of other player is given by the set:

$$(B)^{(j)}(s_{-j}) = \{\hat{s}^{(j)} \in S^j | \pi^{(j)}(\hat{s}^{(j)}, s_{-j}) \leq \pi^{(j)}(\bar{s}^{(j)}, s_{-j}),\ \forall\ \bar{s}^{(j)}\}.$$

Definition 4.3. A strategy profile S is a pure-strategy Nash equilibrium (PNE for short) if each players strategy is a best-response to the strategies of other players ($i.e. S^{(j)} \in B^{(j)}(s_{-j}),\ \forall\ j$).

The following algorithm is used to find the best-response strategy.

Algorithm: Best-response algorithm

(1) If S is a Nash equilibrium then stop. Else, go to Step 2 with S.
(2) Select a player j such that $s^{(j)}$ is the best response to s_{-j} .Update $s^{(j)}$ to a best response of j to s_{-j}.
(3) If S is a Nash equilibrium then stop. Else, go to Step 2 with S.

Definition 4.4. The set of feasible strategies for each player is the set of all paths that begin at a fixed entering node on one side of the grid and conclude at a fixed ending node on the other side of this grid. Furthermore, this set only contain the paths of minimal length, that is, paths with the fewest nodes necessary to transit from the entering node to the ending node. In this game, each player's purpose is to reduce its consumers' time in this network.

In the first case showed in Figure 1, both players can share the entering and ending nodes, each player must pass through five nodes, and it is unclear if there is an equilibrium or which path will be selected. In the second case showed in Figure 2, the players can't share the node of entering and the node of ending but they can share the intermediate nodes. In this case also it is unclear if there is an equilibrium or which path will be selected. Also, if the players want to share the intermediate nodes in this grid or not to share. To answer to these questions, we introduce the following theorems:

Theorem 4.5. *A queuing game on a* 3×3*-grid played by two players as shown in Figure 1 with service rates* μ_i *such that*

$$\lambda^{(1)} + \lambda^{(2)} < \min\{\mu_i\}$$

has a pure-strategy Nash equilibrium.

Proof. Consider the queuing game shown in Figure 1 , player 1 must make a decision to select a strategy from the set $\{S_1^{(1)}, S_2^{(1)}, S_3^{(1)}, S_4^{(1)}, S_5^{(1)}, S_6^{(1)}\}$ which given in Case 1. Player 2 have the same choices and the rate of arrivals for both players are $\lambda^{(1)}, \lambda^{(2)}$ respectively. Let $\Pi^{(j)}(s_i^{(1)}, s_i^{(2)})$ denote the payoff function of player j when this player choose the path i, where $i = 1, 2, \cdots, 6$. Also let$B^{(j)}(s_i^{(m)})$ denote to the player j best response to the path i of player $m \neq j$.

Depending on the Algorithm, the Table 1 which represent the payoff matrix of player 1, Table 5 and Table 6 which represent a numerical examples for this case, we can summarize the best response strategies of player 1 against the strategies $s_i^{(2)}$ of player 2 in the following table:

i	${s_i}^{(2)}$	$B^{(1)}({s_i}^{(2)})$	
		Identical services rate	Different services rate
1	${s_1}^{(2)}$	${s_4}^{(1)}, {s_6}^{(1)}$	${s_4}^{(1)}$
2	${s_2}^{(2)}$	${s_6}^{(1)}$	${s_6}^{(1)}$
3	${s_3}^{(2)}$	${s_1}^{(1)}, {s_5}^{(1)}, {s_6}^{(1)}$	${s_1}^{(1)}, {s_5}^{(1)}$
4	${s_4}^{(2)}$	${s_1}^{(1)}$	${s_1}^{(1)}$
5	${s_5}^{(2)}$	${s_1}^{(1)}, {s_3}^{(1)}, {s_6}^{(1)}$	${s_1}^{(1)}, {s_3}^{(1)}$
6	${s_6}^{(2)}$	${s_1}^{(1)}, {s_2}^{(1)}$	${s_2}^{(1)}$

Table 9: A player 1s best response to the strategies $s_i^{(2)}$.

Using the same approach, the best response strategies of player 2 to the strategies $s_i^{(1)}$ of player 1 in the following table:

i	$s_i^{(1)}$	$B^{(2)}(s_i^{(1)})$	
		Identical services rate	Different services rate
1	$s_1^{(1)}$	$s_4^{(2)}, s_6^{(2)}$	$s_4^{(2)}, s_6^{(2)}$
2	$s_2^{(1)}$	$s_6^{(2)}$	$s_6^{(2)}$
3	$s_3^{(1)}$	$s_1^{(2)}, s_5^{(2)}, s_6^{(2)}$	$s_1^{(2)}, s_5^{(2)}$
4	$s_4^{(1)}$	$s_1^{(2)}$	$s_1^{(2)}$
5	$s_5^{(1)}$	$s_1^{(2)}, s_3^{(2)}, s_6^{(2)}$	$s_3^{(2)}$
6	$s_6^{(1)}$	$s_1^{(2)}, s_2^{(2)}$	$s_1^{(2)},, s_2^{(2)}$

Table 10: A player 2s best response to the strategies $s_i^{(2)}$.

By comparing Table 9 with Table 10, depending on the definition of Nash's equilibrium we see that this game has a PNE points in two cases. In case of identical, the PNE strategies are : $(s_4^{(1)}, s_1^{(2)}), (s_6^{(1)}, s_1^{(2)}), (s_6^{(1)}, s_2^{(2)}), (s_5^{(1)}, s_3^{(2)})$, $(s_2^{(1)}, s_6^{(2)})$ and $(s_3^{(1)}, s_5^{(2)})$. In case of different service rates, the PNE strategies are: $(s_6^{(1)}, s_2^{(2)}), (s_2^{(1)}, s_6^{(2)}), (s_5^{(1)}, s_3^{(2)}), (s_1^{(1)}, s_4^{(2)})$ and $(s_3^{(1)}, s_5^{(2)})$. □

Theorem 4.6. *A queuing game on a 3×3-grid played by two players as shown in Figure 2 with service rates μ_i such that*

$$\lambda^{(1)} + \lambda^{(2)} < min\{\mu_i\}$$

has a pure-strategy Nash equilibrium.

Proof. Using the same approach which used to prove the Theorem 4.5. In this case the both players have a different entering and ending node in their strategies which listed in Case 2. Table 11 and table 12 list all best response strategies for both players 1 and 2.

i	$s_i^{(2)}$	$B^{(1)}(s_i^{(2)})$	
		Identical services rate	Different services rate
1	$s_1^{(2)}$	$s_2^{(1)}, s_3^{(1)}, s_4^{(1)}, s_5^{(1)}$	$s_2^{(1)}, s_3^{(1)}, s_6^{(1)}$
2	$s_2^{(2)}$	$s_1^{(1)}, s_2^{(1)}, s_3^{(1)}, s_4^{(1)}, s_5^{(1)}, s_6^{(1)}$	$s_2^{(1)}, s_3^{(1)}$
3	$s_3^{(2)}$	$s_3^{(1)}$	$s_3^{(1)}$
4	$s_4^{(2)}$	$s_3^{(1)}, s_4^{(1)}, s_5^{(1)}$	$s_2^{(1)}, s_5^{(1)}$
5	$s_5^{(2)}$	$s_4^{(1)}$	$s_2^{(1)}, s_3^{(1)}$
6	$s_6^{(2)}$	$s_2^{(1)}, s_3^{(1)}, s_4^{(1)}, s_5^{(1)}$	$s_2^{(1)}$

Table 11: A player 1s best response to the strategies $s_i^{(2)}$.

i	$s_i^{(1)}$	$B^{(2)}(s_i^{(1)})$	
		Identical services rate	Different services rate
1	$s_1^{(1)}$	$s_2^{(2)}, s_3^{(2)} s_4^{(2)}, s_5^{(2)}$	$s_2^{(2)}, s_3^{(2)} s_4^{(2)}, s_5^{(2)}$
2	$s_2^{(1)}$	$s_1^{(2)}, s_2^{(2)}, s_3^{(2)}, s_4^{(2)}, s_5^{(2)}, s_6^{(2)}$	$s_2^{(2)}, s_3^{(2)} s_4^{(2)}, s_5^{(2)}$
3	$s_3^{(1)}$	$s_3^{(2)}, s_4^{(2)}$	$s_3^{(2)}, s_4^{(2)}$
4	$s_4^{(1)}$	$s_4^{(2)}, s_5^{(2)}$	$s_5^{(2)}$
5	$s_5^{(1)}$	$s_4^{(2)}$	$s_4^{(2)}$
6	$s_6^{(1)}$	$s_2^{(2)}, s_3^{(2)}, s_4^{(2)}, s_5^{(2)}$	$s_2^{(2)}, s_3^{(2)}, s_5^{(2)}$

Table 12: A player 2s best response to the strategies $s_i^{(1)}$.

By comparing Table 11 with Table 12, depending on the definition of Nash's equilibrium, we see that this game has a PNE points in two cases. In case of identical service rates, the PNE strategies are $(s_2^{(1)}, s_1^{(2)}), (s_1^{(1)}, s_2^{(2)}), (s_2^{(1)}, s_2^{(2)})$, $(s_6^{(1)}, s_2^{(2)}), (s_3^{(1)}, s_3^{(2)}), (s_3^{(1)}, s_4^{(2)}), (s_4^{(1)}, s_4^{(2)}), (s_5^{(1)}, s_4^{(2)}), (s_4^{(1)}, s_5^{(2)})$ and $(s_2^{(1)}, s_6^{(2)})$. In case of different service rates, the PNE strategies are:
$(s_2^{(1)}, s_2^{(2)}), (s_2^{(1)}, s_4^{(2)}), (s_2^{(1)}, s_5^{(2)}), (s_3^{(1)}, s_3^{(2)})$ and $(s_5^{(1)}, s_4^{(2)})$. □

5. Existence of PNE for equivalence arrival and service rates

In this case we can assume that there is a relation between the arrival and the service rates. We can assume that all nodes have just two service rates, either μ or $\alpha\mu$ where $\alpha > 1$, and for the relation between the arrival rates we can assume that $\lambda^{(1)} = \beta\lambda^{(2)}$ where $\beta < 1$.

Using a numerical calculation, we can distinguish the conditions for α, β for this game to have a PNE. For that we substitute different values for α and β in payoff matrix of Case 2 for both players (Tables 5,6), we get the following payoff matrices:

Table (13): $\mu_a = \mu_b = \mu_e = \mu_k = 5$, $\mu_c = \mu_g = \mu_d = \mu_f = \mu_h = 4$, $\lambda^{(1)} = 0.5$, $\lambda^{(2)} = 2$, $\alpha = 1.25$, $\beta = 0.25$

	$s_1^{(2)}$	$s_2^{(2)}$	$s_3^{(2)}$	$s_4^{(2)}$	$s_5^{(2)}$	$s_6^{(2)}$
$s_1^{(1)}$	(2.18,2.73)	(2,2.67)	(2,2.67)	(1.80,2.40)	(1.80,2.40)	(1.80,2.47)
$s_2^{(1)}$	(1.73,2.57)	(1.73,2.57)	(1.73,2.57)	(1.53,2.40)	(1.53,2.30)	(1.53,2.30)
$s_3^{(1)}$	(1.73,2.57)	(1.73,2.57)	(1.35,2.40)	(1.91,2.47)	(1.53,2.30)	(1.53,2.30)
$s_4^{(1)}$	(1.80,2.57)	(1.80,2.57)	(1.80,2.40)	(1.80,2.33)	(1.80,2.23)	(1.80,2.40)
$s_5^{(1)}$	(1.80,2.57)	(1.80,2.57)	(2.18,2.57)	(1.42,2.23)	(1.80,2.40)	(1.80,2.30)
$s_6^{(1)}$	(1.66,2.73)	(2.06,2.73)	(2.06,2.67)	(2.06,2.50)	(2.06,2.50)	(2.24,2.57)

In this case, this game has the following PNE strategies: $(s_2^{(1)}, s_5^{(2)}), (s_2^{(1)}, s_6^{(2)})$, $(s_3^{(1)}, s_5^{(2)}), (s_3^{(1)}, s_6^{(2)})$ and $(s_5^{(1)}, s_4^{(2)})$.

Table (14): $\mu_a = \mu_b = \mu_e = \mu_k = 7$, $\mu_c = \mu_g = \mu_d = \mu_f = \mu_h = 4$, $\lambda^{(1)} = 0.5$, $\lambda^{(2)} = 2$, $\alpha = 1.75$, $\beta = 0.25$

	$s_1^{(2)}$	$s_2^{(2)}$	$s_3^{(2)}$	$s_4^{(2)}$	$s_5^{(2)}$	$s_6^{(2)}$
$s_1^{(1)}$	(1.86,2.56)	(1.79,2.53)	(1.79,2.53)	(1.48,2.09)	(1.48,2.09)	(1.48,2.11)
$s_2^{(1)}$	(1.35,2.39)	(1.35,2.39)	(1.35,2.39)	(1.04,2.09)	(1.04,1.94)	(1.04,1.94)
$s_3^{(1)}$	(1.35,2.39)	(1.35,2.39)	(0.97,2.22)	(1.42,2.11)	(1.04,1.94)	(1.04,1.94)
$s_4^{(1)}$	(1.48,2.39)	(1.48,2.39)	(1.48,2.22)	(1.48,2.07)	(1.48,1.92)	(1.48,2.09)
$s_5^{(1)}$	(1.48,2.39)	(1.48,2.39)	(1.86,2.39)	(1.10,1.92)	(1.48,2.09)	(1.48,1.94)
$s_6^{(1)}$	(1.30,2.56)	(1.93,2.53)	(1.93,2.53)	(1.93,2.23)	(1.93,2.23)	(2,2.26)

In this case, there is no dominant strategies for both players, but there are four PNEs for the given game which are $(s_2^{(1)}, s_5^{(2)}), (s_2^{(1)}, s_6^{(2)}), (s_3^{(1)}, s_5^{(2)})$ and

$(s_3^{(1)}, s_6^{(2)})$. From the above calculations in Table 13 and Table 14, we notice that the strategies which a best response will intersect in just at most one node with the strategies of other player, then we must get the following inequality:

$$\frac{1}{\mu - \lambda^{(2)}} < \frac{1}{\alpha\mu - \beta\lambda^{(2)}}.$$

This inequality can be written as $\alpha\mu - \beta\lambda^{(2)} < \mu - \lambda^{(2)}$, which means that $\beta\lambda^{(2)} > (\alpha - 1)\mu$. Also, to guarantee that the players can share a node of least service rate μ, the condition $(\beta + 1)\lambda^{(2)} < \mu$ must hold. From these inequalities, and since $\beta < 1$, then α must hold the condition $1 < \alpha < 1.5$.

6. Conclusion

In this paper, we study the existence of equilibrium play in a specific class of queuing game played on a 3×3-grid as a non-cooperative 2-players game. In this game, both players compete to select, from a set of paths available for each player, just one path to use for their customers and exit from the grid after getting the required service as soon as possible. We have shown the existence of PNE for two cases. First case assumed that both players can share the same entering and ending nodes. Second case assumed that both players cannot share the entering and ending nodes but they can share the intermediate nodes. We discussed and analyzed the existence of PNE for two subclasses of games on 3×3-grid:

(i) A game of 2 players with identical service rates and different arrival rates,

(ii) A game of 2 players with different service rates and different arrival rates.

In both cases, the payoff matrices are computed and we list the best response strategies for both players. We have illustrated via some theorems and different examples that these games have a pure-strategies Nash equilibrium.

In the case that there is a relationship between the service rates of all nodes in this grid such that the service rates are either μ or $\alpha\mu$, and also there is a relationship between the arrival rates such that $\lambda^{(1)} = \beta\lambda^{(2)}$, we found that the conditions on these constants α and β for this game to have a PNE are $1 < \alpha < 1.5$ and $0 < \beta < 1$.

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