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OPTIMAL CONTROL OF DISTRIBUTED PARAMETER SYSTEM GIVEN BY CAHN-HILLIARD EQUATION

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Abstract. In this work, optimal control problem is addressed for distributed systems described by Cahn-Hilliard equation by the means of distributed control, initial control and Neumann boundary control. The existence and uniqueness is provided for weak solution using variational method. Further, existence of optimal control is proved completely, and optimality conditions is established for integral cost and quadratic cost, respectively. Lastly, Bang-Bang principle is deduced.

1. Introduction

In investigating optimal control problem for distributed parameter system described by Cahn-Hilliard (C-H) equation, let Ω be an open bounded set $\mathbf{R}^n$ of x with a piecewise smooth boundary $\Gamma = \partial\Omega$. Let $Q = (0,T)\times\Omega$ and $\Sigma = (0,T)\times\Gamma$ with $T > 0$. Consider the system described by Cahn-Hilliard equation as

$$\begin{cases} \dfrac{\partial y}{\partial t} + \gamma\Delta^2 y - \lambda\Delta f(y) = B^0 v_0 \quad \text{in } \ Q, \\ \dfrac{\partial y}{\partial \eta} = 0, \ \dfrac{\partial(\Delta y)}{\partial \eta} = B^1 v_1 \quad \text{on } \ \Sigma, \\ y(0) = Bw \quad \text{on } \ \Omega, \end{cases} \tag{1.1}$$

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where $\gamma, \lambda > 0$ are constants. Δ denote Laplacian in $\mathbf{R}^n$. y is the concentration function of binary components. B^0, B^1, B are control operators, v_0, v_1, w are control variables, $\dfrac{\partial}{\partial \eta}$ is a unit outer normal derivative. $f(y)$ is a polynomial of order $2p-1$ as follows

$$f(y) = \sum_{j=1}^{2p-1} a_j y^j, \quad p \in \mathbf{N}, \quad p \geq 1. \tag{1.2}$$

Here $\mathbf{N}$ is set of natural numbers, and the leading coefficient of f is positive, i.e., $a_{2p-1} > 0$.

Cahn-Hilliard equation is based on a continuum model for phase transition in binary systems such as alloy, glasses, and polymer-mixtures (cf. [3]). Since Cahn and Hilliard proposed the equation in 1958, contributing research had been reported in many related works (cf. [5, 7, 15]). Paper [4] using free energy method for C-H equation. Article [8] show finite difference scheme for C-H equation with numerical experiment. Further, [11] is a paper on discontinuous finite element method for C-H equation with demonstrations using color graphics visualization. The discontinuous Galerkin method is executed in [22] for Cahn-Hilliard equation to get good experimental agreement. All of these works are focused either on system (analytic, numerical) solution or on system properties. There are few results (cf. [23]) on optimal control problem theoretically and computationally, even nonlinear term in [23] is limited to 3 order polynomial.

To develop optimal control theory as [12] for system (1.1) with integral cost in the form of

$$J(\mathbf{v}) = \Phi(y(T, \mathbf{v})) + \int_0^T L(t, \mathbf{v}, y(t, \mathbf{v}))dt, \quad \forall\, \mathbf{v} \in \mathcal{U}, \tag{1.3}$$

where Φ, L are continuous functions on $t \in [0, T]$, $\mathbf{v} = (v_0, v_1, w)$ is a control vector and $\mathcal{U}$ is a Hilbert space of $\mathbf{v}$, $\mathcal{U} = V_0 \times V_1 \times W$ and V_0, V_1, W are Hilbert spaces of control variables v_0, v_1, w.

Our goal is to find such optimal control $\mathbf{v}^*$ and characterize optimality conditions on $\mathbf{v}^*$ such that $\inf_{\mathbf{v} \in \mathcal{U}_{ad}} J(\mathbf{v}) = J(\mathbf{v}^*)$ for an admissible set $\mathcal{U}_{ad} \subset \mathcal{U}$.

The paper is organized as follows. Section 2 gives the mathematical setting and some preparation. Section 3 states existence and uniqueness of weak solution for free system. Section 4 considers the optimal control problem of system (1.1) with integral performance index (1.3). Section 5 contains conclusions.

2. Mathematical setting

If using [6, 17, 18, 19, 20, 21] to give mathematical setting. Consider the free system with $B^0 v_0 = g$, $B^1 v_1 = k_1$, $Bw = y_0$ in (1.1). Introduce Hilbert space $H = L^2(\Omega)$ as usual. Define Hilbert space

$$V = H^2(\Delta;\Omega) = \big\{\phi \big| \phi \in L^2(\Omega),\ \ \Delta\phi \in L^2(\Omega),\ \ \frac{\partial \phi}{\partial \eta} = 0 \text{ on } \Gamma\big\}$$

with inner product

$$(\phi, \varphi) = \int_\Omega \phi(x)\varphi(x)dx + \int_\Omega \Delta\phi(x)\Delta\varphi(x)dx.$$

V is equipped with the norm $\|\phi\|_V = \|\phi\|_{L^2(\Omega)} + \|\Delta\phi\|_{L^2(\Omega)}$. Neglecting $\phi \equiv C \in \mathbf{R}$, it is clear that the norm $\|\phi\|_V$ is equivalent to $\|\Delta\phi\|_{L^2(\Omega)}$ and $\|\phi\|_{H^2(\Omega)}$ in $V/\mathbf{R}$ (cf. [19, 20, 21]). Let V' be the dual space of V, denote its norm by $\|\cdot\|_{V'}$. The symbol $\langle\cdot,\cdot\rangle$ denotes the dual pair of V and V'. Denote $\phi' = \dfrac{d\phi}{dt}$ to define the solution space by

$$W(0,T;V,V') = \big\{\phi \big| \phi \in L^2(0,T;V),\ \phi' \in L^2(0,T;V')\big\}.$$

According to Neumann boundary condition in (1.1), Hilbert space $H^{\frac{3}{2}}(\Gamma)$ is introduced with norm $\|\cdot\|_{H^{\frac{3}{2}}(\Gamma)}$ defined by

$$\|\phi\|_{H^{\frac{3}{2}}(\Gamma)} = \Big(\int_\Gamma |\phi(s)|^2 ds + \int_\Gamma \sum_{|\alpha|=1} |D^\alpha \phi(s)|^2 ds + \int_{\Gamma\times\Gamma} \frac{|\phi(s)-\phi(\tilde{s})|^2}{|s-\tilde{s}|^{n+1}} ds d\tilde{s} \Big)^{\frac{1}{2}}.$$

Its dual space $H^{-\frac{3}{2}}(\Gamma)$ with norm denoted by $\|\cdot\|_{H^{-\frac{3}{2}}(\Gamma)}$ [1], the symbol $\langle\cdot,\cdot\rangle_{H^{-\frac{3}{2}}(\Gamma),H^{\frac{3}{2}}(\Gamma)}$ denotes the dual pair of $H^{-\frac{3}{2}}(\Gamma)$ and $H^{\frac{3}{2}}(\Gamma)$.

Definition 2.1. A function y is a weak solution (cf. [21]) of (1.1) if $y \in W(0,T;V,V')$ satisfies weak form

$$\begin{cases} \dfrac{d}{dt}(y,v) + \gamma(\Delta y, \Delta v) - \lambda(f(y), \Delta v) \\ = \langle g, v\rangle + \big\langle \gamma k_1, v\big|_\Gamma \big\rangle_{H^{-\frac{3}{2}}(\Gamma),H^{\frac{3}{2}}(\Gamma)}, \quad \forall\, v \in V, \\ y(0) = y_0 \in H, \end{cases} \tag{2.1}$$

in the sense of $\mathcal{D}'(0,T)$, where $\mathcal{D}'(0,T)$ denotes the space of distributions on $(0,T)$.

In order to obtain weaker restrict on nonlinear term as in [21], the assumption on p and n is needed to assure $f(\phi) \in L^2(\Omega)$ for all $\phi \in V$. Define the

$$\text{Assumption } \mathbf{(A)}: \begin{cases} \text{arbitrary } p, & \text{if } \ 1 \le n \le 4. \\ p < \left[\dfrac{n-2}{n-4}\right], & \text{if } \ 5 \le n \le 6. \\ p = 1, & \text{if } \ n \ge 7. \end{cases}$$

Here [ ] denote the Gauss symbol. Citing Gagliardo-Nirenberg inequality in [9, 14, 15].

Lemma 2.2. *Let p_0, r_0, q_0 be integers. If $1 \le r_0 < q_0 \le \infty$ and $p_0 \le q_0$. Then for $\phi(t) \in W^{m_0,p_0}(\Omega) \cap L^{r_0}(\Omega)$ to have*

$$\Big\|\phi(t)\Big\|_{W^{k_0,q_0}(\Omega)} \le C\Big\|\phi(t)\Big\|^{\theta_0}_{W^{m_0,p_0}(\Omega)} \ \Big\|\phi(t)\Big\|^{1-\theta_0}_{L^{r_0}(\Omega)} \tag{2.2}$$

holds with some $C > 0$ and

$$\theta_0 = \left(\frac{k_0}{n} + \frac{1}{r_0} - \frac{1}{q_0}\right) \times \left(\frac{m_0}{n} + \frac{1}{r_0} - \frac{1}{p_0}\right)^{-1}$$

provided that $0 < \theta_0 \le 1$ (assume that $0 < \theta_0 < 1$ if $q_0 = +\infty$). Here in (2.2), k_0, m_0 are the derivatives orders of $\phi(t)$ in space $W^{k_0,q_0}(\Omega)$, $W^{m_0,p_0}(\Omega)$, respectively. $0 \le k_0 < m_0$.

Lemma 2.3. *Let assumption $(\mathbf{A})$ be satisfied, then*

$$\|f(\phi(t))\|_H \le C\Big(\|\phi(t)\|_H^{\frac{4p-2-n(p-1)}{2}} \|\phi(t)\|_V^{\frac{n(p-1)}{2}} + 1\Big)$$

for $\phi(t) \in V/\mathbf{R}$, where $C > 0$ is constant independent of $\phi(t)$.

Taking $r_0 = p_0 = m_0 = 2$, $q_0 = 4p - 2, k_0 = 0$ in (2.2) of Lemma 2.2, then $\theta_0 = \dfrac{n(p-1)}{4p-2} < 1$ to verify Lemma 2.3.

Lemma 2.4. *Let assumption $(\mathbf{A})$ be satisfied, then $f(y(t))$, as a nonlinear mapping: $V \to H$, satisfies the local Lipschitz continuity. That is for some $C > 0$, and $\forall\ \phi, \varphi \in V$ to have*

$$\|f(\phi(t)) - f(\psi(t))\|_H^2 \le C(\|\phi(t)\|_V^{4p-4} + \|\psi(t)\|_V^{4p-4})\|\phi(t) - \psi(t)\|_V^2.$$

One can refer (6.17) in p. 175 of [16] for above Lemma.

3. Existences and uniqueness of weak solution

Consider $y_0 \in L^2(\Omega), y(t)$ for $t \in [0,T]$, then theoretical study is closely related to n and p of the nonlinearity (cf. [21]). If using Gauss symbol [ ], define

$$p_n = \left[1 + \frac{2}{n}\right], n \in \mathbf{N}.$$

Consider $y_0 \in L^2(\Omega)$ and $y(t)$ for $t \in [\eta, T]$, where $\eta > 0$ arbitrary, then the following theorem depends on the conditions of n and p (cf. [21]). One can propose

$$\text{Assumption } \mathbf{(B)}: \begin{cases} \text{arbitrary } p, & \text{if } \ 1 \le n \le 2. \\ p \le \left[\dfrac{n}{n-2}\right], & \text{if } \ 3 \le n \le 4. \\ p = 1, & \text{if } \ n \ge 5. \end{cases}$$

Theorem 3.1. *Let assumption* (**A**) *be satisfied. Assume that* $y_0 \in L^2(\Omega)$ *and* $g \in L^2(0,T;V')$, *then there exists a unique weak solution* $y \in W(0,T;V,V')$ *for free system of* (1.1), *which is belonging to*

$$L^\infty(0,T;H) \cap L^2(0,T;V), \quad \forall\, T > 0. \tag{3.1}$$

For $C > 0$ *to get estimate*

$$\begin{aligned} &\|y\|^2_{L^\infty(0,T;H)} + \|y\|^2_{L^2(0,T;V)} \\ &\le C\Big(1 + \|y_0\|^2_H + \|g\|^2_{L^2(0,T;V')} + \big\|k_1\big\|^2_{L^2(0,T;H^{-\frac{3}{2}}(\Gamma))}\Big). \end{aligned} \tag{3.2}$$

Further, if $p \le p_n$, *then we obtain estimation*

$$\begin{aligned} &\big\|y\big\|^{4p-2}_{L^{4p-2}(0,T;L^{4p-2}(\Omega))} \\ &\le C\Big(1 + \|y_0\|^2_H + \|g\|^2_{L^2(0,T;V')} + \big\|k_1\big\|^2_{L^2(0,T;H^{-\frac{3}{2}}(\Gamma))}\Big)^k, \end{aligned} \tag{3.3}$$

where C *and* k *are constants independent of* y_0, g *and* k_1.
In addition, for any $0 < \eta < T$, *if* n *and* p *satisfy assumption* (**B**), *then the estimation can be given by*

$$\begin{aligned} &\big\|y\big\|^{4p-2}_{L^{4p-2}(\eta,T;L^{4p-2}(\Omega))} \\ &\le C(\eta)\Big(1 + \|y_0\|^2_H + \|g\|^2_{L^2(0,T;V')} + \big\|k_1\big\|^2_{L^2(0,T;H^{-\frac{3}{2}}(\Gamma))}\Big)^k, \end{aligned} \tag{3.4}$$

where $C(\eta)$ *and* k *are independent of* y_0, g *and* k_1.

Proof. Since the boundary Γ of Ω is piecewise smooth, by trace theorem (cf. [13]), it is easy to verify (3.1) and (3.2) (cf. $g = 0$ in [16]). To prove (3.2) and

(3.3), from Lemma 2.3 to deduce that

$$\int_\Omega |y(t)|^{4p-2}dx$$
$$\leq C\Big(1+\|y_0\|_H^2+\|g\|^2_{L^2(0,T;V')}+\big\|k_1\big\|^2_{L^2(0,T;H^{-\frac{3}{2}}(\Gamma))}\Big)^{k'}\|y(t)\|_V^{n(p-1)}, \quad (3.5)$$

where $k'=4p-2-n(p-1)$. If $p\leq p_n$ and $p\neq 1$, using Hölder inequality to get that

$$\int_0^T \|y\|_V^{n(p-1)}dt\leq \Big(\int_0^T\|y\|_V^2dt\Big)^{\frac{n(p-1)}{2}}\ T^{\frac{2-n(p-1)}{2}}. \quad (3.6)$$

By (3.2), (3.5) and (3.6) to deduce the estimate (3.3). To derive (3.4), let F denote the primitive of f vanishing at $y=0$, i.e. $F(y)=\sum_{j=2}^{2p}b_jy^j$ for $jb_j=a_{j-1}$ and $2\leq j\leq 2p$. Define Lyapunov function by

$$Y(y(t))=\frac{\gamma}{2}\|\nabla y(t)\|_H^2+\lambda\int_\Omega F(y(t))dx.$$

Let us quote a result at footnote of p.155 in [16], for arbitrary $\eta>0$,

$$Y(y(t))\leq Y(y(\eta))$$

for $t\in[\eta,T]$ to have

$$y\in L^\infty(\eta,T;H^1(\Omega))\cap L^\infty(\eta,T;L^{2p}(\Omega)).$$

Further, if $y_0\in H^1(\Omega)\cap L^{2p}(\Omega)$, then $Y(y(t))\leq Y(y_0)$ for all $t\geq 0$. Setting $r_0=2p$, $p_0=m_0=2$, $q_0=4p-2, k_0=0$ in Lemma 2.2, then take $\theta_1=\dfrac{n(p-1)}{(2p-1)(4p-n(p-1))}\leq 1$ under assumption (**B**). Applying Gagliardo-Nirenberg inequality (cf. [9, 14]) to $y(t)\in H^2(\Omega)\cap L^{2p}(\Omega)$, by the equivalence of norm $\|y(t)\|_{H^2(\Omega)}$,$\|y(t)\|_V$ and $\|\Delta y(t)\|_H$ in $V/\mathbf{R}$ to deduce that

$$\|y(t)\|_{L^{4p-2}(\Omega)}\leq C\|y(t)\|_V^{\theta_1}\|y(t)\|^{1-\theta_1}_{L^{2p}(\Omega)}<\infty \quad (3.7)$$

for $y(t)\in V\cap L^{2p}(\Omega)$ and some $C>0$. Therefore, by (3.7) to find that

$$\int_\eta^T \|y(t)\|^{4p-2}_{L^{4p-2}(\Omega)}dt$$
$$\leq C\sup_{t\in[0,T]}\|y(t)\|^{1-\theta_1}_{L^{2p}(\Omega)}\int_\eta^T\|y(t)\|_V^{\theta_1(4p-2)}dt\int_\eta^T\|y(t)\|_V^2dt$$

under assumption (**B**). It provides (3.4). Above a priori estimate to prove existence by [6] in a routing way.

The regularity results can be proved by deriving an energy-type equation.

In order to prove uniqueness for $y_0 \in L^2(\Omega)$, we use the following inequalities. Let $f(s)$ be a polynomial defined by (1.2). Then for $b, c \in \mathbf{R}$, there exists C such that

$$|f(b) - f(c)| \leq C|b - c|(1 + |b|^{2p-2} + |c|^{2p-2}). \tag{3.8}$$

By Young inequality (cf. [12]), for $a, b > 0$, it is clear that

$$ab \leq \frac{\varepsilon^{p'}}{p'} a^{p'} + \frac{1}{q'} \frac{b^{q'}}{\varepsilon^{q'}}, \text{ where } \frac{1}{p'} + \frac{1}{q'} = 1. \tag{3.9}$$

Let $y_1(t)$ and $y_2(t)$ be two weak solutions of (1.1) on $[0, T]$ with respect to inputs $g(t)$, $k_1(t)$ and y_0. Set $\bar{y}(t) = y_1(t) - y_2(t)$, then $\bar{y}(t)$ satisfies

$$\begin{cases} \dfrac{d\bar{y}(t)}{dt} + \gamma \Delta^2 \bar{y}(t) - \lambda \Delta (f(y_1(t)) - f(y_2(t))) = 0, \\ \bar{y}(0) = 0, \end{cases} \tag{3.10}$$

in weak sense. Consider $y_1(t), y_2(t), \bar{y}(t) \in V$, and eak form of (2.1), then we have

$$\frac{1}{2} \frac{d}{dt} \|\bar{y}(t)\|_H^2 + \gamma \|\Delta \bar{y}(t)\|_H^2 = \lambda (f(y_1(t)) - f(y_2(t)), \Delta \bar{y}(t)). \tag{3.11}$$

Let us utilize (3.8) to estimate the right hand side of (3.11) as follows

$$\begin{aligned} &|(f(y_1(t)) - f(y_2(t)), \Delta \bar{y}(t))_{L^2(\Omega)}| \\ &\leq \frac{C}{\varepsilon} \|\bar{y}(t)\|_{L^2(\Omega)}^2 + C\varepsilon \|\Delta \bar{y}(t)\|_H^2 + C \int_\Omega |\Delta \bar{y}(t)||\bar{y}(t)||y_1(t)|^{2p-2} dx \\ &\quad + C \int_\Omega |\Delta \bar{y}(t)||\bar{y}(t)||y_2(t)|^{2p-2} dx. \end{aligned}$$

For the exponent of y_i $(i = 1, 2)$, using Hölder inequality, taking

$$p' = \frac{2(4p-2)}{4p - 2 + n(p-1)}, \quad q' = \frac{2(4p-2)}{4p - 2 - n(p-1)}$$

such that $\dfrac{1}{p'} + \dfrac{1}{q'} = 1$ in (3.9), then deduce that

$$\begin{aligned} &\int_\Omega |\Delta \bar{y}(t)||\bar{y}(t)||y_i(t)|^{2p-2} dx \\ &\leq C \frac{\varepsilon^{p'}}{p'} \|\bar{y}(t)\|_V^2 + \frac{C}{q' \varepsilon^{q'}} \big\| y_i(t) \big\|_{L^{4p-2}(\Omega)}^{\frac{(2p-2)(8p-4)}{4p-2-n(p-1)}} \|\bar{y}(t)\|_H^2 \end{aligned} \tag{3.12}$$

for $C > 0$. Setting $k^i(t) = \dfrac{C}{q'\varepsilon^{q'}}\|y_i(t)\|_{L^{4p-2}(\Omega)}^{\frac{(2p-2)(8p-4)}{4p-2-n(p-1)}}$ and $C_i(\varepsilon) = C\dfrac{\varepsilon^{p'}}{p'}$, then obtain

$$\int_\Omega |\Delta\bar{y}(t)||\bar{y}(t)||y_i(t)|^{2p-2}dx \le C_i(\varepsilon)\|\bar{y}(t)\|_V^2 + k^i(t)\|\bar{y}(t)\|_H^2.$$

If $y_0 \in L^2(\Omega)$, one can prove that $k^i(t)$ is integrable on $[0,T]$. In fact

$$\int_0^T \|y_i(t)\|_{L^{4p-2}(\Omega)}^{\frac{(2p-2)(8p-4)}{4p-2-n(p-1)}}dt \le C\|y_i\|_{L^\infty(0,T;H)}^{2p-2}\int_0^T \|y_i(t)\|_V^{\frac{2n(p-1)^2}{4p-2-n(p-1)}}dt$$

for $C > 0$. Since $\dfrac{4p-2-n(p-1)}{n(p-1)^2} > 1$ as $p \le p_n$, by Hölder inequality and for some $k > 0$, one has that

$$\int_0^T \|y_i(t)\|_V^{\frac{2n(p-1)^2}{4p-2-n(p-1)}}dt \le C\Big(\int_0^T \|y_i(t)\|_V^2 dt\Big)^k \text{ as } p \le p_n.$$

This implies that $k^i(t)$ is integrable on $[0,T]$ as $p \le p_n$ for $y_0 \in L^2(\Omega)$. By (3.12), one can rewrite (3.11) as follows

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\bar{y}(t)\|_H^2 + (\gamma - C\varepsilon - C\lambda C(\varepsilon))\|\Delta\bar{y}(t)\|_H^2\\ &\le C\Big(\frac{1}{\varepsilon} + \lambda K(t)\Big)\|\bar{y}(t)\|_H^2,\end{aligned}\tag{3.13}$$

where $C(\varepsilon) = C_1(\varepsilon) + C_2(\varepsilon)$ and $K(t) = k^1(t) + k^2(t)$. Finding ε in (3.13) such that $\gamma - C\varepsilon - C\lambda C(\varepsilon) > 0$, by Bellman-Gronwall theorem (cf. [16]) to have $\|\bar{y}(t)\|_H = 0$ in H for all $t \in [0,T]$ as $p \le p_n$. It completes the proof of uniqueness in Theorem 3.1. □

4. Optimal control problems

As pointed out early, $\mathcal{U} = V_0 \times V_1 \times W$ denotes Hilbert space of control vector $\mathbf{v} = (v_0, v_1, w)$. Let $B^0 \in \mathcal{L}(V_0, L^2(0,T;V'))$, $B^1 \in \mathcal{L}(V_1, L^2(0,T;H^{-\frac{3}{2}}(\Gamma)))$, $B \in \mathcal{L}(W, H)$. For control vector $\mathbf{v} = (v_0, v_1, w)$, by virtue of Theorem 3.1, the solution mapping $\mathbf{v} \to y(\mathbf{v})$ from $\mathcal{U}$ into $W(0,T;V,V')$ is well defined. Here $y(\mathbf{v})$ is called the state of control system (1.1). The integral cost associated with the control system (1.1) is given by (1.3), and we regard

$$\Phi : H \to \mathbf{R}^+, \quad L(t,\mathbf{v},y(t,\mathbf{v})) : [0,T] \times \mathcal{U} \times H \to \mathbf{R}^+.$$

Let $\mathcal{U}_{ad} = V_{ad}^0 \times V_{ad}^1 \times W_{ad}$ be a closed convex (bounded) subset of $\mathcal{U} = V^0 \times V^1 \times W$, which is called the admissible set. Integral cost optimal control problem subject to (1.1) and (1.3) is:

(i) find an element $\mathbf{v}^* = (v_0^*, v_1^*, w^*) \in \mathcal{U}_{ad}$ such that

$$\inf_{\mathbf{v}\in\mathcal{U}_{ad}} J(\mathbf{v}) = J(\mathbf{v}^*);$$

(ii) characterize such $\mathbf{v}^*$.

Such a $\mathbf{v}^*$ is called optimal control for C-H problem (cf. [12]).

4.1. **Existence of optimal control.** Assume $\mathcal{U}_{ad}$ is bounded. Suppose assumptions on Φ and L:

H(Φ) Function $\Phi : H \to \mathbf{R}^+$ is continuous and convex.

H(L) Function $L : [0,T] \times \mathcal{U} \times H \to \mathbf{R}^+$ is an integrand such that

(a) For arbitrary $(\mathbf{v}, y) \in \mathcal{U} \times H$, $L(t, \mathbf{v}, y)$ is measurable in $t \in [0,T]$;

(b) For *a.e.* $t \in [0,T]$, $L(t, \mathbf{v}, y)$ is continuous and convex for each $(\mathbf{v}, y) \in \mathcal{U} \times H$;

(c) For arbitrary bounded set $E \subset \mathcal{U}$, there exists an $m_E \in L^1(0,T)$ s.t.

$$\sup_{\mathbf{v}\in E} |L(t, \mathbf{v}, y(t, \mathbf{v}))| \le m_E(t), \ \textit{a.e.} \text{ in } [0,T]; \tag{4.1}$$

(d) $L(t, \mathbf{v}, y)$ is locally uniformly Lipschitz continuous with respect to y, i.e., for bounded set $K = E \times F \subset \mathcal{U} \times H$, there exists an $m_K \in L^2(0,T)$ s.t.

$$|L(t, \boldsymbol{\phi}, y) - L(t, \boldsymbol{\phi}, z)| \le m_K(t)\|y - z\|_H. \tag{4.2}$$

Theorem 4.1. *Let* $\mathbf{H}(\Phi)$ *and* $\mathbf{H}$*(L) be satisfied and assume* $y_0 \in L^2(\Omega)$. *Consider* $y(t), t \in [0,T]$ *as* $p \le p_n$ *and* $y(t), t \in [\eta, T]$ *under assumption* ($\mathbf{B}$). *If* $\mathcal{U}_{ad}$ *is closed convex (bounded) subset, then there exists at least one optimal control* $\mathbf{v}^* \in \mathcal{U}_{ad}$ *such that* $\mathbf{v}^*$ *minimizes the cost* (1.3).

Proof. By virtue of Theorem 3.1, there exists a weak solution $y(\mathbf{v})$ of equation (1.1) with $B^0 v_0 \in L^2(0,T;V')$, $B^1 v_1 \in L^2(0,T;H^{-\frac{3}{2}}(\Gamma))$ and $Bw \in H$. $y(\mathbf{v})$ is uniformly bounded for $\mathbf{v} \in E$, i.e., $\sup\{|y(t,\mathbf{v})|_H : \mathbf{v} \in E, t \in [0,T]\} < +\infty$. By (c) of $\mathbf{H}$(L) to know that $J(\mathbf{v})$ is bounded on E, it means that $J(\mathbf{v})$ makes sense for any $\mathbf{v} \in \mathcal{U}$. Since $\mathcal{U}_{ad}$ is non-empty, there exists a sequence $\{\mathbf{v}_n\}$, $\mathbf{v}_n = (v_0^n, v_1^n, w^n)$, in $\mathcal{U}$ such that $\inf\limits_{\mathbf{v}\in\mathcal{U}_{ad}} J(\mathbf{v}) = \lim\limits_{n\to\infty} J(\mathbf{v}_n) = J$. Here y_n is the trajectory corresponding to $\mathbf{v}_n$, that is $y_n = y(t, \mathbf{v}_n)$. Because $\mathcal{U}_{ad}$ is bounded, convexity and closed, one can choose a subsequence $\{\mathbf{v}_m\}$, $\mathbf{v}_m = (v_0^m, v_1^m, w^m)$, of $\{\mathbf{v}_n\}$ and find a $\mathbf{v}^* = (v_0^*, v_1^*, w^*) \in \mathcal{U}_{ad}$ such that

$$\mathbf{v}_m \to \mathbf{v}^* \text{ weakly in } \mathcal{U}, \text{ as } m \to \infty. \tag{4.3}$$

Analogous to the proof of Theorem 2.1, there exists a subsequence (rewritten as $\{y(\mathbf{v}_m)\}$) of $\{y(\mathbf{v}_m)\}$ and $z \in W(0,T;V,V')$ such that

$$y(\mathbf{v}_m) \to z \qquad \text{weakly} * \text{in} \ \ L^\infty(0,T;H), \tag{4.4}$$

$$y(\mathbf{v}_m) \to z \qquad \text{weakly in} \ \ W(0,T;V,V'), \tag{4.5}$$

$$y(\mathbf{v}_m) \to z \qquad \text{strongly in} \ \ L^2(0,T;H). \tag{4.6}$$

Because of convergences (4.4), (4.5) and Gagliardo-Nirenberg inequality (cf. [9, 14]), one sees that $y(\mathbf{v}_m)$ belongs to $L^{4p-2}(0,T;L^{4p-2}(\Omega))$ for any m as $p \le p_n$. This implies that $f(y(\mathbf{v}_m))$ belongs to $L^2(Q)$ as $p \le p_n$. By (4.6) and the continuity of $f(y)$ with respect to y, we know that

$$f(y(\mathbf{v}_m)(t,x)) \to f(z(t,x)) \text{ a.e. in } Q \text{ as } p \le p_n.$$

Noticing the boundedness of $f(y(\mathbf{v}_m))$ for any m, by the same argument in subsection 4.2 to deduce that

$$f(y(\mathbf{v}_m)(t,x)) \to f(z(t,x)) \text{ weakly in } L^2(Q) \tag{4.7}$$

as $m \to \infty$. The weak form is expressed as

$$\begin{aligned} &\frac{d}{dt}(y(t,\mathbf{v}_m),v) + \gamma(\Delta y(t,\mathbf{v}_m),\Delta v) - \lambda \int_\Omega f(y(t,\mathbf{v}_m))\Delta v dx \\ &= \langle B^0 v_0^m(t), v\rangle + \big\langle \gamma B^1 v_1^m(t), v|_\Gamma \big\rangle_{H^{-\frac{3}{2}}(\Gamma),H^{\frac{3}{2}}(\Gamma)}, \end{aligned} \tag{4.8}$$

for $v \in V$. By (4.3), (4.4), (4.5), (4.6) and (4.7), taking the limit in (4.8) yields (2.1). One can obtain $z = y(\mathbf{v}^*)$ via uniqueness, that is, function $y(\mathbf{v}^*)$ is the state corresponding to control variable $\mathbf{v}^*$. Therefore from (4.5) to have

$$y(t,\mathbf{v}_m) \to y(t,\mathbf{v}^*) \ \text{ weakly in } H, \ a.e. \text{ in } \ [0,T]. \tag{4.9}$$

It is well known that continuity plus convexity imply weak lower semi-continuity (cf. [12]), then from $\mathbf{H}(\Phi)$ and (4.9) with $t = T$ to get

$$\varliminf_{m\to\infty} \Phi(y(T,\mathbf{v}_m)) \ge \Phi(y(T,\mathbf{v}^*)). \tag{4.10}$$

Since $\mathbf{v}_m \to \mathbf{v}$ weakly in $\mathcal{U}$, there exists a constant $M > 0$ such that $\|\mathbf{v}_m\|_\mathcal{U} \le M$, this means there exists a bounded set $E = \{\mathbf{v} : \|\mathbf{v}_m\|_\mathcal{U} \le M\} \subset \mathcal{U}$. Then by the estimate in Theorem 3.1, we have

$$N = \sup\Big\{|y(t,\mathbf{v}_m)|_H : \|\mathbf{v}_m\|_\mathcal{U} \le M, \ t \in [0,T]\Big\} < +\infty.$$

This deduces that there exists a bounded set $F = \{y : |y(t,\mathbf{v}_m)|_H \le N\} \subset H$. For E and F, there exists a bounded set

$$\begin{aligned} K &= E \times F \\ &= \big\{\mathbf{v} : \|\mathbf{v}_m\|_\mathcal{U} \le M\big\} \times \big\{y : |y(t,\mathbf{v}_m)|_H \le N\big\} \subset \mathcal{U} \times H. \end{aligned} \tag{4.11}$$

By local uniformly Lipschitz continuity assumption (d) with (4.2) of $\mathbf{H}$(L), for the bounded set K there exists an $m_K \in L^2(0,T)$ such that

$$\begin{aligned}&\lim_{m\to\infty}\int_0^T |L(t,\mathbf{v}_m,y(t,\mathbf{v}_m)) - L(t,\mathbf{v}_m,y(t,\mathbf{v}^*))|dt\\ &\le \|m_K\|_{L^2(0,T)}\lim_{m\to\infty}\|y(t,\mathbf{v}_m) - y(t,\mathbf{v}^*)\|_{L^2(0,T;H)}=0. \qquad (4.12)\end{aligned}$$

By assumption (c) with (4.1) of $\mathbf{H}$(L), for the bounded set E, there exists $m_E \in L^1(0,T)$ s.t.

$$\sup_m \int_0^T |L(t,\mathbf{v}_m,y(t,\mathbf{v}^*))|dt \le \int_0^T m_E(t)dt < +\infty.$$

Also because continuity plus convexity imply weak lower semi-continuity, the assumption (b) of $\mathbf{H}$(L) and (4.3), it follows from the Lebesgue-Fatou lemma (cf. [13]) that

$$\varliminf_{m\to\infty}\int_0^T L(t,\mathbf{v}_m,y(t,\mathbf{v}^*))dt \ge \int_0^T L(t,\mathbf{v}^*,y(t,\mathbf{v}^*))dt. \qquad (4.13)$$

Therefore, from (4.10), (4.12) and (4.13) to get $J = \liminf_{m\to\infty} J(\mathbf{v}_m) \ge J(\mathbf{v}^*)$, then $J(\mathbf{v}^*) = \inf_{\mathbf{v}\in\mathcal{U}_{ad}} J(\mathbf{v})$, i.e., $\mathbf{v}^*$ is an optimal control for cost (1.3). It finish proof of Theorem 4.1. □

4.2. **Necessary conditions for optimality.** It is well known that the optimality condition in [12] for $\mathbf{v}^*$ is given by the variational inequality

$$J'(\mathbf{v}^*)(\mathbf{v}-\mathbf{v}^*) \ge 0, \quad \forall\, \mathbf{v} \in \mathcal{U}_{ad},$$

where $J'(\mathbf{v}^*)$ denotes the Gâteaux derivative of $J(\mathbf{v})$ in (1.3) at $\mathbf{v}^*$. In order to derive the optimality conditions, the following assumptions are posed.

A(Φ) The function $\Phi : H \to \mathbf{R}^+$ is Gâteaux differentiable, and $\Phi'(y)$ is continuous on H.

A(L) The function $L : [0,T]\times\mathcal{U}\times H \to \mathbf{R}^+$ satisfies

(a) For fixed $(t,\mathbf{v}) \in [0\ ,\ T] \times \mathcal{U}$, there exists Gâteaux derivative $L'_y(t,\mathbf{v},y)$, which is continuous in $(\mathbf{v},y) \in \mathcal{U}\times H$. For bounded set $K = E\times F \subset \mathcal{U}\times H$, there exists $m_k^1(t) \in L^1(0,T)$ s.t.

$$\sup_{(\mathbf{v},y)\in K} \|L'_y(t,\mathbf{v},y)\|_{L(H)} \le m_k^1(t) \ a.e. \text{ in } [0,T];$$

(b) For fixed $(t,y) \in [0,T]\times H$, there exists Gâteaux derivative $L'_{\mathbf{v}}(t,\mathbf{v},y)$, which is continuous in $\mathbf{v}\in\mathcal{U}$. For bounded set $K = E\times F \subset \mathcal{U}\times H$, there exists an $m_k^2(t) \in L^2(0,T)$ s.t.

$$\sup_{(\mathbf{v},y)\in K} \|L'_{\mathbf{v}}(t,\mathbf{v},y)\|_{L(\mathcal{U})} \le m_k^2(t) \ \ a.e. \text{ in } [0,T].$$

The Gâteaux derivative of the cost $J(\mathbf{v})$ can be calculated by assumption $\mathbf{A}(\Phi)$ and $\mathbf{A}(\mathrm{L})$. It means that the cost $J(\mathbf{v})$ is weak Gâteaux differentiable at $\mathbf{v}^*$ in the direction $\mathbf{v}-\mathbf{v}^*$. Therefore optimality condition $J'(\mathbf{v}^*)(\mathbf{v}-\mathbf{v}^*) \geq 0$ can be rewritten as

$$\begin{aligned} &\Phi'(y(T,\mathbf{v}^*))z(T) + \int_0^T L'_y(t,\mathbf{v}^*,y(t,\mathbf{v}^*))z(t)dt \\ &+ \int_0^T L'_{\mathbf{v}^*}(t,\mathbf{v}^*,y(t,\mathbf{v}^*))(\mathbf{v}-\mathbf{v}^*)dt \geq 0 \end{aligned} \tag{4.14}$$

for all $\mathbf{v} \in \mathcal{U}_{ad}$. Here $z(t) = Dy(\mathbf{v}^*)(\mathbf{v}-\mathbf{v}^*)$ is Gâteaux differential of $y(\mathbf{v})$ at $\mathbf{v}^*$ in the direction $\mathbf{v}-\mathbf{v}^*$.

In general, solve problem (ii) by introducing adjoint system, forming (4.14) to derive theorem on optimality condition.

Theorem 4.2. *Let $\boldsymbol{H}(\Phi)$, $\boldsymbol{H}(L)$, $\boldsymbol{A}(\Phi)$ and $\boldsymbol{A}(L)$ be satisfied, and assume that $y_0 \in L^2(\Omega)$. If supposing $y(t), t \in [0,T]$ as $p \leq p_n$, or $y(t), t \in [\eta, T]$ under assumption $(\mathbf{B})$. Then the optimal control $\mathbf{v}^* = (v_0^*, v_1^*, w^*) \in \mathcal{U}_{ad}$ for (1.3) is characterized by optimality system*

$$\begin{cases} \dfrac{\partial y}{\partial t} + \gamma\Delta^2 y - \lambda\Delta f(y) = B^0 v_0^* \quad \text{in} \quad Q, \\ \dfrac{\partial y}{\partial \eta} = 0, \ \dfrac{\partial(\Delta y)}{\partial \eta} = B^1 v_1^* \quad \text{on} \quad \Sigma, \\ y(0) = Bw^* \quad \text{on} \quad \Omega. \end{cases}$$

$$\begin{cases} -\dfrac{\partial \mathbf{p}(\mathbf{v}^*)}{\partial t} + \gamma\Delta^2\mathbf{p}(\mathbf{v}^*) - \lambda f'_y(y(\mathbf{v}^*))\Delta\mathbf{p}(\mathbf{v}^*) = L'_y(t,\mathbf{v}^*,y(\mathbf{v}^*)) \quad \text{in} \quad Q, \\ \dfrac{\partial \mathbf{p}(\mathbf{v}^*)}{\partial \eta} = 0, \quad \dfrac{\partial(\Delta\mathbf{p}(\mathbf{v}^*))}{\partial \eta} = 0 \quad \text{on } \Sigma, \\ \mathbf{p}(T,\mathbf{v}^*) = \Phi'(y(T,\mathbf{v}^*)) \quad \text{on } \Omega. \end{cases}$$

$$\begin{aligned} &\int_0^T \langle B^0(v_0 - v_0^*), \mathbf{p}(\mathbf{v}^*)\rangle dt + \int_0^T \left\langle \gamma B^1(v_1 - v_1^*), \mathbf{p}(\mathbf{v}^*)|_\Gamma \right\rangle_{H^{-\frac{3}{2}}(\Gamma), H^{\frac{3}{2}}(\Gamma)} dt \\ &+ (B(w-w^*), \mathbf{p}(0,\mathbf{v}^*))_H + \int_0^T L'_{\mathbf{v}}(t,\mathbf{v}^*,y(\mathbf{v}^*))(\mathbf{v}-\mathbf{v}^*)dt \geq 0, \end{aligned}$$

for all $\mathbf{v} = (v_0, v_1, w) \in \mathcal{U}_{ad}$.

Consider the distributed observation $z_1(\mathbf{v}) = \mathcal{C}_1 y(\mathbf{v})$ and terminal value observation $z_2(\mathbf{v}) = \mathcal{C}_2 y(T,\mathbf{v})$, where $\mathcal{C}_1 \in \mathcal{L}(W(0,T;\, V, V'), M_1)$ and $\mathcal{C}_2 \in \mathcal{L}(H, M_2)$ are operators, also called the observers, and M_1, M_2 are observation spaces. The cost function associated with (1.1) is given by

$$\begin{aligned} J(\mathbf{v}) = &\|\mathcal{C}_1 y(\mathbf{v}) - z_d\|^2_{M_1} \\ &+\|\mathcal{C}_2 y(T,\mathbf{v}) - z_d^T\|^2_{M_2} + (\mathbf{N}\mathbf{v},\mathbf{v})_{\mathcal{U}}, \ \ \forall\, \mathbf{v} \in \mathcal{U}. \end{aligned} \tag{4.15}$$

Here $z_d \in M_1, z_d^T \in M_2$ are desired values of $z_1(\mathbf{v})$ and $z_2(\mathbf{v})$ at t and T, respectively. $\mathbf{N} = (N_0, N_1, N) \in \mathcal{L}(\mathcal{U},\mathcal{U})$ is symmetric and positive operator,

$$(\mathbf{N}\mathbf{v},\mathbf{v})_{\mathcal{U}} = (N_0 v_0, v_0)_{V_0} + (N_1 v_1, v_1)_{V_1} + (Nw, w)_W.$$

Theorem 4.3. *Let the assumption in Theorem* 4.1 *be satisfied. Then the optimal control* $\mathbf{v}^* = (v_0^*, v_1^*, w^*) \in \mathcal{U}_{ad}$ *for cost function* (4.15) *is characterized by the following equalities and inequality, optimality system*

$$\begin{cases} \dfrac{\partial y}{\partial t} + \gamma\Delta^2 y - \lambda\Delta f(y) = B^0 v_0^* \quad \text{in} \ \ Q, \\ \dfrac{\partial y}{\partial \eta} = 0, \ \dfrac{\partial(\Delta y)}{\partial \eta} = B^1 v_1^* \quad \text{on} \ \ \Sigma, \\ y(0) = Bw^* \quad \text{on} \ \ \Omega, \end{cases}$$

$$\begin{cases} -\dfrac{\partial \mathbf{p}(\mathbf{v}^*)}{\partial t} + \gamma\Delta^2\mathbf{p}(\mathbf{v}^*) - \lambda f'(y(\mathbf{v}^*))\Delta\mathbf{p}(\mathbf{v}^*) \\ = \mathcal{C}_1^*\Lambda_{M_1}(\mathcal{C}_1 y(\mathbf{v}^*) - z_d) \quad \text{in}\ Q, \\ \dfrac{\partial \mathbf{p}(\mathbf{v}^*)}{\partial \eta} = 0, \dfrac{\partial(\Delta\mathbf{p}(\mathbf{v}^*))}{\partial \eta} = 0 \quad \text{on}\ \Sigma, \\ \mathbf{p}(T,\mathbf{v}^*) = \mathcal{C}_2^*\Lambda_{M_2}(\mathcal{C}_2 y(T,\mathbf{v}^*) - z_d^T) \quad \text{on}\ \Omega. \end{cases}$$

$$\begin{aligned} &\int_0^T \langle B^0(v_0 - v_0^*), \mathbf{p}(\mathbf{v}^*)\rangle dt \\ &+ \int_0^T \left\langle \gamma B^1(v_1 - v_1^*), \mathbf{p}(\mathbf{v}^*)|_\Gamma \right\rangle_{H^{-\frac{3}{2}}(\Gamma), H^{\frac{3}{2}}(\Gamma)} dt \\ &+(B(w - w^*), \mathbf{p}(0,\mathbf{v}^*))_H + (\mathbf{N}\mathbf{v}^*, \mathbf{v} - \mathbf{v}^*)_{\mathcal{U}} \geq 0, \end{aligned} \tag{4.16}$$

for all $\mathbf{v} = (v_0, v_1, w) \in \mathcal{U}_{ad}$.

If specifying that $\mathcal{U} = V^0 \times V^1 \times W = L^2(Q) \times L^2(\Sigma) \times L^2(\Omega)$, and

$$\begin{aligned} \mathcal{U}_{ad} &= V^0_{ad} \times V^1_{ad} \times W_{ad} \\ &= \{v_0 \mid v_0^a \leq v_0 \leq v_0^b,\ a.e. \text{ on } Q\} \times \{v_1 \mid v_1^a \leq v_1 \leq v_1^b,\ a.e. \text{ on } \Sigma\} \\ &\quad \times\{w \mid w^a \leq w \leq w^b,\ a.e. \text{ on } \Omega\} \end{aligned}$$

with $v_0^a,\ v_0^b \in L^\infty(Q),\ v_1^a,\ v_1^b \in L^\infty(\Sigma)$ and $w^a,\ w^b \in L^\infty(\Omega)$. Assume that $N = 0, B = B^0 = B^1 = I$ and $\mathcal{U}_{ad}$ is nonempty. Since $\mathcal{U}_{ad}$ is closed and convex in $\mathcal{U}$, then from the optimality condition (4.16), we have

$$\int_0^T \langle v_0(t) - v_0^*(t), \mathbf{p}(t, \mathbf{v}^*)\rangle dt + (w - w^*, \mathbf{p}(0, \mathbf{v}^*))_H + \int_0^T \Big\langle \gamma(v_1(t) - v_1^*(t)), \mathbf{p}(t, \mathbf{v}^*)|_\Gamma \Big\rangle_{H^{-\frac{3}{2}}(\Gamma), H^{\frac{3}{2}}(\Gamma)} dt \ \geq \ 0 \qquad (4.17)$$

for all $\mathbf{v} \in \mathcal{U}_{ad}$, where $\mathbf{v}^* = (v_0^*, v_1^*, w^*) \in \mathcal{U}_{ad}$. By setting $(v_0^*, v_1^*, w) \in \mathcal{U}_{ad}$ in (4.17), we get

$$(\mathbf{p}(0, \mathbf{v}^*), w - w^*)_{L^2(\Omega)} \geq 0, \quad \forall\, w \in W_{ad}.$$

By Lebesgue convergence theorem in [12] and (4.17), we have for a.e. $t \in [0, T]$,

$$(\mathbf{p}(t, \mathbf{v}^*), v_0(t) - v_0^*(t))_{L^2(\Omega)} \geq 0, \quad \forall\, v_0 \in V_{ad}^0,$$
$$(\mathbf{p}(t, \mathbf{v}^*)|_\Gamma, \gamma(v_1(t) - v_1^*(t)))_{L^2(\Gamma)} \geq 0, \quad \forall\, v_1 \in V_{ad}^1.$$

Then one can convert the following property of $\mathbf{v}^*$:

i) $w^*(x) = w^a(x)$ if $\mathbf{p}(0, \mathbf{v}^*, x) > 0$;
$w^*(x) = w^b(x)$ if $\mathbf{p}(0, \mathbf{v}^*, x) < 0$ for $x \in \Omega$.

ii) $v_0^*(t, x) = v_0^a(t, x)$ if $\mathbf{p}(t, \mathbf{v}^*, x) > 0$;
$v_0^*(t, x) = v_0^b(t, x)$ if $\mathbf{p}(t, \mathbf{v}^*, x) < 0$ for $(t, x) \in Q$.

iii) $v_1^*(t, \xi) = v_1^a(t, \xi)$ if $\mathbf{p}(t, \mathbf{v}^*, \xi) > 0$;
$v_1^*(t, \xi) = v_1^b(t, \xi)$ if $\mathbf{p}(t, \mathbf{v}^*, \xi) < 0$ for $(t, \xi) \in \Sigma$.

As is well known, it is Bang-Bang principle (cf. [12]) of optimal control $\mathbf{v}^* = (v_0^*, v_1^*, w^*)$.

5. Conclusions

In this paper, surveyed optimal control problem of distributed parameter system given by Cahn-Hilliard equation by the means of distributed control, initial control and Neumann boundary control. Further, for integral and quadratic cost function, existences of optimal control is straitforwardly proved under rational assumptions, and necessary optimality conditions are considered, respectively.

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