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SOME CONDITIONS FOR FUNCTIONS DEFINED BY BINOMIAL DISTRIBUTION TO BE IN A COMPREHENSIVE SUBFAMILY OF ANALYTIC FUNCTIONS

Basem Aref Frasin[1], Tariq Al-Hawary[2] and Jamal Salah[3]

[1]Faculty of Science, Department of Mathematics, Al al-Bayt University,
Mafraq, Jordan
e-mail: bafrasin@aabu.edu.jo

[2]Department of Applied Science, Ajloun College, Al Balqa Applied University,
Ajloun 26816, Jordan
e-mail: tariq_amh@bau.edu.jo

[3]Jamal Salah: College of Applied and Health Sciences, A'Sharqiyah University,
Post Box No. 42, Post Code No. 400 Ibra, Sultanate of Oman
e-mail: damous73@yahoo.com

Abstract. In this paper, considering the exciting recent results on binomial distribution, we examine an extensive subfamily $\Theta_{\varpi}(C_1, C_2, C_3, C_4)$ of univalent functions. Also, we give some conditions for the functions $F(\beta, \delta, \beth)$, $F(\beta, \delta, \beth)h$ and the operator $L(\beta, \delta, \beth)$ defined by the binomial distribution to be in this subfamily. The novelty of this research to inspire researchers of further studies to find new conditions to be in the subfamily $\Theta_{\varpi}(C_1, C_2, C_3, C_4)$, not only for binomial distribution but also for many other special functions. Moreover, we summarizes many previous studies.

1. Preliminaries

A basic discrete probability distribution getting exactly t successes out of β Bernoulli trials is given by the binomial distribution (assuming that each

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[0]Corresponding author: T. Al-Hawary(tariq_amh@bau.edu.jo).

Bernoulli trial's outcome is true with probability δ and false with probability $1-\delta$), and is widely used in probability theory and statistics, the binomial distribution is relevant to the fields of biology, health, social sciences, quality control, finance, and the outcomes of surveys or experiments involving binary responses.

Let the binomial distribution $g(\beta,\delta)$ defined by

$$g(\beta,\delta)=\Pr(Y=t)=\binom{\beta}{t}\delta^t(1-\delta)^{\beta-t}\equiv\frac{\beta!}{(\beta-t)!t!}\delta^t(1-\delta)^{\beta-t}, t=0,1,\cdots,\beta.$$

If $\beta>t$, then $g(\beta,\delta)=0$.

The Poisson distribution and the binomial distribution are connected when $\beta\delta$ is moderate such that δ is small and β is large, also Bernoulli distribution and the binomial distribution are connected when $\beta=1$.

Let the family of analytic and univalent functions Δ of the form:

$$h(\varkappa)=\varkappa+\sum_{t=2}^{\infty}\alpha_t\varkappa^t,\quad |\varkappa|<1 \tag{1.1}$$

such that $h(0)=h'(0)-1=0$.

Let us define a power series as follows:

$$\Pi(\beta,\delta,\varkappa)=\varkappa+\sum_{t=2}^{\infty}\frac{(\beta-1)!}{(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\varkappa^t.$$

Also, consider the series

$$F(\beta,\delta,\varkappa)=\varkappa-\sum_{t=2}^{\infty}\frac{(\beta-1)!}{(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\varkappa^t. \tag{1.2}$$

Moreover, by the convolution $(*)$, let a linear operator $F(\beta,\delta,\varkappa)h:\Delta\to\Delta$ be defined as:

$$F(\beta,\delta,\varkappa)h=F(\beta,\delta,\varkappa)*h(\varkappa)=\varkappa-\sum_{t=2}^{\infty}\frac{(\beta-1)!}{(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\alpha_t\varkappa^t.$$

Definition 1.1. A function $h\in\Delta$ is said to be in subfamily $\Theta_{\varpi}(C_1,C_2,C_3,C_4)$ if

$$\sum_{t=2}^{\infty}\left(C_1t^3+C_2t^2+C_3t+C_4\right)|\alpha_t|\le\varpi,$$

where C_1,C_2,C_3 and C_4 are real numbers and $\varpi>0$.

Through specialization for the coefficients C_1,C_2,C_3,C_4 and ϖ in Definition 1.1, we get that:

(1) A function $h \in \mathcal{T}(\delta, \kappa) \equiv \Theta_{1-\delta}(0, 0, 1-\delta\kappa, \delta\kappa-\delta)$ ([5]) if

$$\sum_{t=2}^{\infty} (t - \delta\kappa t - \delta + \delta\kappa) |\alpha_t| \leq 1 - \delta.$$

(2) A function $h \in \mathcal{C}(\delta, \kappa) \equiv \Theta_{1-\delta}(0, 1-\delta\kappa, \delta\kappa-\delta, 0)$ ([5]) if

$$\sum_{t=2}^{\infty} t\,(t - \delta\kappa t - \delta + \delta\kappa) |\alpha_t| \leq 1 - \delta.$$

(3) A function $h \in \wp^*_{\eta_3}(\eta_1, \eta_2) \equiv \Theta_{1-\eta_1}(0, 0, \eta_2 + 1, -\eta_3(\eta_1 + \eta_2))$ ([12]) if

$$\sum_{t=2}^{\infty} (t(\eta_2 + 1) - \eta_3(\eta_1 + \eta_2)) |\alpha_t| \leq 1 - \eta_1.$$

(4) A function $h \in \mathcal{C}^*_{\eta_3}(\eta_1, \eta_2) \equiv \Theta_{1-\eta_1}(0, \eta_2 + 1, -\eta_3(\eta_1 + \eta_2), 0)$ ([12]) if

$$\sum_{t=2}^{\infty} t\,(t(\eta_2 + 1) - \eta_3(\eta_1 + \eta_2)) |\alpha_t| \leq 1 - \eta_1.$$

Very recently, several researchers fined a connections between various subfamily of analytic and univalent functions, by using hypergeometric functions, see for example [3], [14] and [17] and by distribution series, see for example [9], [10] and [15].

This paper aims to introduce the comprehensive subfamily $\Theta_{\varpi}(C_1, C_2, C_3, C_4)$ of the family Δ, that generalize many previous subfamilies of analytic functions in open unit disk. Also, we give some conditions for the functions $F(\beta, \delta, \varkappa)$, $F(\beta, \delta, \varkappa)h$ and the operator $L(\beta, \delta, \varkappa)$ to be in this subfamily.

Numerous authors have written about the geometric properties of different special functions(see [1]-[7]).

2. Main results

In this section, we give necessary condition for the function $F(\beta, \delta, \varkappa)$ to be in the subfamily $\Theta_{\varpi}(C_1, C_2, C_3, C_4)$.

Theorem 2.1. *The function $F(\beta, \delta, \varkappa)$ is in subfamily $\Theta_{\varpi}(C_1, C_2, C_3, C_4)$ if*

$$C_1\delta^3(\beta-1)(\beta-2)(\beta-3) + (6C_1 + C_2)\delta^2(\beta-1)(\beta-2)$$
$$+(7C_1 + 3C_2 + C_3)\delta(\beta-1) + (C_1 + C_2 + C_3 + C_4)\Gamma_0(\beta, \delta) \leq \varpi,$$

where

$$\Gamma_0(\beta, \delta) = \sum_{t=1}^{\infty} \frac{(\beta-1)!}{(\beta-t-1)!t!} \delta^t (1-\delta)^{\beta-t-1}. \tag{2.1}$$

Proof. From Definition 1.1 and the function

$$F(\beta,\delta,\varkappa)=\varkappa-\sum_{t=2}^{\infty}\frac{(\beta-1)!}{(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\varkappa^{t},$$

it suffices to prove that

$$\sum_{t=2}^{\infty}\left(C_1t^3+C_2t^2+C_3t+C_4\right)\frac{(\beta-1)!}{(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\leq\varpi.$$

We know that

$$\begin{cases} t=(t-1)+1; \\ t^2=(t-1)(t-2)+3(t-1)+1; \\ t^3=(t-1)(t-2)(t-3)+6(t-1)(t-2)+7(t-1)+1. \end{cases} \tag{2.2}$$

Hence we get

$$\begin{aligned}
&\sum_{t=2}^{\infty}\left(C_1t^3+C_2t^2+C_3t+C_4\right)\frac{(\beta-1)!}{(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\\
&\quad=\sum_{t=2}^{\infty}\{C_1(t-1)(t-2)(t-3)+(6C_1+C_2)(t-1)(t-2)\\
&\qquad+(7C_1+3C_2+C_3)(t-1)+C_1+C_2+C_3+C_4\}\\
&\qquad\times\frac{(\beta-1)!}{(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\\
&\quad=C_1\sum_{t=4}^{\infty}\frac{(\beta-1)!}{(\beta-t)!(t-4)!}\delta^{t-1}(1-\delta)^{\beta-t}\\
&\qquad+(6C_1+C_2)\sum_{t=3}^{\infty}\frac{(\beta-1)!}{(\beta-t)!(t-3)!}\delta^{t-1}(1-\delta)^{\beta-t}\\
&\qquad+(7C_3+3C_2+C_3)\sum_{t=2}^{\infty}\frac{(\beta-1)!}{(\beta-t)!(t-2)!}\delta^{t-1}(1-\delta)^{\beta-t}\\
&\qquad+(C_1+C_2+C_3+C_4)\sum_{t=2}^{\infty}\frac{(\beta-1)!}{(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}
\end{aligned}$$

$$= C_1 \sum_{t=0}^{\infty} \frac{(\beta-1)!}{t!(\beta-t-4)!} \delta^{t+3}(1-\delta)^{\beta-t-4}$$
$$+ (6C_1 + C_2) \sum_{t=0}^{\infty} \frac{(\beta-1)!}{t!(\beta-t-3)!} \delta^{t+2}(1-\delta)^{\beta-t-3}$$
$$+ (7C_1 + 3C_2 + C_3) \sum_{t=0}^{\infty} \frac{(\beta-1)!}{t!(\beta-t-2)!} \delta^{t+1}(1-\delta)^{\beta-t-2}$$
$$+ (C_1 + C_2 + C_3 + C_4) \sum_{t=1}^{\infty} \frac{(\beta-1)!}{t!(\beta-t-1)!} \delta^{t}(1-\delta)^{\beta-t-1}$$
$$= C_1\delta^3(\beta-1)(\beta-2)(\beta-3) \sum_{t=0}^{\infty} \frac{(\beta-4)!}{t!(\beta-t-4)!} \delta^{t}(1-\delta)^{\beta-t-4}$$
$$+ (6C_1 + C_2)\delta^2(\beta-1)(\beta-2) \sum_{t=0}^{\infty} \frac{(\beta-3)!}{t!(\beta-t-3)!} \delta^{t}(1-\delta)^{\beta-t-3}$$
$$+ (7C_1 + 3C_2 + C_3)\delta(\beta-1) \sum_{t=0}^{\infty} \frac{(\beta-2)!}{t!(\beta-t-2)!} \delta^{t}(1-\delta)^{\beta-t-2}$$
$$+ (C_1 + C_2 + C_3 + C_4) \sum_{t=1}^{\infty} \frac{(\beta-1)!}{t!(\beta-t-1)!} \delta^{t}(1-\delta)^{\beta-t-1}$$
$$= C_1\delta^3(\beta-1)(\beta-2)(\beta-3) + (6C_1 + C_2)\delta^2(\beta-1)(\beta-2)$$
$$+ (7C_1 + 3C_2 + C_3)\delta(\beta-1) + (C_1 + C_2 + C_3 + C_4)\Gamma_0(\beta,\delta)$$
$$\leq \varpi.$$

This completes the proof. □

3. Inclusion properties

For $\varepsilon_1 \in (0,1]$, $\varepsilon_2 < 1$ and $\varepsilon_3 \in \mathbb{C} - \{0\}$, a function $h \in \Delta$ is said to be in the subfamily $\mathcal{G}^{\varepsilon_3}(\varepsilon_1, \varepsilon_2)$ if it satisfies

$$\left| \frac{(1-\varepsilon_1)\frac{h(\varkappa)}{\varkappa} + \varepsilon_1 h'(\varkappa) - 1}{2\varepsilon_3(1-\varepsilon_2) + (1-\varepsilon_1)\frac{h(\varkappa)}{\varkappa} + \varepsilon_1 h'(\varkappa) - 1} \right| < 1, \quad |\varkappa| < 1.$$

Lemma 3.1. ([16]) *If $h \in \mathcal{G}^{\varepsilon_3}(\varepsilon_1, \varepsilon_2)$ of the form* (1.1), *then*

$$|\alpha_t| \leq \frac{2\,|\varepsilon_3|\,(1-\varepsilon_2)}{\varepsilon_1(t-1)+1}, \quad t \in \mathbb{N} - \{1\}.$$

Theorem 3.2. *For* $h \in \mathcal{G}^{\varepsilon_3}(\varepsilon_1, \varepsilon_2)$, *if the inequality*

$$\left\{\delta^2 C_1(\beta-1)(\beta-2)+\delta(3C_1+C_2)(\beta-1)+(C_1+C_2+C_3)\Gamma_0(\beta,\delta)\right\} \leq \frac{\varepsilon_1 \varpi}{2\left|\varepsilon_3\right|(1-\varepsilon_2)}$$

is satisfied where $\Gamma_0(\beta,\delta)$ *is in* (2.1), *then* $F(\beta,\delta,\varkappa)h \in \Theta_{\varpi}(C_1,C_2,C_3,0)$.

Proof. Let $h \in \mathcal{G}^{\varepsilon_3}(\varepsilon_1, \varepsilon_2)$, by Definition 1.1 it suffices to prove that

$$\sum_{t=2}^{\infty}\left(C_1t^3+C_2t^2+C_3t\right)\frac{(\beta-1)!}{(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\left|\alpha_t\right| \leq \varpi.$$

Since $h \in \mathcal{G}^{\varepsilon_3}(\varepsilon_1, \varepsilon_2)$, then by Lemma 3.1

$$\begin{aligned}&\sum_{t=2}^{\infty}\left(C_1t^3+C_2t^2+C_3t\right)\frac{(\beta-1)!}{(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\left|\alpha_t\right|\\ &\quad\leq 2\left|\varepsilon_3\right|(1-\varepsilon_2)\sum_{t=2}^{\infty}\frac{\left(C_1t^3+C_2t^2+C_3t\right)(\beta-1)!}{\left(\varepsilon_1(t-1)+1\right)(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}.\end{aligned}$$

Also, since $\varepsilon_1(t-1)+1 \geq t\varepsilon_1$, we get

$$\begin{aligned}&\sum_{t=2}^{\infty}\left(C_1t^3+C_2t^2+C_3t\right)\frac{(\beta-1)!}{(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\left|\alpha_t\right|\\ &\quad\leq \frac{2\left|\varepsilon_3\right|(1-\varepsilon_2)}{\varepsilon_1}\sum_{t=2}^{\infty}\frac{\left(C_1t^2+C_2t+C_3\right)(\beta-1)!}{(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\\ &\quad= \frac{2\left|\varepsilon_3\right|(1-\varepsilon_2)}{\varepsilon_1}\sum_{t=2}^{\infty}\left(C_1(t-1)(t-2)+(3C_1+C_2)(t-1)+C_1+C_2+C_3\right)\\ &\qquad\times\frac{(\beta-1)!}{(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\\ &\quad= \frac{2\left|\varepsilon_3\right|(1-\varepsilon_2)}{\varepsilon_1}\left\{C_1\sum_{t=3}^{\infty}\frac{(\beta-1)!}{(\beta-t)!(t-3)!}\delta^{t-1}(1-\delta)^{\beta-t}\right.\\ &\qquad+(3C_1+C_2)\sum_{t=2}^{\infty}\frac{(\beta-1)!}{(\beta-t)!(t-2)!}\delta^{t-1}(1-\delta)^{\beta-t}\\ &\qquad\left.+(C_1+C_2+C_3)\sum_{t=2}^{\infty}\frac{(\beta-1)!}{(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\right\}\end{aligned}$$

$$= \frac{2|\varepsilon_3|(1-\varepsilon_2)}{\varepsilon_1}\left\{C_1\sum_{t=0}^{\infty}\frac{(\beta-1)!}{t!(\beta-t-3)!}\delta^{t+2}(1-\delta)^{\beta-t-3}\right.$$

$$+(3C_1+C_2)\sum_{t=0}^{\infty}\frac{(\beta-1)!}{t!(\beta-t-2)!}\delta^{t+1}(1-\delta)^{\beta-t-2}$$

$$\left.+(C_1+C_2+C_3)\sum_{t=1}^{\infty}\frac{(\beta-1)!}{t!(\beta-t-1)!}\delta^{t}(1-\delta)^{\beta-t-1}\right\}$$

$$= \frac{2|\varepsilon_3|(1-\varepsilon_2)}{\varepsilon_1}\left\{\delta^2C_1(\beta-1)(\beta-2)\sum_{t=0}^{\infty}\frac{(\beta-3)!}{t!(\beta-t-3)!}\delta^{t}(1-\delta)^{\beta-t-3}\right.$$

$$+\delta(3C_1+C_2)(\beta-1)\sum_{t=0}^{\infty}\frac{(\beta-2)!}{t!(\beta-t-2)!}\delta^{t}(1-\delta)^{\beta-t-2}$$

$$\left.+(C_1+C_2+C_3)\sum_{t=1}^{\infty}\frac{(\beta-1)!}{t!(\beta-t-1)!}\delta^{t-1}(1-\delta)^{\beta-t-1}\right\}$$

$$= \frac{2|\varepsilon_3|(1-\varepsilon_2)}{\varepsilon_1}\{\delta^2C_1(\beta-1)(\beta-2)+\delta(3C_1+C_2)(\beta-1)$$

$$+(C_1+C_2+C_3)\Gamma_0(\beta,\delta)\}.$$

The proof of Theorem 3.2 is completed with the last equation, which is bounded above by ϖ. □

4. An integral operator $L(\beta,\delta,\varkappa)$

Theorem 4.1. *If the integral operator $L(\beta,\delta,\varkappa)$ is given by*

$$L(\beta,\delta,\varkappa)=\int_0^{\varkappa}\frac{F(\beta,\delta,u)}{u}du,\ |\varkappa|<1, \tag{4.1}$$

then $L(\beta,\delta,\varkappa)\in\Theta_{\varpi}(C_1,C_2,C_3,0)$ if

$$\delta^2C_1(\beta-1)(\beta-2)+\delta(3C_1+C_2)(\beta-1)+(C_1+C_2+C_3)\Gamma_0(\beta,\delta)\le\varpi,$$

where $\Gamma_0(\beta,\delta)$ is given by (2.1).

Proof. Since

$$L(\beta,\delta,\varkappa)=\varkappa-\sum_{t=2}^{\infty}\frac{(\beta-1)!}{t(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\varkappa^t,$$

then by Definition 1.1, it suffices to prove that

$$\begin{aligned}&\sum_{t=2}^{\infty}\left(C_1t^3+C_2t^2+C_3t\right)\frac{(\beta-1)!}{t(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\\&=\sum_{t=2}^{\infty}\left(C_1t^2+C_2t+C_3\right)\frac{(\beta-1)!}{(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\\&\le\varpi.\end{aligned}$$

Similar to the proof of Theorem 3.2, we have

$$\begin{aligned}&\sum_{t=2}^{\infty}\left(C_1t^3+C_2t^2+C_3t\right)\frac{(\beta-1)!}{t(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\\&\le\delta^2C_1(\beta-1)(\beta-2)+\delta(3C_1+C_2)(\beta-1)+(C_1+C_2+C_3)\Gamma_0(\beta,\delta)\\&\le\varpi.\end{aligned}\tag{4.2}$$

□

Theorem 4.2. *If the operator $L(\beta,\delta,\varkappa)$ is given by* (4.1), *then $L(\beta,\delta,\varkappa)\in\Theta_{\varpi}(C_1,C_2,C_3,C_4)$ if*

$$\begin{aligned}&C_1\Gamma_1(\beta,\delta)+(6C_1+C_2)\Gamma_2(\beta,\delta)+(7C_1+3C_2+C_3)\Gamma_3(\beta,\delta)\\&+(C_1+C_2+C_3+C_4)\Gamma_4(\beta,\delta)\le\varpi,\end{aligned}$$

where

$$\begin{aligned}\Gamma_1(\beta,\delta)&=\sum_{t=0}^{\infty}\frac{(\beta-1)!}{t!(\beta-t-4)!(t+4)}\delta^{t+3}(1-\delta)^{\beta-t-4},\\\Gamma_2(\beta,\delta)&=\sum_{t=0}^{\infty}\frac{(\beta-1)!}{t!(\beta-t-3)!(t+3)}\delta^{t+2}(1-\delta)^{\beta-t-3},\\\Gamma_3(\beta,\delta)&=\sum_{t=0}^{\infty}\frac{(\beta-1)!}{t!(\beta-t-2)!(t+2)}\delta^{t+1}(1-\delta)^{\beta-t-2}\end{aligned}\tag{4.3}$$

and

$$\Gamma_4(\beta,\delta)=\sum_{t=2}^{\infty}\frac{(\beta-1)!}{t!(\beta-t)!}\delta^{t-1}(1-\delta)^{\beta-t}.\tag{4.4}$$

Proof. Since

$$L(\beta,\delta,\varkappa)=\varkappa-\sum_{t=2}^{\infty}\frac{(\beta-1)!}{t(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\varkappa^t,$$

then by Definition 1.1, it suffices to prove that

$$\begin{aligned}
&\sum_{t=2}^{\infty}\left(C_1t^3+C_2t^2+C_3t+C_4\right)\frac{(\beta-1)!}{t(\beta-t)!(t-1)!}\delta^{t-1}(1-\delta)^{\beta-t}\\
&\quad=\sum_{t=2}^{\infty}\left(C_1t^3+C_2t^2+C_3t+C_4\right)\frac{(\beta-1)!}{t!(\beta-t)!}\delta^{t-1}(1-\delta)^{\beta-t}\\
&\quad\leq\varpi.
\end{aligned}$$

So

$$\begin{aligned}
&\sum_{t=2}^{\infty}\left(C_1t^3+C_2t^2+C_3t+C_4\right)\frac{(\beta-1)!}{t!(\beta-t)!}\delta^{t-1}(1-\delta)^{\beta-t}\\
&=C_1\sum_{t=4}^{\infty}\frac{(\beta-1)!}{t(\beta-t)!(t-4)!}\delta^{t-1}(1-\delta)^{\beta-t}\\
&\quad+(6C_1+C_2)\sum_{t=3}^{\infty}\frac{(\beta-1)!}{t(\beta-t)!(t-3)!}\delta^{t-1}(1-\delta)^{\beta-t}\\
&\quad+(7C_1+3C_2+C_3)\sum_{t=2}^{\infty}\frac{(\beta-1)!}{t(\beta-t)!(t-2)!}\delta^{t-1}(1-\delta)^{\beta-t}\\
&\quad+(C_1+C_2+C_3+C_4)\sum_{t=2}^{\infty}\frac{(\beta-1)!}{t!(\beta-t)!}\delta^{t-1}(1-\delta)^{\beta-t}\\
&=C_1\sum_{t=0}^{\infty}\frac{(\beta-1)!}{t!(\beta-t-4)!(t+4)}\delta^{t+3}(1-\delta)^{\beta-t-4}\\
&\quad+(6C_1+C_2)\sum_{t=0}^{\infty}\frac{(\beta-1)!}{t!(\beta-t-3)!(t+3)}\delta^{t+2}(1-\delta)^{\beta-t-3}\\
&\quad+(7C_1+3C_2+C_3)\sum_{t=0}^{\infty}\frac{(\beta-1)!}{t!(\beta-t-2)!(t+2)}\delta^{t+1}(1-\delta)^{\beta-t-2}\\
&\quad+(C_1+C_2+C_3+C_4)\sum_{t=2}^{\infty}\frac{(\beta-1)!}{t!(\beta-t)!}\delta^{t-1}(1-\delta)^{\beta-t}\\
&\leq\varpi.
\end{aligned}$$

This completes the proof □

5. Corollaries

By specializing the coefficients C_1, C_2, C_3, C_4 and ϖ in our Theorems, we obtain many results studied by many authors, as an illustration:

Let $C_1 = 0$, $C_2 = 0$, $C_3 = 1-\delta\kappa$, $C_4 = \delta\kappa - \delta$ and $\varpi = 1-\delta$ in Theorem 2.1 we get the following corollary, which is due to [13] in Theorem 2.1.

Corollary 5.1. ([13]) *The function* $F(\beta,\delta,\varkappa)$ *is in the subfamily* $\mathcal{T}(\delta,\kappa)$ *if*

$$\delta(\beta-1)(1-\delta\kappa) + (1-\delta)\Gamma_0(\beta,\delta) \leq 1-\delta.$$

Let $C_1 = 0$, $C_2 = 1-\delta\kappa$, $C_3 = \delta\kappa-\delta$, $C_4 = 0$ and $\mu = 1-\delta$ in Theorem 2.1, we conclude the following corollary, which is due to [13] in Theorem 2.2.

Corollary 5.2. ([13]) *The function* $F(\beta,\delta,\varkappa)$ *is in the subfamily* $\mathcal{C}(\delta,\kappa)$ *if*

$$\delta^2(\beta-1)(\beta-2)(1-\delta\kappa) + \delta(\beta-1)(3-2\delta\kappa-\delta) + (1-\delta)\Gamma_0(\beta,\delta) \leq 1-\delta.$$

Let $C_1 = 0$, $C_2 = 1-\delta\kappa$, $C_3 = \delta\kappa-\delta$, $C_4 = 0$ and $\mu = 1-\delta$ in Theorem 3.2, we conclude the following corollary.

Corollary 5.3. *For* $h \in \mathcal{G}^{\varepsilon_3}(\varepsilon_1,\varepsilon_2)$, *if the inequality*

$$\delta(\beta-1)(1-\delta\kappa) + (1-\delta)\Gamma_0(\beta,\delta) \leq \frac{\varepsilon_1(1-\delta)}{2\left|\varepsilon_3\right|(1-\varepsilon_2)}$$

is satisfied, then $F(\beta,\delta,\varkappa)h \in \mathcal{C}(\delta,\kappa)$.

Let $C_1 = 0$, $C_2 = 1-\delta\kappa$, $C_3 = \delta\kappa-\delta$, $C_4 = 0$ and $\mu = 1-\delta$ in Theorem 4.1, we conclude the following corollary, which is due to [13] in Theorem 2.3.

Corollary 5.4. ([13]) *If the integral operator* $L(\beta,\delta,\varkappa)$ *is given by* (4.1), *then* $L(\beta,\delta,\varkappa) \in \mathcal{C}(\delta,\kappa)$ *if*

$$\delta(1-\delta\kappa)(\beta-1) + (1-\delta)\Gamma_0(\beta,\delta) \leq 1-\delta.$$

Let $C_1 = 0$, $C_2 = 0$, $C_3 = 1-\delta\kappa$, $C_4 = \delta\kappa-\delta$ and $\varpi = 1-\delta$ in Theorem 4.2, we conclude the following corollary, which is due to [13] in Theorem 2.4.

Corollary 5.5. ([13]) *If integral operator* $L(\beta,\delta,\varkappa)$ *is given by* (4.1), *then* $L(\beta,\delta,\varkappa) \in \mathcal{T}(\delta,\kappa)$ *if*

$$(1-\delta\kappa)\Gamma_3(\beta,\delta) + (1-\delta)\Gamma_4(\beta,\delta) \leq 1-\delta,$$

where $\Gamma_3(\beta,\delta)$ *given by* (4.3) and $\Gamma_4(\beta,\delta)$ *given by* (4.4).

Let $C_1 = 0$, $C_2 = 0$, $C_3 = \eta_2+1$, $C_4 = -\eta_3(\eta_1+\eta_2)$ and $\varpi = 1-\eta_1$ in Theorem 2.1 we get the following corollary, which is due to [4] in Theorem 2.1.

Corollary 5.6. ([4]) *The function* $F(\beta,\delta,\varkappa)$ *is in the subfamily* $\wp^*_{\eta_3}(\eta_1,\eta_2)$ *if*

$$(\eta_2+1)\delta(\beta-1) + (\eta_2-\eta_3(\eta_1+\eta_2)+1)\Gamma_0(\beta,\delta) \leq 1-\eta_1.$$

Let $C_1 = 0$, $C_2 = \eta_2+1$, $C_3 = -\eta_3(\eta_1+\eta_2)$, $C_4 = 0$ and $\mu = 1-\eta_1$ in Theorem 2.1, we conclude the following corollary, which is due to [4] in Theorem 2.2.

Corollary 5.7. ([4]) *The function* $F(\beta,\delta,\varkappa)$ *is in the subfamily* $\mathcal{C}^*_{\eta_3}(\eta_1,\eta_2)$ *if*

$$(\eta_2+1)\delta^2(\beta-1)(\beta-2)+(3(\eta_2+1)-\eta_3(\eta_1+\eta_2))\delta(\beta-1)$$
$$+(\eta_2-\eta_3(\eta_1+\eta_2)+1)\Gamma_0(\beta,\delta)\leq 1-\eta_1.$$

Let $C_1=0$, $C_2=\eta_2+1$, $C_3=-\eta_3(\eta_1+\eta_2)$, $C_4=0$ and $\mu=1-\eta_1$ in Theorem 3.2, we conclude the following corollary, which is due to [4] in Theorem 3.2.

Corollary 5.8. ([4]) *For* $h\in\mathcal{G}^{\varepsilon_3}(\varepsilon_1,\varepsilon_2)$, *if the inequality*

$$\delta(\beta-1)(1-\delta\kappa)+(1-\delta)\Gamma_0(\beta,\delta)\leq\frac{\varepsilon_1(1-\eta_1)}{2\left|\varepsilon_3\right|(1-\varepsilon_2)}$$

holds, then $F(\beta,\delta,\varkappa)h\in\mathcal{C}^*_{\eta_3}(\eta_1,\eta_2)$.

Let $C_1=0$, $C_2=\eta_2+1$, $C_3=-\eta_3(\eta_1+\eta_2)$, $C_4=0$ and $\mu=1-\eta_1$ in Theorem 4.1, we conclude the following corollary, which is due to [4] in Theorem 4.1.

Corollary 5.9. ([4]) *If the integral operator* $L(\beta,\delta,\varkappa)$ *is given by* (4.1), *then* $L(\beta,\delta,\varkappa)\in\mathcal{C}^*_{\eta_3}(\eta_1,\eta_2)$ *if*

$$\delta(1-\delta\kappa)(\beta-1)+(1-\delta)\Gamma_0(\beta,\delta)\leq 1-\eta_1.$$

Let $C_1=0$, $C_2=0$, $C_3=\eta_2+1$, $C_4=-\eta_3(\eta_1+\eta_2)$ and $\varpi=1-\eta_1$ in Theorem 4.2, we conclude the following corollary.

Corollary 5.10. *If integral operator* $L(\beta,\delta,\varkappa)$ *is given by* (4.1), *then* $L(\beta,\delta,\varkappa)\in\wp^*_{\eta_3}(\eta_1,\eta_2)$ *if*

$$(1-\delta\kappa)\Gamma_3(\beta,\delta)+(1-\delta)\Gamma_4(\beta,\delta)\leq 1-\eta_1,$$

where $\Gamma_3(\beta,\delta)$ *given by* (4.3) and $\Gamma_4(\beta,\delta)$ *given by* (4.4).

6. Conclusions

Using of the binomial distribution, we find necessary conditions for the functions $F(\beta,\delta,\varkappa)$, $F(\beta,\delta,\varkappa)h$ and the operator $L(\beta,\delta,\varkappa)$ defined by binomial distribution to be in the inclusive subfamily $\Theta_\varpi(C_1,C_2,C_3,C_4)$. Furthermore, this paper inspire researchers of further studies to find new conditions to be in the subfamily $\Theta_\varpi(C_1,C_2,C_3,C_4)$, not only for binomial distribution but for many other special functions.

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