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APPROXIMATE OF HOMOMORPHISM AND DERIVATIONS ON BANACH ALGEBRA VIA DIRECT AND FIXED POINT METHOD

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Abstract. This study examines the approximation of homomorphism and derivations associated with the functional equation:

$$f(2x+y) + f(2x-y) = f(x+y) + f(x-y) + 2f(2x) + 2f(x).$$

In the context of Banach algebras spaces, utilizing the direct and fixed-point methods.

1. Introduction

A functional equation F is considered stable if any solution f is approximately close to an exact solution.

Starting from Ulam's question on stability posed in 1940 [17] during a mathematical colloquium at the University of Wisconsin. Hyers [10] was the first author to provide an answer to Ulam's question in Banach spaces. The result of Hyers was extended by Aoki [6] and also by Rassias [14] by considering the unbounded Cauchy differences. Following this, many mathematicians began conducting studies and achieving results related to the subject due to its importance and applications in various fields such as functional analysis, algebra,

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number theory, as well as practical applications in engineering, physics, and economics.

Using different methods, the most prominent methods of stability are the direct method and the fixed-point method. Mathematicians have proved the stability of functional equations in multiple spaces and obtained generalized and interesting results. Referring to both methods, we have selected some papers that present results related to the subject [1, 2, 4, 5, 8, 11, 12, 16, 18] and [15].

In this paper, we studied the stability of the additive functional equation

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+2f(2x)+2f(x). \tag{1.1}$$

Furthermore, we discussed the homomorphism and derivation of the previous functional equation in a Banach algebra space in the case of the function being odd, and we reached valuable results and good conclusions, whether by the direct method or the fixed point method.

2. Preliminaries

Definition 2.1. ([3]) Let X be a real vector space, which is called Banach algebra if the following axioms are satisfied:

(1) X is a Banach space,
(2) X is an algebra,
(3) there exist $a \in X$ such that $ax = xa = x$ for all $x \in X$ and $\|a\| = 1$,
(4) $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in X$.

Definition 2.2. ([7]) Let X, Y be Banach algebras. A real linear mapping $H : X \to Y$ is said to be a homomorphism if $H(xy) = H(x)H(y)$ for all $x, y \in X$.

Definition 2.3. ([7]) Let X, Y be Banach algebras. A real linear mapping $\delta : X \to Y$ is said to be a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in X$.

Definition 2.4. ([9]) Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if,

(1) $d(p, q) = 0$ if and only if $p = q$,
(2) $d(p, q) = d(q, p)$ for all $p, q \in X$,
(3) $d(p, s) \leq d(p, q) + d(q, s)$ for all $p, q, s \in X$.

Theorem 2.5. ([9]) *Let $(X,\ d)$ be a complete generalized metric space, and let $J : X \to X$ be a strictly contractive mapping with the Lipschitz constant $L < 1$. Then, for each $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = +\infty$$

for all $n \geq 0$ or there exists a natural number n_0 such that

(1) $d\left(J^n x, J^{n+1} x\right) < +\infty$ *for all* $n \geq n_0$,
(2) *The sequence* $\{J^n x\}$ *is convergent to a fixed point* y^* *of* J,
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X, d\left(J^{n_0} x, y\right) < \infty\}$,
(4) $d\left(y, y^*\right) \leq \frac{1}{1-L} d\left(y, Jy\right)$ *for all* $x, y \in Y$.

3. Stability of functional equation (1.1) using the direct method

Theorem 3.1. *Let A,B be two Banach algebra spaces, and let $f : A \to B$ be an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$. Suppose the functions $\theta : A^2 \to [0, \infty)$ and $\zeta : A^2 \to [0, \infty)$ such that*

$$\sum_{i=0}^{\infty} \frac{\theta\left(3^i x, 3^i y\right)}{3^i} < \infty \ for\ all\ x, y \in A, \tag{3.1}$$

$$\lim_{n \to \infty} \frac{\theta(3^n x, 3^n y)}{3^n} = 0\ for\ all\ x, y \in A, \tag{3.2}$$

$$\|D_f(x, y)\|_B \leq \theta(x, y)\ \ for\ all\ x, y\ \in A, \tag{3.3}$$

where

$$D_f(x, y) = f(2x + y) + f(2x - y) - f(x + y) - f(x - y) - 2f(2x) - 2f(x),$$

$$\|f(xy) - f(x) f(y)\|_B \leq \zeta(x, y), for\ all\ x, y \in A, \tag{3.4}$$

$$\lim_{m \to \infty} \frac{1}{3^{2m}} \zeta(3^m x, 3^m y) = 0. \tag{3.5}$$

Then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{1}{3} \sum_{i=0}^{\infty} \frac{\theta\left(3^i x, 3^i x\right)}{3^i}. \tag{3.6}$$

Proof. If $x = y$ in (3.3)

$$D_f(x, x) = f(3x) - 3f(x),$$

we get

$$\|D_f(x, x)\| = \|f(3x) - 3f(x)\| \leq \theta(x, x)$$

and

$$\left\|\frac{f(3x)}{3} - f(x)\right\| \leq \frac{1}{3}\ \theta(x, x). \tag{3.7}$$

Since

$$\frac{f(3^n x)}{3^n} - f(x) = \sum_{i=0}^{n-1} \left(\frac{f\left(3^{i+1} x\right)}{3^{i+1}} - \frac{f\left(3^i x\right)}{3^i}\right),$$

$$\begin{aligned}\left\|\frac{f\left(3^{n}x\right)}{3^{n}}-f\left(x\right)\right\| &\leq \left\|\frac{f\left(3x\right)}{3}-f\left(x\right)\right\|+\left\|\frac{f\left(3^{2}x\right)}{3^{2}}-\frac{f\left(x\right)}{3}\right\| \\ &\quad+\ldots+\left\|\frac{f\left(3^{n}x\right)}{3^{n}}-\frac{f\left(3^{n-1}x\right)}{3^{n-1}}\right\| \\ &\leq \frac{1}{3}\ \theta\left(x,x\right)+\frac{1}{3^{2}}\theta\left(3x,3x\right)+\ldots+\frac{1}{3^{n}}\ \theta\left(3^{n-1}x,3^{n-1}x\right).\end{aligned}$$

Hence, we have

$$\left\|\frac{f\left(3^{n}x\right)}{3^{n}}-f\left(x\right)\right\| \leq \frac{1}{3}\sum_{i=0}^{n-1}\frac{\theta\left(3^{i}x,3^{i}x\right)}{3^{i}}. \tag{3.8}$$

Now, there exist $k\in\mathbb{N}$ such that for all $m\geq n\geq k$,

$$\begin{aligned}\left\|\frac{f\left(3^{n+m}x\right)}{3^{n+m}}-\frac{f\left(3^{m}x\right)}{3^{m}}\right\| &= \frac{1}{3^{m}}\left\|\frac{f\left(3^{n+m}x\right)}{3^{n}}-f\left(3^{m}x\right)\right\| \\ &\leq \frac{1}{3}\sum_{i=0}^{n-1}\frac{\theta\left(3^{i+m}x,3^{i+m}x\right)}{3^{i+m}}.\end{aligned} \tag{3.9}$$

Letting $m\to\infty$, we have

$$\left\|\frac{f\left(3^{n+m}x\right)}{3^{n+m}}-\frac{f\left(3^{m}x\right)}{3^{m}}\right\|\to 0.$$

Hence, $\left\{\frac{f(3^{m}x)}{3^{m}}\right\}$ is a Cauchy sequence in B. Since B is a complete space, the sequence $\left\{\frac{f(3^{m}x)}{3^{m}}\right\}$ is convergent. Set $H\left(x\right)=\lim_{m\to\infty}\frac{f(3^{m}x)}{3^{m}}$. By [11], f is an additive mapping and also is homogenous. Then f is linear and by (3.4) is approximately homomorphism.

$$\begin{aligned}H\left(x+y\right) &= \lim_{m\to\infty}\frac{f\left(3^{m}(x+y)\right)}{3^{m}} \\ &= \lim_{m\to\infty}\left(\frac{f\left(3^{m}x\right)}{3^{m}}+\frac{f\left(3^{m}y\right)}{3^{m}}\right) \\ &= \lim_{m\to\infty}\frac{f\left(3^{m}x\right)}{3^{m}}+\lim_{m\to\infty}\frac{f\left(3^{m}y\right)}{3^{m}} \\ &= H\left(x\right)+H\left(y\right),\end{aligned}$$

$$H\left(\lambda x\right)=\lim_{m\to\infty}\frac{f\left(3^{m}\lambda x\right)}{3^{m}}=\lambda\lim_{m\to\infty}\frac{f\left(3^{m}x\right)}{3^{m}}=\lambda H(x)$$

and

$$
\begin{aligned}
&\left\|H\left(xy\right)-H\left(x\right)H\left(y\right)\right\| \\
&\quad=\left\|\lim_{m\to\infty}\left(\frac{f\left(3^{2m}\left(xy\right)\right)}{3^{2m}}\right)-\lim_{m\to\infty}\left(\frac{f\left(3^{m}x\right)}{3^{m}}\right)\lim_{m\to\infty}\left(\frac{f\left(3^{m}y\right)}{3^{m}}\right)\right\| \\
&\quad=\left\|\lim_{m\to\infty}\left(\frac{f\left(3^{2m}\left(xy\right)\right)}{3^{2m}}-\frac{f\left(3^{m}x\right)}{3^{m}}\frac{f\left(3^{m}y\right)}{3^{m}}\right)\right\| \\
&\quad=\left\|\lim_{m\to\infty}\frac{1}{3^{2m}}\left(f\left(3^{2m}\left(xy\right)\right)-f\left(3^{m}x\right)f\left(3^{m}y\right)\right)\right\| \\
&\quad=\left\|\lim_{m\to\infty}\frac{1}{3^{2m}}\left(f\left(3^{2m}\left(xy\right)\right)-f\left(3^{m}x\right)f\left(3^{m}y\right)\right)\right\| \\
&\quad\leq\lim_{m\to\infty}\frac{1}{3^{2m}}\zeta\left(3^{m}x,3^{m}x\right) \\
&\quad=0.
\end{aligned}
\tag{3.10}
$$

Therefore, $\|H\left(xy\right)-H\left(x\right)H\left(y\right)\|=0$, it implies that $H\left(xy\right)=H\left(x\right)H\left(y\right)$. Hence H is homomorphism. In (3.8), taking $n\to\infty$

$$\|H\left(x\right)-f(x)\|\leq\frac{3}{2}\sum_{i=0}^{\infty}\theta(3^{i}x,3^{i}x).$$

Since

$$\frac{3}{2}=\sum_{i=0}^{\infty}\frac{1}{3^{i}},$$

$$
\begin{aligned}
&\|H\left(2x+y\right)+H\left(2x-y\right)-H\left(x+y\right)-H\left(x-y\right)-2H\left(2x\right)-2H\left(x\right)\| \\
&\quad=\lim_{n\to\infty}\left\|\frac{f(3^{n}(2x+y))}{3^{n}}+\frac{f(3^{n}(2x-y))}{3^{n}}-\frac{f(3^{n}(x+y))}{3^{n}}\right. \\
&\qquad\left.-\frac{f(3^{n}(x-y))}{3^{n}}-\frac{2f(3^{n}2x)}{3^{n}}-\frac{2f(3^{n}x)}{3^{n}}\right\| \\
&\quad\leq\lim_{n\to\infty}\frac{1}{3^{n}}\theta\left(3^{n}x,3^{n}x\right) \\
&\quad=0.
\end{aligned}
\tag{3.11}
$$

We get H satisfies the functional equation (1.1).

Now, we want to prove that H is a unique homomorphism. Assume that there exists another one denoted by $\acute{H}:A\to B$ such that $\acute{H}$ satisfies the functional equations (1.1) and (3.8).

Let
$$H\acute{}(x) = \lim_{m\to\infty} \frac{g(3^m x)}{3^m}.$$
Then, we have
$$\begin{aligned}\left\|H(x) - H\acute{}(x)\right\| &= \left\|\frac{H(3^m x)}{3^m} - \frac{H\acute{}(3^m x)}{3^m}\right\| \\ &= \frac{1}{3^m}\left\|H(3^m x) - H\acute{}(3^m x)\right\| \\ &\leq \frac{1}{3^m}\left\|\left(H(3^m x) - f(3^m x)\right\| + \left\|f(3^m x) - H\acute{}(3^m x)\right\|\right).\end{aligned}$$
Letting $m \to \infty$,
$$\|H(x) - H\acute{}(x)\| \leq \frac{1}{3}\left(\sum_{i=0}^{\infty} \frac{\theta\left(3^{i+m}x, 3^{i+m}x\right)}{3^{i+m}} + \sum_{i=0}^{\infty} \frac{\theta\left(3^{i+m}x, 3^{i+m}x\right)}{3^{i+m}}\right) \to 0. \tag{3.12}$$
Hence, $\|H(x) - H\acute{}(x)\| = 0$, this implies $H(x) = H\acute{}(x)$. □

Corollary 3.2. *Let λ, β be nonnegative real numbers. Let odd function $f : A \to B$ satisfies the inequality $\|D_f(x,x)\|_B \leq \lambda$ for all $x, y \in A$ and $\|f(xy) - f(x) f(y)\|_B \leq \beta$ for all $x, y \in A$. Then there exists a unique homomorphism*
$$\|H(x) - f(x)\| \leq \frac{\lambda}{2} \quad \textit{for all } x \in A.$$

Proof. In the Theorem 3.1, taking
$$\theta(x,y) = \lambda,$$
$$\zeta(x,y) = \beta \text{ for all } x, y \in A,$$
then the result is immediate. □

Corollary 3.3. *Let p, δ be nonnegative real numbers. Let an odd function $f : A \to B$ satisfies the inequality $\|D_f(x,y)\| \leq \delta(\|x\|^p + \|x\|^p)$ for all $x, y \in A$, and $\|f(xy) - f(x) f(y)\| \leq x^p + y^p$ for all $x, y \in A$. Then there exists a unique homomorphism $H : A \to B$ such that*
$$\|H(x) - f(x)\| \leq \frac{2\delta}{3 - 3^p} \;\; \textit{for all } x \in A.$$

Proof. In the Theorem 3.1, taking
$$\theta(x,y) = \delta(\|x\|^p + \|y\|^p) \text{ for } p < 1,$$
$$\zeta(x,y) = \|x\|^p + \|y\|^p,\; p < 2 \text{ for all } x, y \in A,$$
then the result is immediate. □

Theorem 3.4. *Let A, B be two Banach algebra spaces, and let $f : A \to B$ be an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$. Suppose the functions $\theta : A^2 \to [0, \infty)$ and $\zeta : A^2 \to [0, \infty)$ such that*

$$\sum_{i=0}^{\infty} \frac{\theta\left(3^i x, 3^i y\right)}{3^i} < \infty \ for\ all\ x, y \in A, \tag{3.13}$$

$$\|D_f(x, y)\|_B \leq \ \theta(x, y) \ \ for\ all\ x, y \in A, \tag{3.14}$$

$$\lim_{n\to\infty} \frac{\theta(3^n x, 3^n y)}{3^n} = 0, \tag{3.15}$$

$$\|f(xy) - xf(y) - yf(x)\|_B \leq \zeta(x, y) \ \ for\ all\ x, y \in A, \tag{3.16}$$

$$\lim_{m\to\infty} \frac{1}{3^{2m}} \zeta(3^m x, 3^m y) = 0. \tag{3.17}$$

Then there exists a unique derivation $\delta : A \to B$ such that

$$\|f(x) - \delta(x)\|_B \leq \frac{1}{3} \sum_{i=0}^{\infty} \frac{\theta\left(3^i x, 3^i x\right)}{3^i}. \tag{3.18}$$

Proof. We will suffice with the proof derivation and the rest is easy according to the Theorem 3.1,

$$\begin{aligned}
&\|\delta(xy) - x\delta(y) - y\delta(x)\| \\
&= \left\| \lim_{m\to\infty} \left(\frac{f\left(3^{2m}(xy)\right)}{3^{2m}} - \frac{3^m x f(3^m y)}{3^{2m}} - \frac{3^m y f(3^m x)}{3^{2m}} \right) \right\| \\
&= \lim_{m\to\infty} \frac{1}{3^{2m}} \left\| f\left(3^{2m}(xy)\right) - 3^m x f(3^m y) - 3^m y f(3^m x) \right\| \\
&\leq \lim_{m\to\infty} \frac{1}{3^{2m}} \zeta(3^m x, 3^m y) \\
&= 0.
\end{aligned} \tag{3.19}$$

Hence, $\|\delta(xy) - x\delta(y) - y\delta(x)\| = 0$, it implies $\delta(xy) = x\delta(y) + y\delta(x)$. So, we have δ is derivation. □

Theorem 3.5. *Let A, B be two Banach algebra spaces, and let $f : A \to B$ be an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in R$. Suppose the functions $\theta : A^2 \to [0, \infty)$ and $\zeta : A^2 \to [0, \infty)$ such that*

$$\sum_{i=1}^{\infty} 3^i \theta\left(\frac{x}{3^i}, \frac{y}{3^i}\right) < \infty \ for\ all\ x, y \in A, \tag{3.20}$$

$$\lim_{n\to\infty} 3^n\theta\left(\frac{x}{3^n},\frac{y}{3^n}\right)=0 \ for\ all\ x,y\in A, \tag{3.21}$$

$$\|D_f\ (x,y)\|_B \leq\ \theta\left(x,y\right)\ for\ all\ x,y\ \in A, \tag{3.22}$$

$$\|f(xy)-f(x)f(y)\|_B \leq \zeta\left(x,y\right)\ for\ all\ x,y\ \in A, \tag{3.23}$$

$$\lim_{m\to\infty} 3^{2m}\zeta\left(\frac{x}{3^m},\frac{y}{3^m}\right)=0.$$

Then there exists a unique homomorphism $H : A \to B$ such that

$$f\left(x\right)-H(x)\leq\sum_{i=1}^{\infty}3^i\theta\left(\frac{x}{3^i},\frac{x}{3^i}\right). \tag{3.24}$$

Proof. This theorem can be readily demonstrated utilizing the same way to the Theorem 3.1. □

Theorem 3.6. *Let A,B be two Banach algebra spaces, and let $f : A \to B$ be an odd mapping, $f\left(0\right)=0$, $f\left(\lambda x\right)=\lambda f\left(x\right)$ for all $x\in A$, $\lambda\in R$. Suppose the functions $\theta : A^2\to[0,\infty)$ and $\zeta : A^2\to[0,\infty)$ such that*

$$\sum_{i=1}^{\infty}3^i\theta\left(\frac{x}{3^i},\frac{y}{3^i}\right)<\infty\ for\ all\ x,y\in A, \tag{3.25}$$

$$\lim_{n\to\infty} 3^n\theta\left(\frac{x}{3^n},\frac{y}{3^n}\right)=0\ for\ all\ x,y\in A, \tag{3.26}$$

$$\|D_f\ (x,y)\|_B \leq\ \theta\left(x,y\right)\ for\ all\ x,y\in A, \tag{3.27}$$

$$\|f\left(xy\right)-xf\left(y\right)-yf(x)\|_B \leq \zeta\left(x,y\right)\ for\ all\ x,y\ \in A, \tag{3.28}$$

$$\lim_{m\to\infty} 3^{2m}\zeta\left(\frac{x}{3^m},\frac{y}{3^m}\right)=0.$$

Then there exists a unique derivation $\delta : A\to B$ such that

$$\|f\left(x\right)-\delta(x)\|_B\leq\sum_{i=1}^{\infty}3^i\theta\left(\frac{x}{3^i},\frac{x}{3^i}\right). \tag{3.29}$$

Proof. This theorem can be readily demonstrated utilizing the same way to the Theorem 3.4. □

4. Stability of functional equation (1.1) using fixed point method

Theorem 4.1. *Let A, B be two Banach algebra spaces, and let $f : A \to B$ be an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$. Suppose functions $\theta : A^2 \to [0, \infty)$ and $\zeta : A^2 \to [0, \infty)$ such that for all $x \in A, m \in \mathbb{R}$ there exists $L < 1$ with $\theta(mx, mx) \leq mL\theta(x, x)$,*

$$\|D_f(x, y)\|_B \leq \theta(x, y) \text{ for all } x, y \in A, \tag{4.1}$$

where

$$\begin{aligned} D_f(x, y) = & f(2x + y) + f(2x - y) - f(x + y) \\ & - f(x - y) - 2f(2x) - 2f(x), \end{aligned}$$

$$\|f(xy) - f(x) f(y)\|_B \leq \zeta(x, y) \text{ for all } x, y \in A, \tag{4.2}$$

$$\lim_{n\to\infty} \frac{1}{3^{2n}} \zeta(3^n x, 3^n y) = 0. \tag{4.3}$$

Then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{1}{3 - 3L} \theta(x, x). \tag{4.4}$$

Proof. If $x = y$ in (4.1), then

$$D_f(x, x) = f(3x) - 3f(x),$$

we get

$$\|D_f(x, x)\| = \|f(3x) - 3f(x)\| \leq \theta(x, x)$$

and

$$\left\| \frac{f(3x)}{3} - f(x) \right\| \leq \frac{1}{3} \theta(x, x). \tag{4.5}$$

Let the set $X = \{g : A \to B\}$ and introduce the generalized metric on X,

$$d(p, q) = \inf\left\{c \in \mathbb{R}^+ : \|p(x) - q(x)\|_B \leq c\theta(x, x),\ \forall x \in A\right\}.$$

Then (X, d) is complete by [8]. Let $J : X \to X$ be a linear mapping such that

$$J(p(x)) = \frac{p(3x)}{3} \text{ for all } x \in A$$

and

$$\|J(f(x)) - f(x)\|_B \leq \frac{1}{3} \theta(x, x) \text{ for all } x \in A.$$

Then

$$d(Jf, f) \leq \frac{1}{3}.$$

Let $f, H \in X$. Then,

$$\|f(x) - H(x)\|_B \leq \theta(x, x),$$

$$d\left(f,H\right)=1,$$

$$\|J(f\left(x\right))-J(H\left(x\right))\|_{B}=\left\|\frac{f\left(3x\right)}{3}-\frac{H\left(3x\right)}{3}\right\|.$$

Therefore,

$$\begin{aligned}\frac{1}{3}\|f\left(3x\right)-H\left(3x\right)\| &\leq \frac{1}{3}\theta\left(3x,3x\right)\\ &\leq \frac{1}{3}3L\theta\left(x,x\right)\\ &= L\theta\left(x,x\right).\end{aligned}$$

Then $d\left(Jf,JH\right)\leq L$ it implies

$$d\left(Jf,JH\right)\leq Ld\left(f,H\right). \tag{4.6}$$

(1) H is fixed point of J, that is,

$$\begin{aligned}J\left(H\left(x\right)\right)&=H\left(x\right),\\ \frac{H\left(3x\right)}{3}&=\ H\left(x\right),\\ H\left(3x\right)&=3H\left(x\right).\end{aligned} \tag{4.7}$$

Hence, for all $x\in A$, the H is a unique fixed point of J in the set

$$Y=\left\{g\in X:d\left(f,g\right)<\infty\right\}.$$

Therefore, there exists $c\in(0,\infty)$ such that $d\left(f,H\right)<c$. Hence,

$$f\left(x\right)-H\left(x\right)_{B}\leq c\theta\left(x,x\right) \text{ for all } x\in A. \tag{4.8}$$

(2) $d\left(J^{n}f,H\right)\to 0$ as $n\to\infty$ this implies the equality

$$\lim_{n\to\infty}\frac{f(3^{n}x)}{3^{n}}=H\left(x\right) \text{ for all } x\in A. \tag{4.9}$$

(3)

$$\begin{aligned}d\left(f,H\right)&\leq\frac{1}{1-L}d\left(f,Jf\right),\\ d\left(f,H\right)&\leq\frac{1}{1-L}\frac{1}{3},\end{aligned}$$

that is,

$$d(f,H)\leq\frac{1}{3-3L}.$$

Hence,

$$\|f\left(x\right)-H\left(x\right)\|_{B}\leq\frac{1}{3-3L}\theta\left(x,x\right). \tag{4.10}$$

Using the same method as in Theorem 3.1, H satisfies the functional equation (1.1). □

Corollary 4.2. *Assume that $r < 1$, $m > 1$ and let $f : A \to B$ be an odd mapping, $f(0) = 0, f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$ such that*

$$\|D_f(x, y)\|_B \leq \|x\|^r + \|y\|^r, r < 1 \text{ for all } x, y \in A$$

and

$$\|f(xy) - f(x) f(y)\|_B \leq \|x\|^r . \|y\|^r, \ r < 1 \text{ for all } x, y \in A.$$

Then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{2}{3 - 3L} \|x\|^r.$$

Proof. In the Theorem 4.1, if we choose

$$\theta(x, y) = \|x\|^r + \|y\|^r, \ r < 1 \text{ for all } x, y \in A,$$

$$\zeta(x, y) = \|x\|^r \cdot \|y\|^r, \ r < 1 \text{ for all } x, y \in A,$$

then the result will be achieved when $L \geq m^{r-1}, \quad m > 1, \quad r < 1.$ □

Corollary 4.3. *Let $m > 1$ and $f : A \to B$ be an odd mapping, and $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$ such that*

$$\|D_f(x, y)\|_B \leq \frac{1}{\|x\| + \|y\|} \text{ for all } x, y \in A,$$

and

$$\|f(xy) - f(x) f(y)\|_B \leq \frac{1}{\|x\| + \|y\|} \text{ for all } x, y \in A.$$

Then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{1}{3 - 3L} \frac{1}{2\|x\|}.$$

Proof. In the Theorem 4.1, if we choose

$$\theta(x, y) = \frac{1}{\|x\| + \|y\|} \text{ for all } x, y \in A,$$

$$\zeta(x, y) = \frac{1}{\|x\| . \|y\|} \text{ for all } x, y \in A,$$

then the result will be achieved when $L \geq \frac{1}{m^2}, m > 1.$ □

Corollary 4.4. *Let $r > 0$, $m > 1$ and let $f : A \to B$ be an odd mapping, $f(0) = 0, f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$ such that*

$$\|D_f(x, y)\|_B \leq \frac{\|x\|}{\|y\|^r}, \ r > 0 \ \text{ for all } x, y \in A,$$

$$\|f(xy) - f(x) f(y)\|_B \leq \frac{\|x\|}{\|y\|^r}, \ r > 0 \ \text{ for all } x, y \in A.$$

Then there exists a unique homomorphism $H : A \to B$ *such that*

$$\|f(x) - H(x)\|_B \leq \frac{1}{3-3L}\frac{\|x\|}{\|y\|^r}.$$

Proof. In the Theorem 4.1, if we choose

$$\theta(x,y) = \frac{\|x\|}{\|y\|^r}, r > 0 \text{ for all } x, y \in A,$$

$$\zeta(x,y) = \frac{\|x\|}{\|y\|^r}, r > 0 \text{ for all } x, y \in A,$$

then the result will be achieved when $L \geq \frac{1}{m^r}, m > 1, r > 0$. □

Corollary 4.5. *Assume that* $r < 1$, $0 < m < 1$ *and let* $f : A \to B$ *be an odd mapping,* $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ *for all* $x \in A$, $\lambda \in \mathbb{R}$ *such that*

$$\|D_f(x,y)\|_B \leq \|x\|.\|y\| \textit{ for all } x, y \in A.$$

$$\|f(xy) - f(x)f(y)\|_B \leq \|x\|^r.\|y\|^r, \ r < 1 \textit{ for all } x, y \in A.$$

Then there exists a unique homomorphism $H : A \to B$ *such that*

$$\|f(x) - H(x)\|_B \leq \frac{1}{3-3L}\|x\|^2.$$

Proof. In the Theorem 4.1, if we choose

$$\theta(x,y) = \|x\|.\|y\| \ \text{ for all } x, y \in A,$$

$$\zeta(x,y) = \|x\|^r.\|y\|^r, r < 1 \ \text{ for all } x, y \in A,$$

then the result will be achieved when $L \geq m^2$, $0 < m < 1$. □

Corollary 4.6. *Let* $m > 1$, *and* $f : A \to B$ *be an odd mapping,* $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ *for all* $x \in A$, $\lambda \in \mathbb{R}$ *such that*

$$\|D_f(x,y)\|_B \leq \frac{\|x\|.\|y\|}{\|x\|^r + \|y\|^r}, \ r > 1 \ \textit{ for all } x, y \in A,$$

$$\|f(xy) - f(x)f(y)\|_B \leq \frac{\|x\| + \|y\|}{\|x\|^r + \|y\|^r}, \ r > 0 \ \textit{ for all } x, y \in A.$$

Then there exists a unique homomorphism $H : A \to B$ *such that*

$$\|f(x) - H(x)\|_B \leq \frac{1}{3-3L}\frac{\|x\|^2}{2\|x\|^r}.$$

Proof. In the Theorem 4.1, if we choose

$$\theta(x,y) = \frac{\|x\|.\|y\|}{\|x\|^r + \|y\|^r}, \ r > 1 \ \text{ for all } x, y \in A,$$

$$\zeta(x,y) = \frac{\|x\| + \|y\|}{\|x\|^r + \|y\|^r}, \ r > 0 \ \text{ for all } x, y \in A,$$

then the result will be achieved when $L \geq \frac{1}{m^{r-1}}, m > 1, r > 1$, because

$$\frac{\|mx\|.\|mx\|}{\|mx\|^r + \|mx\|^r} \leq mL\frac{\|x\|^2}{2\|x\|^r},$$

$$\frac{1}{m^{r-1}} \leq L$$

and

$$\lim_{n\to\infty} \frac{1}{3^{2n}}\zeta(3^n x, 3^n y) = \lim_{n\to\infty} \frac{1}{3^{2n}} \frac{\|3^n x\| + \|3^n y\|}{\|3^n x\|^r + \|3^n y\|^r} = 0.$$

□

Theorem 4.7. *Let A, B be two Banach algebra spaces, and $f : A \to B$ be an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A, \lambda \in \mathbb{R}$. Suppose functions $\theta : A^2 \to [0,\infty)$ and $\zeta : A^2 \to [0,\infty)$ such that for all $x \in A, m \in \mathbb{R}$, there exists $L < 1$, where $\theta(mx, mx) \leq mL\theta(x,x)$,*

$$\|D_f(x,y)\|_B \leq \theta(x,y) \ \textit{ for all } x, y \in A, \tag{4.11}$$

$$\|f(xy) - xf(y) - yf(x)\|_B \leq \zeta(x,y) \ \textit{ for all } x, y \in A \tag{4.12}$$

and

$$\lim_{n\to\infty} \frac{1}{3^{2n}}\zeta(3^n x, 3^n y) = 0. \tag{4.13}$$

Then there exists a unique derivation $\delta : A \to B$ such that

$$\|f(x) - \delta(x)\|_B \leq \frac{1}{3 - 3L}\theta(x,x). \tag{4.14}$$

Proof. Same method as the previous theorem, only we prove that δ is derivation.

$$\begin{aligned}
&\|\delta(xy) - x\delta(y) - y\delta(x)\| \\
&= \left\| \lim_{n\to\infty} \left(\frac{f\left(3^{2n}(xy)\right)}{3^{2n}} - \frac{3^n x f(3^n y)}{3^{2n}} - \frac{3y^n f(3^n x)}{3^{2n}} \right) \right\| \\
&= \lim_{n\to\infty} \frac{1}{3^{2n}} \left\| \left(f\left(3^{2n}(xy)\right) - 3^n x f(3^n y) - 3y^n f(3^n x) \right) \right\| \\
&\leq \lim_{n\to\infty} \frac{1}{3^{2n}} \zeta(3^n x, 3^n x) = 0.
\end{aligned} \tag{4.15}$$

Hence, we have

$$\delta(xy) = x\delta(y) + y\delta(x).$$

Then δ is derivation. □

Corollary 4.8. *Assume that $r < 1$, $m > 1$ and $f : A \to B$ be an odd mapping, and $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$ such that*

$$\|D_f(x, y)\|_B \leq \|x\|^r + \|y\|^r, \ r < 1 \quad for\ all\ x, y \in A,$$

$$\|f(xy) - xf(y) - yf(x)\|_B \leq \|x\|^r . \|y\|^r, \ r < 1 \quad for\ all\ x, y \in A.$$

Then there exists a unique derivation $\delta : A \to B$ such that

$$\|f(x) - \delta(x)\|_B \leq \frac{2}{3 - 3L}\|x\|^r.$$

Proof. In the Theorem 4.7, if we choose

$$\theta(x, y) = \|x\|^r + \|y\|^r, \ r < 1 \text{ for all } x, y \in A,$$

$$\zeta(x, y) = \|x\|^r . \|y\|^r, \ r < 1 \text{ for all } x, y \in A,$$

then the result will be achieved when $L \geq m^{r-1}, m > 1, r < 1$. □

Corollary 4.9. *Let $m > 1$, and $f : A \to B$ be an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A, \lambda \in \mathbb{R}$ such that*

$$\|D_f(x, y)\|_B \leq \frac{1}{\|x\| + \|y\|} \quad for\ all\ x, y \in A,$$

$$\|f(xy) - xf(y) - yf(x)\|_B \leq \frac{1}{\|x\| + \|y\|} \quad for\ all\ x, y \in A.$$

Then there exists a unique derivation $\delta : A \to B$ such that

$$\|f(x) - \delta(x)\|_B \leq \frac{1}{3 - 3L}\frac{1}{2x}.$$

Proof. In the Theorem 4.7, if we choose

$$\theta(x, y) = \frac{1}{\|x\| + \|y\|} \text{ for all } x, y \in A,$$

$$\zeta(x, y) = \frac{1}{\|x\| . \|y\|} \text{ for all } x, y \in A,$$

then the result will be achieved when $L \geq \frac{1}{m^2}, m > 1$. □

Corollary 4.10. *Let $m > 1$ and $f : A \to B$ be an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$ such that*

$$\|D_f(x,y)\|_B \leq \frac{\|x\|}{\|y\|^r}, \ r > 0 \ \ for\ all\ x, y \in A$$

and

$$\|f(xy) - xf(y) - yf(x)\|_B \leq \frac{\|x\|}{\|y\|^r}, \ r < 1 \ \ for\ all\ x, y \in A.$$

Then there exists a unique derivation $\delta : A \to B$ such that

$$\|f(x) - \delta(x)\|_B \leq \frac{1}{3-3L}\frac{\|x\|}{\|y\|^r}.$$

Proof. In the Theorem 4.7, if we choose

$$\theta(x,y) = \frac{\|x\|}{\|y\|^r}, \ r > 0 \ \text{ for all } x, y \in A,$$

$$\zeta(x,y) = \frac{\|x\|}{\|y\|^r}, \ r < 1 \ \text{ for all } x, y \in A,$$

then the result will be achieved when $L \geq \frac{1}{m^r}, m > 1, r > 0$. □

Corollary 4.11. *Assume that $r < 1$, $0 < m < 1$, and $f : A \to B$ be an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$ such that*

$$\|D_f(x,y)\|_B \leq \|x\|\|y\| \ for\ all\ x, y \in A,$$

$$\|f(xy) - xf(y) - yf(x)\|_B \leq \|x\|^r\|y\|^r, \ r < 1 \ for\ all\ x, y \in A.$$

Then there exists a unique derivation $\delta : A \to B$ such that

$$\|f(x) - \delta(x)\|_B \leq \frac{1}{3-3L}\|x\|^2.$$

Proof. In the Theorem 4.7, if we choose

$$\theta(x,y) = \|x\|\|y\| \ \text{ for all } x, y \in A,$$

$$\zeta(x,y) = \|x\|^r\|y\|^r, \ r < 1 \ \text{ for all } x, y \in A,$$

then the result will be achieved when $L \geq m^2$, $0 < m < 1$. □

Corollary 4.12. *Let $m > 1$ and $f : A \to B$ be an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$ such that*

$$\|D_f(x,y)\|_B \leq \frac{\|x\|.\|y\|}{\|x\|^r + \|y\|^r}, \ r > 1 \ \ for\ all\ x, y \in A$$

and

$$\|f(xy) - xf(y) - yf(x)\|_B \leq \frac{\|x\| + \|y\|}{\|x\|^r + \|y\|^r}, \ r > 0 \ \ for\ all\ x, y \in A.$$

Then there exists a unique derivation $\delta : A \to B$ such that

$$\|f(x) - \delta(x)\|_B \leq \frac{1}{3-3L} \frac{\|x\|^2}{2\|x\|^r}.$$

Proof. In the Theorem 4.7, if we choose

$$\theta(x,y) = \frac{\|x\|.\|y\|}{\|x\|^r + \|y\|^r}, \; r > 1 \text{ for all } x, y \in A,$$

$$\zeta(x,y) = \frac{\|x\| + \|y\|}{\|x\|^r + \|y\|^r}, \; r > 0 \text{ for all } x, y \in A,$$

then the result will be achieved when $L \geq \frac{1}{m^{r-1}}, m > 1, r > 1$. □

Theorem 4.13. *Let A, B be two Banach algebra spaces, and $f : A \to B$ be an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in R$, suppose functions $\theta : A^2 \to [0, \infty)$ and $\zeta : A^2 \to [0, \infty)$ such that for all $x \in A, m \in \mathbb{R}$, there exists $L < 1$, where $\theta\left(\frac{x}{m}, \frac{x}{m}\right) \leq mL\theta(x,x)$,*

$$\|D_f(x,y)\|_B \leq \theta(x,y) \quad for\ all\ x, y \in A, \tag{4.16}$$

$$\|f(xy) - f(x)f(y)\|_B \leq \zeta(x,y) \quad for\ all\ x, y \in A, \tag{4.17}$$

$$\lim_{n\to\infty} 3^{2n}\zeta\left(\frac{x}{3^n}, \frac{y}{3^n}\right) = 0. \tag{4.18}$$

Then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{1}{1-L}\theta(x,x). \tag{4.19}$$

Proof. This theorem can be demonstrated using the same way as in the Theorem 4.1. □

Corollary 4.14. *Let $m > 1$ and $f : A \to B$ be an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$ such that*

$$\|D_f(x,y)\|_B \leq \; \|x\|^r + \|y\|^r, \; r > 0 \;\; for\ all\ x, y \in A,$$

$$\|f(xy) - f(x)f(y)\|_B \leq \|x\|^r.\|y\|^r, \; r > 1 \;\; for\ all\ x, y \in A.$$

Then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{2}{1-L}\|x\|^r.$$

Proof. In the Theorem 4.13, if we choose

$$\theta(x,y) = \|x\|^r + \|y\|^r, \; r > 0 \text{ for all } x, y \in A,$$

$$\zeta(x,y) = \|x\|^r.\|y\|^r, \; r > 1 \text{ for all } x, y \in A,$$

then the result will be achieved when $L \geq \frac{1}{m^{r+1}}, m > 1, r > 0$. □

Corollary 4.15. *Assume that $m > 1$ and $f : A \to B$ be an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$ such that*

$$\|D_f(x,y)\|_B \leq \frac{\|x\|^r}{\|y\|},\ r > 0 \ \text{ for all } x, y \in A$$

and

$$\|f(xy) - f(x) f(y)\|_B \leq \frac{\|x\|^r}{\|y\|},\ r > 3 \ \text{ for all } x, y \in A.$$

Then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{1}{1-L} \frac{\|x\|^r}{\|y\|}.$$

Proof. In the Theorem 4.13, if we choose

$$\theta(x,y) = \frac{\|x\|^r}{\|y\|},\ r > 0 \ \text{ for all } x, y \in A,$$

$$\zeta(x,y) = \frac{\|x\|^r}{\|y\|},\ r > 3 \ \text{ for all } x, y \in A,$$

then the result will be achieved when $L \geq \frac{1}{m^r}, m > 1, r > 0$, because

$$\frac{\|\frac{x}{m}\|^r}{\|\frac{x}{m}\|} \leq mL \frac{\|x\|^r}{\|x\|},$$

that is,

$$\frac{1}{m^r} \leq L$$

and

$$\lim_{n\to\infty} 3^{2n} \zeta\left(\frac{x}{3^n}, \frac{y}{3^n}\right) = \lim_{n\to\infty} 3^{n(3-r)} \left(\frac{\|x\|^r}{\|y\|}\right) = 0.$$

□

Corollary 4.16. *Assume that $m > 1$ and $f : A \to B$ is an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$ such that*

$$\|D_f(x,y)\|_B \leq \frac{\|x\|}{\|y\|^r},\ r < 2 \ \text{ for all } x, y \in A$$

and

$$\|f(xy) - f(x) f(y)\|_B \leq \frac{\|x\|}{\|y\|^r},\ r < -1 \ \text{ for all } x, y \in A.$$

Then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{1}{1-L} \frac{\|x\|}{\|y\|^r}\ .$$

Proof. In the Theorem 4.13, if we choose

$$\theta(x,y) = \frac{\|x\|}{\|y\|^r},\ r<2 \ \text{ for all } x,y \in A,$$

$$\zeta(x,y) = \frac{\|x\|}{\|y\|^r},\ r<-1 \ \text{ for all } x,y \in A,$$

then the result will be achieved when $L \geq \frac{1}{m^{r+1}}$, $m>1, r<2$. □

Corollary 4.17. *Let $r>1$, $m>1$ and $f: A \to B$ be an odd mapping, $f(0)=0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$ such that*

$$\|D_f(x,y)\|_B \leq \|x\|\|y\| \ \text{ for all } x,y \in A$$

and

$$\|f(xy) - f(x)f(y)\|_B \leq \|x\|^r\|y\|^r,\ r>1 \ \text{ for all } x,y \in A.$$

Then there exists a unique homomorphism $H: A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{1}{1-L}\|x\|^2.$$

Proof. In the Theorem 4.13, if we choose

$$\theta(x,y) = \|x\|\|y\| \ \text{ for all } x,y \in A,$$

$$\zeta(x,y) = \|x\|^r\|y\|^r,\ r>1 \ \text{ for all } x,y \in A,$$

then the result will be achieved when $L \geq \frac{1}{m^3}$, $m>1$. □

Corollary 4.18. *Let $m>1$ and $f: A \to B$ be an odd mapping, $f(0)=0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$ such that*

$$\|D_f(x,y)\|_B \leq \frac{\|x\|.\|y\|}{\|x\|^r + \|y\|^r},\ r<3 \ \text{ for all } x,y \in A$$

and

$$\|f(xy) - f(x)f(y)\|_B \leq \frac{\|x\| + \|y\|}{\|x\|^r + \|y\|^r},\ r<-1 \ \text{ for all } x,y \in A.$$

Then there exists a unique homomorphism $H: A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{1}{1-L}\frac{\|x\|^2}{2\|x\|^r}.$$

Proof. In the Theorem 4.13, if we choose

$$\theta(x,y) = \frac{\|x\|.\|y\|}{\|x\|^r + \|y\|^r},\ r<3 \ \text{ for all } x,y \in A,$$

$$\zeta(x,y) = \frac{\|x\| + \|y\|}{\|x\|^r + \|y\|^r},\ r<-1 \ \text{ for all } x,y \in A,$$

then the result will be achieved when $L \geq m^{r-3}$, $m>1, r<3$. □

Theorem 4.19. *Let A, B be two Banach algebra spaces, and $f : A \to B$ be an odd mapping, $f(0) = 0, f(\lambda x) = \lambda f(x)$ for all $x \in A, \lambda \in \mathbb{R}$, suppose functions $\theta : A^2 \to [0, \infty)$ and $\zeta : A^2 \to [0, \infty)$ such that for all $x \in A$, $m \in \mathbb{R}$, there exists $L < 1$, where $\theta\left(\frac{x}{m}, \frac{x}{m}\right) \le mL\theta(x, x)$,*

$$\|D_f(x, y)\|_B \le \theta(x, y) \text{ for all } x, y \in A, \tag{4.20}$$

$$\|f(xy) - xf(y) - yf(x)\|_B \le \zeta(x, y) \text{ for all } x, y \in A, \tag{4.21}$$

and

$$\lim_{n\to\infty} 3^{2n}\zeta\left(\frac{x}{3^n}, \frac{y}{3^n}\right) = 0. \tag{4.22}$$

Then there exists a unique derivation $\delta : A \to B$ such that

$$\|f(x) - \delta(x)\|_B \le \theta(x, x). \tag{4.23}$$

Proof. This theorem can be demonstrated using the same way as in the Theorem 4.7. □

Corollary 4.20. *Assume that $m > 1$ and $f : A \to B$ is an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$ such that*

$$\|D_f(x, y)\|_B \le \|x\|^r + \|y\|^r, \ r > 0 \ \text{ for all } x, y \in A$$

and

$$\|f(xy) - f(x) f(y)\|_B \le \|x\|^r.\|y\|^r, \ r > 1 \ \text{ for all } x, y \in A.$$

Then there exists a unique derivation $\delta : A \to B$ such that

$$\|f(x) - \delta(x)\|_B \le \frac{2}{1 - L}\|x\|^r.$$

Proof. In the Theorem 4.19, if we choose

$$\theta(x, y) = \|x\|^r + \|y\|^r, r > 0 \ \text{ for all } x, y \in A,$$

$$\zeta(x, y) = \|x\|^r.\|y\|^r, r > 1 \ \text{ for all } x, y \in A,$$

then the result will be achieved when $L \ge \frac{1}{m^{r+1}}, m > 1, r > 0$. □

Corollary 4.21. *Assume that $m > 1$ and $f : A \to B$ is an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$ such that*

$$D_f(x, y)_B \le \frac{\|x\|^r}{\|y\|}, \ r > 0 \ \text{ for all } x, y \in A$$

and

$$\|f(xy) - xf(y) - yf(x)\|_B \le \frac{\|x\|^r}{\|y\|}, \ r > 3 \ \text{ for all } x, y \in A.$$

Then there exists a unique derivation $\delta : A \to B$ such that

$$\|f(x) - \delta(x)\|_B \leq \frac{1}{1-L} \frac{\|x\|^r}{\|y\|}.$$

Proof. In the Theorem 4.19, if we choose

$$\theta(x,y) = \frac{\|x\|^r}{\|y\|},\ r > 0 \quad \text{for all } x, y \in A,$$

$$\zeta(x,y) = \frac{\|x\|^r}{\|y\|},\ r > 3 \ \text{ for all } x, y \in A,$$

then the result will be achieved when $L \geq \frac{1}{m^r}, m > 1, r > 0$. □

Corollary 4.22. *Assume that $m > 1$ and $f : A \to B$ is an odd mapping, $f(0) = 0, f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$ such that*

$$\|D_f(x,y)\|_B \leq \frac{\|x\|}{\|y\|^r},\ r < 2 \ \text{ for all } x, y \in A$$

and

$$\|f(xy) - xf(y) - yf(x)\|_B \leq \frac{\|x\|}{\|y\|^r},\ r < -1 \ \text{ for all } x, y \in A.$$

Then there exists a unique derivation $\delta : A \to B$ such that

$$\|f(x) - \delta(x)\|_B \leq \frac{1}{1-L} \frac{\|x\|}{\|y\|^r}.$$

Proof. In the Theorem 4.19, if we choose

$$\theta(x,y) = \frac{\|x\|}{\|y\|^r},\ r < 2 \ \text{ for all } x, y \in A,$$

$$\zeta(x,y) = \frac{\|x\|}{\|y\|^r},\ r < -1 \ \text{ for all } x, y \in A,$$

then the result will be achieved when $L \geq m^{r-1}, m > 1, r < 2$. □

Corollary 4.23. *Let $r > 1$, $m > 1$ and $f : A \to B$ be an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A, \lambda \in \mathbb{R}$ such that*

$$\|D_f(x,y)\|_B \leq \|x\|\|y\| \ \text{ for all } x, y \in A$$

and

$$\|f(xy) - xf(y) - yf(x)\|_B \leq \|x\|^r\|y\|^r,\ r > 1 \ \text{ for all } x, y \in A.$$

Then there exists a unique derivation $\delta : A \to B$ such that

$$\|f(x) - \delta(x)\|_B \leq \frac{1}{1-L}\|x\|^2.$$

Proof. In the Theorem 4.19, if we choose

$$\theta(x,y) = \|x\|\|y\| \text{ for all } x,y \in A,$$
$$\zeta(x,y) = \|x\|^r\|y\|^r, \ r > 1 \text{ for all } x,y \in A,$$

then the result will be achieved when $L \geq \frac{1}{m^3}, m > 1$. □

Corollary 4.24. *Let $m > 1$ and $f : A \to B$ be an odd mapping, $f(0) = 0$, $f(\lambda x) = \lambda f(x)$ for all $x \in A$, $\lambda \in \mathbb{R}$ such that*

$$\|D_f(x,y)\|_B \leq \frac{\|x\|.\|y\|}{\|x\|^r + \|y\|^r}, \ r < 3 \ \text{ for all } x,y \in A$$

and

$$\|f(xy) - xf(y) - yf(x)\|_B \leq \frac{\|x\| + \|y\|}{\|x\|^r + \|y\|^r}, \ r < -1 \ \text{ for all } x,y \in A.$$

Then there exists a unique derivation $\delta : A \to B$ such that

$$\|f(x) - \delta(x)\|_B \leq \frac{1}{1-L}\frac{\|x\|^2}{2\|x\|^r}.$$

Proof. In the Theorem 4.19, if we choose

$$\theta(x,y) = \frac{\|x\|.\|y\|}{\|x\|^r + \|y\|^r}, \ r < 3 \text{ for all } x,y \in A,$$
$$\zeta(x,y) = \frac{\|x\| + \|y\|}{\|x\|^r + \|y\|^r}, \ r < -1 \text{ for all } x,y \in A,$$

then the result will be achieved when $L \geq m^{r-3}$, $m > 1$, $r < 3$. □

5. Conclusion

We have come to conclusion that the approximate of homomorphism and derivation of additive functional equation in Banach algebra. Moreover, we can get other results in various spaces and for many other functional equations

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