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ANALYTICAL SOLUTION OF ABEL'S INTEGRAL EQUATION VIA NEW TRANSFORM SO-CALLED SADIK TRANSFORM WITH APPLICATIONS

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Abstract. Abel's integral equation is an important type of integral equation that arises in many areas of mathematics, physics, and engineering. For example, in tomography (specifically, in X-ray tomography or CT scans), the reconstruction of a function from its projection data can often be formulated as an Abel's integral equation. The equation helps reconstruct the radial density distribution of an object from its projections at various angles. So, the objective of this study is to improve and develop the solve Abel's integral equation by using a new transform so-called Sadik Transform. The main contributions include establishing and deriving sufficient conditions to solve Abel's integral equation and generalized Abel's integral equation by recent Sadik Transform. In addition, some applications are made to explain the solution procedure of Abel's integral equation by Sadik Transform.

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1. Introduction

Integral equations are defined as equations that involve an unknown function situated beneath an integral sign. These equations have found utility in various problems across multiple disciplines due to their relationship with differential equations [11, 13, 20]. A notable example is Abel's integral equation, which is a significant singular integral equation derived by Abel from a mechanical problem known as the tautochrone problem. This equation and its variants are utilized across a wide range of fields, including heat transfer between solid materials and gases in non-linear boundary scenarios, the study of superfluidity, subsolutions related to nonlinear diffusion challenges, shock wave movement in gas conduits, microscopy, seismology, radio astronomy, satellite photometry of atmospheric glows, electron emission, atomic scattering, radar measurement, optical fiber assessment, X-ray imaging, and flame and plasma diagnostics [9, 21]. Integral transforms can be very useful for solving integral equations by simplifying the problem, often converting it into an algebraic equation that is easier to handle like Laplace Transform, Fourier Transform, Z-Transform and so on, finally Sadik Transform (ST) which is family transform, developed by Sadik [3, 15].

Abel discovered an important singular integral equation known as Abel's integral equation while working on the tautochrone problem in mechanics. This problem is considered to be the first application of fractional calculus to an engineering problem [1], where the problem given by

$$f(\varsigma) = \int_0^{\varsigma} \frac{\varkappa(\tau)}{\sqrt{\varsigma - \tau}} d\tau.$$

During the examination of the movement of a particle as it slides down an unfamiliar path in a vertical plane [21]. Abel's integral equation, along with several of its variants, has been utilized in various fields, including heat transfer between solids and gases under non-linear boundary conditions [10], water percolation [5], subsolutions for non-linear diffusion problems, and the propagation of shock waves in gas field tubes [12], the theory of superfluidity [8]. Additionally, it has applications in microscopy, seismology, radio astronomy, satellite photometry of airglows, electron emission, atomic scattering, radar ranging, optical fiber assessment, X-ray radiography, and diagnostics of flames and plasma [19].

Integral transformations are mathematical techniques used to simplify problems, solve various types of differential equations, or analyze functions by transforming them into a different form. The differential equations convert into algebraic equations by using integral transforms, algebraic equations are easier than differential equations. In the literature, there are different integral transforms and all of them are suitable to resolve many types of differential equations. lately, several new integral transforms were introduced, see [2, 4, 6, 7].

Recently, Shaikh [15, 16, 17, 18] proposed a novel integral transform referred to as ST. This transform serves as a unification of several established transforms, including the Laplace transform, Tarig and Laplace-Carson transform, Kamal transform, Elzaki transform, and Sumudu transform. For instance, in [15], Shaikh outlined various properties of this transform, including the duality theorem and the existing theorem of ST.

Furthermore, the author demonstrated that the aforementioned transforms are specific instances of ST. In [16], Shaikh established properties of ST concerning the derivatives of functions and the shifting theorem for ST. Additionally, the author derived the transfer function of a dynamical system within control theory using ST. He also addressed several applications in control theory through the use of ST. Overall, the integral transform method proves to be a valuable and effective tool for solving fractional differential equations.

1.1. **Why the Sadik transform? Reasons and motivation.** The ST, which is named after the mathematician Sadik [14, 15], represents a recent addition to the integral transforms family. Its creation was driven by the necessity to overcome specific limitations of existing transforms and to provide fresh insights and techniques for function analysis and differential equations resolution. Unlike traditional integral transforms such as the Laplace and Fourier transforms, which are mainly suitable for linear differential equations, the ST was designed to effectively handle certain types of nonlinear differential equations, offering solutions in cases where conventional methods may encounter difficulties. By generalizing the concepts of other integral transforms, the ST introduces a new viewpoint that can be particularly valuable in unique scenarios where other transforms may not be directly applicable or may require complex adjustments. It introduces novel tools for fields where traditional transforms are less efficient, finding applications in signal processing, control theory, and other domains where classical transforms may not deliver optimal performance.

1.2. **Construction of the paper.** The paper is structured as follows: Section 2 provides the necessary background, including basic de nitions, and a supporting lemma. Building on this foundation, Section 3 is the core of the work, where the authors utilize ST to solve Abel's integral equation and generalized Abel's integral equation. The ndings from this theoretical analysis are then illustrated through relevant some applications are made to explain the solution procedure of Abel's integral equation by ST in Section 4.

2. PRELIMINARIES

In this section, we recall some notions, definitions and lemmas that used through this paper. Let $[a,b] \subset \mathbb{R}^+$ and $\mathcal{C}[a,b]$ be the space of all continuous functions $\varkappa : [a,b] \longrightarrow \mathbb{R}$ with the norm $\|\varkappa\|_\infty = \max\{|\varkappa(\varsigma)| : \varsigma \in [a,b]\}$ for any $\varkappa \in \mathcal{C}[a,b]$. Denote $L^1[a,b]$ the Lebesgue integrable functions with the norm $\|\varkappa\|_{L^1} = \int_a^b |\varkappa(\varsigma)|\, d\varsigma < \infty$.

Definition 2.1. ([15], Sadik Transform) Assume that $\varkappa$ is piecewise continuous on the interval $[0,A]$ for any $A > 0$ and satisfies $|\varkappa(\varsigma)| \leq Le^{a\varsigma}$ when $\varsigma \geq N$, for any real constant a, and some positive constants L and N. Then the ST of $\varkappa(\varsigma)$ is defined by

$$X(\upsilon, q_1, q_2) = \mathcal{S}[\varkappa(\varsigma)] = \frac{1}{\upsilon^{q_2}} \int_0^\infty e^{-\varsigma \upsilon^{q_1}} \varkappa(\varsigma) d\varsigma,$$

where υ is complex variable, q_1 is any non zero real number, and q_2 is any real number.

Proposition 2.2. ([15]) *Let $X(\upsilon, q_1, q_2)$ be a ST of $\varkappa(\varsigma)$, that is, $\mathcal{S}[\varkappa(\varsigma)] = X(\upsilon, q_1, q_2)$. Then*

(1) *If $\varkappa(\varsigma) = 1$, then $\mathcal{S}[1] = \frac{1}{\upsilon^{q_1+q_2}}$.*
(2) *If $\varkappa(\varsigma) = \varsigma^n$, then $\mathcal{S}[\varsigma^n] = \frac{n!}{\upsilon^{nq_1+(q_1+q_2)}}$.*
(3) *If $\varkappa(\varsigma) = e^{a\varsigma}$, then $\mathcal{S}[e^{a\varsigma}] = \frac{\upsilon^{-q_2}}{\upsilon^{q_1}-a}$.*
(4) *If $\varkappa(\varsigma) = \sin(a\varsigma)$, then $\mathcal{S}[\sin(a\varsigma)] = \frac{a\upsilon^{-q_2}}{\upsilon^{2q_1}+a^2}$.*

Lemma 2.3. ([15]) *Let $\varkappa_1$ and $\varkappa_2$ two functions belong to $L^1[a,b]$ the usuall convolution product is given by*

$$(\varkappa_1 * \varkappa_2)(\varsigma) = \int_{-\infty}^{\infty} \varkappa_1(\tau)\varkappa_2(\varsigma - \tau) d\tau, \ \varsigma > 0.$$

Lemma 2.4. ([15]) *Let $X_1(\upsilon, q_1, q_2)$ and $X_2(\upsilon, q_1, q_2)$ are STsof $\varkappa_1(\varsigma)$ and $\varkappa_2(\varsigma)$ respectively, and $(\varkappa_1 * \varkappa_2)(\varsigma)$ is a convolution of $\varkappa_1(\varsigma)$ and $\varkappa_2(\varsigma)$. Then, ST of $(\varkappa_1 * \varkappa_2)(\varsigma)$ is*

$$\mathcal{S}[(\varkappa_1 * \varkappa_2)(\varsigma)] = \upsilon^{q_2} X_1(\upsilon, q_1, q_2).X_2(\upsilon, q_1, q_2),$$

where $*$ denotes convolution.

Lemma 2.5. ([15]) *Let $\mathcal{S}[\varkappa(\varsigma)] = X(\upsilon, q_1, q_2)$. Then*

$$\mathcal{S}[\varsigma^n \varkappa(\varsigma)] = (-1)^n \left(\frac{1}{q_1 \upsilon^{q_1 - 1}} \frac{d}{d\upsilon} + \frac{q_2}{q_1 \upsilon^{q_1}} \right)^n X(\upsilon, q_1, q_2).$$

3. Main results

In this section, we study Abel's integral equation and generalized Abel's integral equation by using recent transform so-called Sadik transform (ST).

Lemma 3.1. *Let $\varkappa(\varsigma) = \varsigma^{-\frac{1}{2}}$. Then ST of $\varkappa(\varsigma)$ given by*

$$\mathcal{S}[\varkappa(\varsigma)] = \mathcal{S}[\varsigma^{-\frac{1}{2}}] = \frac{\sqrt{\pi}}{\upsilon^{\frac{q_1}{2}+q_2}}.$$

Proof. Applying ST definition to $\varkappa(\varsigma)$, we get

$$\begin{aligned} \mathcal{S}[\varkappa(\varsigma)] &= \mathcal{S}[\varsigma^{-\frac{1}{2}}] \\ &= \frac{1}{\upsilon^{q_2}} \int_0^\infty e^{-\varsigma\upsilon^{q_1}} \frac{1}{\varsigma^{\frac{1}{2}}} d\varsigma. \end{aligned} \tag{3.1}$$

Applying the change of variable $\varsigma = \varpi^2, d\varsigma = 2\varpi d\varpi$, to eq. (3.1),

$$\begin{aligned} \mathcal{S}[\varsigma^{-\frac{1}{2}}] &= \frac{1}{\upsilon^{q_2}} \int_0^\infty e^{-\varpi^2\upsilon^{q_1}} \frac{2\varpi}{\varpi} d\varpi \\ &= \frac{2}{\upsilon^{q_2}} \int_0^\infty e^{-\varpi^2\upsilon^{q_1}} d\varpi. \end{aligned} \tag{3.2}$$

Applying the change of variable $\varpi\upsilon^{\frac{q_1}{2}} = \omega$, $d\varpi = \upsilon^{\frac{-q_1}{2}} d\omega$, to eq. (3.2), we get

$$\begin{aligned} \mathcal{S}[\varsigma^{-\frac{1}{2}}] &= \frac{2}{\upsilon^{q_2}} \int_0^\infty e^{\omega^2} \upsilon^{\frac{-q_1}{2}} d\omega \\ &= \frac{2}{\upsilon^{\frac{q_1}{2}+q_2}} \int_0^\infty e^{\omega^2} d\omega \\ &= \frac{\sqrt{\pi}}{\upsilon^{\frac{q_1}{2}+q_2}}. \end{aligned}$$

□

Theorem 3.2. **(The ST Method for Abel integral equation)** *Consider the Abel integral equation*

$$f(\varsigma) = \int_0^\varsigma \frac{\varkappa(\tau)}{\sqrt{\varsigma - \tau}} d\tau. \tag{3.3}$$

Then the solution for it by using ST given by

$$\varkappa(\varsigma) = \frac{1}{\pi} \frac{d}{d\varsigma} \int_0^\varsigma \frac{f(\tau)}{(\varsigma - \tau)^{\frac{q_1}{2}}} d\tau.$$

Proof. Apply ST for Eq.(3.3) and using the convolution theorem, we get

$$\begin{aligned}
\mathcal{S}\left[f(\varsigma)\right] &= \upsilon^{q_2}\mathcal{S}\left[(\varsigma^{\frac{-1}{2}})\right]\mathcal{S}\left[\varkappa(\varsigma)\right],\\
F(\upsilon,q_1,q_2) &= \upsilon^{q_2}X(\upsilon,q_1,q_2)\left[\frac{\Gamma(\frac{1}{2})}{\upsilon^{\frac{1}{2}q_1+q_2}}\right]\\
&= \frac{\Gamma(\frac{1}{2})}{\upsilon^{\frac{1}{2}q_1}}X(\upsilon,q_1,q_2)\\
&= \frac{\sqrt{\pi}}{\upsilon^{\frac{1}{2}q_1}}X(\upsilon,q_1,q_2),
\end{aligned}$$

this gives

$$X(\upsilon,q_1,q_2) = \frac{\upsilon^{\frac{1}{2}q_1}}{\sqrt{\pi}}F(\upsilon,q_1,q_2),$$

where Γ is the gamma function, and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. The last equation can be rewritten as

$$\mathcal{S}\left[\varkappa(\varsigma)\right] = \frac{\upsilon^{q_1}}{\pi}\left[\sqrt{\pi}\upsilon^{\frac{-1}{2}q_1}F(\upsilon,q_1,q_2)\right] = \frac{\upsilon^{q_1}}{\pi}\mathcal{S}\left[y(\varsigma)\right], \tag{3.4}$$

where

$$y(\varsigma) = \int_0^{\varsigma}(\varsigma-\tau)^{\frac{-1}{2}}f(\tau)d\tau.$$

We know that

$$\mathcal{S}\left[y'(\varsigma)\right] = \upsilon^{q_1}\mathcal{S}\left[y(\varsigma)\right] - y(0),$$

using this in Eq.(3.4), we obtain

$$\mathcal{S}\left[\varkappa(\varsigma)\right] = \frac{1}{\pi}\mathcal{S}\left[y'(\varsigma)\right].$$

Applying inverse of ST in above equation, we get

$$\varkappa(\varsigma) = \frac{1}{\pi}\frac{d}{d\varsigma}\int_0^{\varsigma}\frac{f(\tau)}{(\varsigma-\tau)^{\frac{q_1}{2}}}d\tau.$$

□

When $q_1 = 1$ and $q_2 = 0$, we get the solution for it by using Laplace transform as

$$\varkappa(\varsigma) = \frac{1}{\pi}\frac{d}{d\varsigma}\int_0^{\varsigma}\frac{f(\tau)}{(\varsigma-\tau)^{\frac{1}{2}}}d\tau.$$

Theorem 3.3. *Let the generalized Abel's integral equation*

$$f(\varsigma) = \int_0^{\varsigma}\frac{\varkappa(\tau)}{(\varsigma-\tau)^{\gamma}}d\tau,\ 0<\gamma<1 \tag{3.5}$$

and $\mathcal{S}[f(\varsigma)] = F(\upsilon, q_1, q_2)$*. Then the solution for it by using ST given by*

$$\varkappa(\varsigma) = \frac{\sin(\gamma\pi)}{\pi}\frac{d}{d\varsigma}\int_0^{\varsigma}\frac{f(\tau)}{(\varsigma-\tau)^{(1-\gamma)q_1}}d\tau.$$

Proof. Taking STs of both sides of Eq.(3.5), we leads to

$$\begin{aligned}
\mathcal{S}[f(\varsigma)] &= \upsilon^{q_2}\mathcal{S}\left[(\varsigma^{-\gamma})\right]\mathcal{S}[\varkappa(\varsigma)], \\
F(\upsilon, q_1, q_2) &= \upsilon^{q_2}X(\upsilon, q_1, q_2)\left[\frac{\Gamma(1-\gamma)}{\upsilon^{-\gamma q_1+q_1+q_2}}\right] \\
&= X(\upsilon, q_1, q_2)\left[\frac{\Gamma(1-\gamma)}{\upsilon^{(1-\gamma)q_1}}\right],
\end{aligned}$$

this tends to

$$\begin{aligned}
X(\upsilon, q_1, q_2) &= \left[\frac{\upsilon^{(1-\gamma)q_1}}{\Gamma(1-\gamma)}\right]F(\upsilon, q_1, q_2) \\
&= \frac{\upsilon^{q_1}}{\Gamma(\gamma)\Gamma(1-\gamma)}\left[\Gamma(\gamma)\upsilon^{-\gamma q_1}F(\upsilon, q_1, q_2)\right], \\
\mathcal{S}[\varkappa(\varsigma)] &= \frac{\upsilon^{q_1}}{\Gamma(\gamma)\Gamma(1-\gamma)}\mathcal{S}[y(\varsigma)], \qquad (3.6)
\end{aligned}$$

where

$$y(\varsigma) = \int_0^{\varsigma}\frac{f(\tau)}{(\varsigma-\tau)^{(1-\gamma)}}d\tau,$$

since

$$\begin{aligned}
\mathcal{S}[y(\varsigma)] &= \upsilon^{q_2}\mathcal{S}\left[\varsigma^{(\gamma-1)}\right]\mathcal{S}[f(\varsigma)] = \upsilon^{q_2}\left[\frac{\Gamma(\gamma-1+1)}{\upsilon^{(\gamma-1)q_1+q_1+q_2}}\right]F(\upsilon, q_1, q_2) \\
&= \Gamma(\gamma)\upsilon^{-\gamma q_1}F(\upsilon, q_1, q_2).
\end{aligned}$$

Now using the fact

$$\mathcal{S}\left[y'(\varsigma)\right] = \upsilon^{q_1}\mathcal{S}[y(\varsigma)] - y(0), \text{ where } y(0) = 0$$

and

$$\Gamma(\gamma)\Gamma(1-\gamma) = \frac{\pi}{\sin\gamma\pi},$$

in Eq.(3.6), we get

$$\mathcal{S}[\varkappa(\varsigma)] = \frac{\sin\gamma\pi}{\pi}\mathcal{S}\left[y'(\varsigma)\right].$$

Applying $\mathcal{S}^{-1}$in above equation, we obtain

$$\varkappa(\varsigma) = \frac{\sin(\gamma\pi)}{\pi}\frac{d}{d\varsigma}\int_0^{\varsigma}\frac{f(\tau)}{(\varsigma-\tau)^{(1-\gamma)q_1}}d\tau.$$

□

When $q_1 = 1$ and $q_2 = 0$, we get the solution for it by using Laplace transform as

$$\varkappa(\varsigma) = \frac{\sin(\gamma\pi)}{\pi}\frac{d}{d\varsigma}\int_0^{\varsigma}\frac{f(\tau)}{(\varsigma-\tau)^{(1-\gamma)}}d\tau.$$

4. Applications

In this section, some applications are made to explain the solution procedure of the Abel's integral equation with ST.

Example 4.1. Consider the following Abel's integral equation

$$\varsigma = \int_0^{\varsigma}\frac{\varkappa(\tau)}{\sqrt{\varsigma-\tau}}d\tau. \tag{4.1}$$

Taking ST of both sides of eq. (4.1), we get

$$\mathcal{S}(\varsigma) = \mathcal{S}\left[\int_0^{\varsigma}\frac{\varkappa(\tau)}{\sqrt{\varsigma-\tau}}d\tau\right],$$

$$\frac{1}{\upsilon^{2q_1+q_2}} = \mathcal{S}[\varsigma^{\frac{-1}{2}} * \varkappa(\varsigma)]. \tag{4.2}$$

Using convolution theorem of ST on eq. (4.2), and using Proposition 2.2-(2), we deduce

$$\begin{aligned}\frac{1}{\upsilon^{2q_1+q_2}} &= \upsilon^{q_2}\mathcal{S}\left[\varsigma^{\frac{-1}{2}}\right]\mathcal{S}\left[\varkappa(\varsigma)\right] \\ &= \upsilon^{q_2}\left[\frac{\sqrt{\pi}}{\upsilon^{\frac{q_1}{2}+q_2}}\right]\mathcal{S}\left[\varkappa(\varsigma)\right],\end{aligned}$$

which gives,

$$\mathcal{S}\left[\varkappa(\varsigma)\right] = \frac{1}{\sqrt{\pi}}\frac{1}{\upsilon^{\frac{3}{2}q_1+q_2}}. \tag{4.3}$$

Applying inverse ST on both sides of eq. (4.3), we will get the the required solution of eq. (4.1) which given by

$$\begin{aligned}\varkappa(\varsigma) &= \frac{1}{\sqrt{\pi}}\mathcal{S}^{-1}\left[\frac{1}{\upsilon^{\frac{3}{2}q_1+q_2}}\right] \\ &= \frac{2}{\pi}\sqrt{\varsigma}.\end{aligned}$$

Example 4.2. Consider the following Abel's integral equation

$$\varsigma + \varsigma^3 = \int_0^{\varsigma}\frac{\varkappa(\tau)}{\sqrt{\varsigma-\tau}}d\tau. \tag{4.4}$$

Taking ST of both sides of eq. (4.4), and applying Proposition 2.2-(2)

$$\mathcal{S}(\varsigma+\varsigma^3) = \mathcal{S}\left[\int_0^{\varsigma}\frac{\varkappa(\tau)}{\sqrt{\varsigma-\tau}}d\tau\right],$$

$$\frac{1}{\upsilon^{2q_1+q_2}} + \frac{6}{\upsilon^{4q_1+q_2}} = \mathcal{S}[\varsigma^{\frac{-1}{2}} * \varkappa(\varsigma)]. \tag{4.5}$$

Using convolution theorem of ST on eq. (4.5), we get

$$\begin{aligned} \frac{1}{\upsilon^{2q_1+q_2}} + \frac{6}{\upsilon^{4q_1+q_2}} &= \upsilon^{q_2}\mathcal{S}\left[\varsigma^{\frac{-1}{2}}\right]\mathcal{S}\left[\varkappa(\varsigma)\right] \\ &= \upsilon^{q_2}\left[\frac{\sqrt{\pi}}{\upsilon^{\frac{q_1}{2}+q_2}}\right]\mathcal{S}\left[\varkappa(\varsigma)\right], \end{aligned}$$

$$\mathcal{S}\left[\varkappa(\varsigma)\right] = \frac{1}{\sqrt{\pi}}\left(\frac{1}{\upsilon^{\frac{3}{2}q_1+q_2}}\right) + \frac{6}{\sqrt{\pi}}\left(\frac{1}{\upsilon^{\frac{7}{2}q_1+q_2}}\right). \tag{4.6}$$

Applying inverse ST on both sides of eq. (4.6),

$$\begin{aligned} \varkappa(\varsigma) &= \frac{1}{\sqrt{\pi}}\mathcal{S}^{-1}\left[\frac{1}{\upsilon^{\frac{3}{2}q_1+q_2}}\right] + \frac{6}{\sqrt{\pi}}\mathcal{S}^{-1}\left[\frac{1}{\upsilon^{\frac{7}{2}q_1+q_2}}\right] \\ &= \frac{2}{\pi}\left(\sqrt{\varsigma} + \frac{8}{5}\varsigma^{\frac{5}{2}}\right), \end{aligned}$$

which is the required solution of eq. (4.4).

Example 4.3. Consider the following Abel's integral equation

$$\pi\left(\varsigma+1\right) = \int_0^{\varsigma} \frac{\varkappa(\tau)}{\sqrt{\varsigma-\tau}}d\tau. \tag{4.7}$$

Taking ST of both sides of eq.4.7, and applying Proposition 2.2-(2), we deduce

$$\pi\mathcal{S}(\varsigma+1) = \mathcal{S}\left[\int_0^{\varsigma} \frac{\varkappa(\tau)}{\sqrt{\varsigma-\tau}}d\tau\right],$$

$$\begin{aligned} \pi\mathcal{S}(\varsigma) + \pi\mathcal{S}(1) &= \mathcal{S}[\varsigma^{\frac{-1}{2}} * \varkappa(\varsigma)], \\ \pi\left(\frac{1}{\upsilon^{2q_1+q_2}}\right) + \pi\left(\frac{1}{\upsilon^{q_1+q_2}}\right) &= \mathcal{S}[\varsigma^{\frac{-1}{2}} * \varkappa(\varsigma)]. \end{aligned} \tag{4.8}$$

Using convolution theorem of ST on eq. (4.8),

$$\pi\left(\frac{1}{\upsilon^{2q_1+q_2}}\right) + \pi\left(\frac{1}{\upsilon^{q_1+q_2}}\right) = \upsilon^{q_2}\left[\frac{\sqrt{\pi}}{\upsilon^{\frac{q_1}{2}+q_2}}\right]\mathcal{S}\left[\varkappa(\varsigma)\right],$$

which gives

$$\mathcal{S}\left[\varkappa(\varsigma)\right] = \frac{\sqrt{\pi}}{\upsilon^{\frac{3}{2}q_1+q_2}} + \frac{\sqrt{\pi}}{\upsilon^{\frac{1}{2}q_1+q_2}}. \tag{4.9}$$

Applying inverse ST on both sides of eq. (4.9), we will get the the required solution of eq. (4.7) which given by

$$\begin{aligned} \varkappa(\varsigma) &= \sqrt{\pi}\mathcal{S}^{-1}\left[\frac{1}{\upsilon^{\frac{3}{2}q_1+q_2}}\right] + \sqrt{\pi}\mathcal{S}^{-1}\left[\frac{1}{\upsilon^{\frac{1}{2}q_1+q_2}}\right] \\ &= 2\varsigma^{\frac{1}{2}} + \varsigma^{-\frac{1}{2}}. \end{aligned}$$

Example 4.4. Consider the following Abel's integral equation

$$\frac{3\pi}{8}\varsigma^2 = \int_0^{\varsigma} \frac{\varkappa(\tau)}{\sqrt{\varsigma - \tau}} d\tau. \tag{4.10}$$

Taking ST of both sides of eq. (4.10),

$$\frac{3\pi}{8}\mathcal{S}(\varsigma^2)=\mathcal{S}\left[\int_0^{\varsigma} \frac{\varkappa(\tau)}{\sqrt{\varsigma - \tau}} d\tau\right],$$

$$\frac{3\pi}{8}\left(\frac{2}{\upsilon^{3q_1+q_2}}\right) = \mathcal{S}[\varsigma^{\frac{-1}{2}} * \varkappa(\varsigma)] \ .$$

Using convolution theorem of ST on the above equation and using Proposition 2.2-(2), we get

$$\mathcal{S}\left[\varkappa(\varsigma)\right] = \frac{3\sqrt{\pi}}{4}\left(\frac{1}{\upsilon^{\frac{5}{2}q_1+q_2}}\right). \tag{4.11}$$

Applying inverse ST on both sides of eq. (4.11),

$$\begin{aligned} \varkappa(\varsigma) &= \frac{3\sqrt{\pi}}{4}\mathcal{S}^{-1}\left[\frac{1}{\upsilon^{\frac{5}{2}q_1+q_2}}\right] \\ &= \varsigma^{\frac{3}{2}}, \end{aligned}$$

which is the required solution of eq. (4.10).

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