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FIXED POINT THEOREMS FOR ADMISSIBLE HYBRID CONTRACTION IN G-METRIC SPACES

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Abstract. In the present manuscript, we introduce a new notion of $(G-\beta-\psi)$-admissible hybrid contraction in G-metric spaces. In addition to this, some fixed point results are also proved for such type of contraction. An example is also provided to support the validity of our result. We further examine Ulam-type stability and well-posedness for the new contraction proposed herein. As an application of an integral equation is also solved by making use of our results.

1. Introduction

Fixed point theory simply deals with solution of the equation $fx = x$ where f is self-map on a nonempty set X. The fixed point problems first appeared in the solution of an initial value problem. In 1837, Liouville et al. [12] and in 1890, Picard et al. [17] solved the problem using successive approximation

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method and which also provided the solution of the fixed point equation. Before 1922, there was no any direct method to evaluate the fixed point of a map. Then in 1922, Banach [3] was the first one who introduced the contraction principle to evaluate the fixed point.

A new concept of hybrid contraction was recently presented by [9], combining and unifying some of the existing linear and nonlinear contractions in metric spaces. with related findings with generalized contractions, please refer to [10, 16] and its citations. However, Mustafa [13] established an expansion of metric space known as generalized metric space, or G-metric space, and established some fixed point results for contraction mappings of the Banach type. The idea was brought to the limelight by Mustafa and Sims [15]. Many scholars in the field of fixed point theory were subsequently drawn to Mustafa [14] fixed point results for Lipschitzian-type mappings on G-metric space [4, 6, 11, 20]. We observe that hybrid fixed point findings in G-metric spaces are not well explored, in accordance with the literature that is currently available. Jleli and Samet [7] noted that if a G-metric can be reduced to a quasi-metric, then the related fixed point results become the known fixed point results in the context of a quasi-metric space.

Motivated by the concepts in [5, 8, 9], we there by show several related fixed point theorems and introduce a novel notion of admissible hybrid $(G-\beta-\psi)$-contraction in G-metric space. To prove the validity of our result and that the major concepts gained here do not reduce to any known result in metric spaces, an example is shown. Finally, Ulam-type stability and well-posedness of this type of hybrid contraction in G-metric space are demonstrated.

In this section, we will present some fundamental notations and results that will be applied subsequently in our manuscript. Throughout, every set X is considered nonempty, $\mathbb{N}$ is the set of natural numbers, $\mathbb{R}$ represents the set of real numbers and $\mathbb{R}_+$ is the set of nonnegative real numbers.

2. Preliminaries

Definition 2.1. ([15]) Let X be a nonempty set and let $G : X \times X \times X \to \mathbb{R}_+$ be a function satisfying:

(1) $G(x, y, z) = 0$ if $x = y = z$;
(2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;
(4) $G(x, y, z) = G(x, z, y) = G(y, x, z)$(symmetry in all three variables);
(5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangular inequality).

The function G is called a generalized metric, or more generally, a G-metric and (X, G) is said to be a G-metric space.

Definition 2.2. ([14]) Let (X, G) be G-metric space and let $\{x_n\}_{n\in N}$ be G-convergent to x if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is, for any $\epsilon > 0$, there exist $n_0 \in N$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq n_0$. We refer to x as the limit of sequence $\{x_n\}_{n\in N}$.

Definition 2.3. ([14]) Let (X, G) be a G-metric space. Then the following are equivalent:

(1) $\{x_n\}_{n\epsilon N}$ is G-convergent to x;
(2) $G(x, x_n, x_m) \to 0$ as $n, m \to \infty$;
(3) $G(x_n, x, x) \to 0$ as $n \to \infty$;
(4) $G(x_n, x_n, x) \to 0$ as $n \to \infty$.

Definition 2.4. ([14]) Let (X, G) be a G-metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called G-Cauchy if given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq n_0$. That is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Definition 2.5. ([14]) In a G-metric space (X, G), the following are equivalent:

(1) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is G-Cauchy.
(2) For every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq n_0$.

Definition 2.6. ([14]) A G-metric space (X, G) is said to be G-complete (or complete G-metric) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Theorem 2.7. ([13]) *Let (X, G) be a complete G-metric space, and let $T : X \to X$ be a mapping satisfying the following condition*

$$G(Tx, Ty, Tz) \leq kG(x, y, z) \tag{2.1}$$

for all $x, y, z \in X$, where $0 < k < 1$. Then T has a unique fixed point (say u, that is, $Tu = u$), and T is G-continuous at u.

Consistent with [19], let ψ be the set of all function of Ψ such that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function with $\lim_{n\to\infty} \psi^n(t) = 0$ for all $n \in (0, \infty)$. If $\psi \in \Psi$, then ψ is called a Ψ-map. Let $\psi \in \Psi$ be a Ψ-map such that there exist $n_0 \in \mathbb{N}$, $k \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{n=1}^{\infty} v_n$ satisfying $\psi^{n+1}(t) \leq k\psi^n(t) + v_n$ for $n \geq n_0$ and $t > 0$. Then ψ is called a (c)-comparison function [1].

Lemma 2.8. ([1]) *If $\psi \in \Psi$, then the following hold:*

(1) $\{\psi^n(t)\}_{n\in\mathbb{N}}$ *converges to* 0 *as* $n \to \infty$ *for* $t \to 0$.
(2) $\psi(t) < t$ *for any* $t \in \mathbb{R}_+$;
(3) ψ *is continuous at* 0.

Popescu [18] gave the following definitions in the setting of metric spaces:

Definition 2.9. ([18]) Let $\beta : X \times X \to \mathbb{R}_+$ be a function. A self-mapping $T : X \to X$ is called β-orbital admissible if for all $x \in X$, $\beta(x, Tx) \geq 1$ implies $\beta(Tx, T^2x) \geq 1$.

Definition 2.10. ([18]) Let $\beta : X \times X \to \mathbb{R}_+$ be a function. A self-mapping $T : X \to X$ is called triangular β-orbital admissible if for all $x \in X$, T is β-orbital admissible and $\beta(x, y) \geq 1$ and $\beta(y, Ty) \geq 1$ implies $\beta(x, Ty) \geq 1$.

We modify the above definitions in the framework of G-metric space as follows:

Definition 2.11. Let $\beta : X \times X \times X \to \mathbb{R}_+$ be a function. A self-mapping $T : X \to X$ is called β_G-orbital admissible if for all $x \in X$, $\beta(x, Tx, T^2x) \geq 1$ implies $\beta(Tx, T^2x, T^3x) \geq 1$.

Definition 2.12. Let $\beta : X \times X \times X \to \mathbb{R}_+$ be a function. A self-mapping $T : X \to X$ is called triangular β_G-orbital admissible if for all $x \in X$, T is β_G-orbital admissible and $\beta(x, y, Ty) \geq 1$ and $\beta(y, Ty, T^2y) \geq 1$ implies $\beta(x, Ty, T^2y) \geq 1$.

Lemma 2.13. *Let $T : X \to X$ be a triangular β_G-orbital admissible mapping. If there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, Tx_0) \geq 1$, then*

$$\beta(x_n, x_m, x_l) \geq 1, \quad \forall\, n, m, l \in \mathbb{N}, \tag{2.2}$$

where the sequence $\{x_n\}_{n\in\mathbb{N}}$ is defined by $x_{n+1} = Tx_n, n \in \mathbb{N}$.

Proof. Since T is β_G-orbital admissible mapping and $\beta(x_0, Tx_0, T^2x_0) \geq 1$, then we deduce that $\beta(x_1, x_2, x_3) = \beta(Tx_0, Tx_1, Tx_2) \geq 1$. Continuing in this manner, we obtain $\beta(x_n, x_{n+1}, x_{n+2}) \geq 1$ for all $n \geq 1$. Assume that $\beta(x_n, x_m, x_{m+1}) \geq 1$, where $m > n$. Since T is triangular β_G-orbital admissible mapping and $\beta(x_m, x_{m+1}, x_{m+2}) \geq 1$, then clearly, $\beta(x_n, x_{m+1}, x_{m+2}) \geq 1$ for all $m, n \in \mathbb{N}$. This validates our assumption that $\beta(x_n, x_m, x_{m+1} \geq 1$. Letting $l = m + 1$ completes the proof. □

Definition 2.14. ([2]) Let $\beta : X \times X \times X \to \mathbb{R}_+$ be a mapping. The set X is called regular with respect to β if for a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\beta(x_n, x_{n+1}, x_{n+2}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, we have $\beta(x_n, x, x) \geq 1$ for all n.

3. Main results

We begin this section by defining the notion of admissible hybrid (β_G, ψ)-contraction in G metric space and then we shall prove some fixed point result by making use of this new contraction.

Definition 3.1. Let (X, G) be a G-metric spaces. A self-mapping $T : X \to X$ is called an admissible hybrid (β_G, ψ)-contraction. If there exist $\psi \in \Psi$ and $\beta : X \times X \times X \to \mathbb{R}_+$ such that

$$\beta(x, y, Ty)G(Tx, Ty, T^2y) \leq \psi(M(x, y, Ty)), \tag{3.1}$$

where

$$M(x,y,Ty) = \begin{cases} \left[\alpha_1\left(\frac{G(x,Tx,T^2x)G(y,Ty,T^2y)}{G(x,y,Ty)}\right)^s + \alpha_2(G(x,y,Ty))^s\right]^{\frac{1}{s}}, \\ \qquad for \quad s > 0, x, y \in X, \\ (G(x,Tx,T^2x))^{\alpha_1}(G(y,Ty,T^2y))^{\alpha_2}, \\ \qquad for \quad s = 0, x, y \in Fix(T))\}, \end{cases} \tag{3.2}$$

such that $\alpha_1 + \alpha_2 = 1$ and $\mathrm{Fix}(T) = \{x \in X : Tx = x\}$.

Theorem 3.2. *Let (X, G) be a complete G-metric space and Let $T : X \to X$ be an admissible hybrid (β_G, ψ)-contraction. Assume further that*

(1) *T is triangular β_G-orbital admissible; there exist $x_0 \in X$ such that $\beta(x, Tx_0, T^2x_0) \geq 1$;*
(2) *either T is continuous or T^3 is continuous;*
(3) *$\beta(x, Tx, T^2x) \geq 1$ for any $x \in Fix\ (T^3)$.*

Then T has a fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point and define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X by $x_n = T^nx_0$ for all $n \in \mathbb{N}$. Assume there exists some $m \in \mathbb{N}$ such that $Tx_m = x_{m+1} = x_m$. Then clearly, x_m is a fixed point of T. Assume now that $x_n \neq x_{n-1}$ for any $n \in \mathbb{N}$. Since T is an admissible hybrid (β_G, ψ)-contraction, then we have from (3.1) that

$$\beta(x_{n-1}, x_n, Tx_n)G(Tx_{n-1}, Tx_n, T^2x_n) \leq \psi(M(x_{n-1}, x_n, Tx_n)). \tag{3.3}$$

Owing to the fact that T is triangular β_G-orbital admissible together with Lemma 2.13 and equation (3.1), we have

$$\begin{aligned} G(x_n, x_{n+1}, Tx_{n+1}) &= G(x_n, x_{n+1}, x_{n+2}) \\ &\leq \beta(x_{n-1}, x_n, x_{n+1})G(Tx_{n-1}, Tx_n, Tx_{n+1}) \\ &< \psi\left(M\left(x_{n-1}, x_n, x_{n+1}\right)\right). \end{aligned} \tag{3.4}$$

We now consider the following cases:

Case 1: For $s > 0$, we have

$$
\begin{aligned}
M(x_{n-1}, x_n, Tx_n) &= \Big[\alpha_1\Big(\frac{(G(x_{n-1}, Tx_{n-1}, T^2x_{n-1})G(x_n, Tx_n, T^2x_n))}{G(x_{n-1}, x_n, Tx_n)}\Big)^s \\
&\quad + \alpha_2(G(x_{n-1}, x_n, Tx_n))^s\Big]^{\frac{1}{s}} \\
&= \Big[\alpha_1\Big(\frac{(G(x_{n-1}, x_n, x_{n+1})G(x_n, x_{n+1}, x_{n+2}))}{G(x_{n-1}, x_n, x_{n+1})}\Big)^s \\
&\quad + \alpha_2(G(x_{n-1}, x_n, Tx_n))^s\Big]^{\frac{1}{s}} \\
&= [\alpha_1(G(x_{n-1}, x_n, x_{n+1}))^s]^{\frac{1}{s}} x_{n+1}, x_{n+2}))^s \\
&\quad + \alpha_2(G(x_{n-1}, x_n, x_{n+1}))^s]^{\frac{1}{s}}.
\end{aligned}
$$

Since ψ is nondecreasing, if we assume that

$$G(x_{n-1}, x_n, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+2})$$

so from (3.4), we have

$$
\begin{aligned}
G(x_n, x_{n+1}, x_{n+2}) &\leq \beta(x_{n-1}, x_n, x_{n+1})G(Tx_{n-1}, Tx_n, Tx_{n+1}) \\
&\leq \psi([(\alpha_1 + \alpha_2)G(x_n, x_{n+1}, x_{n+2})^s]^{\frac{1}{s}}) \\
&= \psi((\alpha_1 + \alpha_2)^{\frac{1}{s}} G(x_n, x_{n+1}, x_{n+2})) \\
&< (\alpha_1 + \alpha_2)^{\frac{1}{s}} G(x_n, x_{n+1}, x_{n+2}) \\
&\leq G(x_n, x_{n+1}, x_{n+2}),
\end{aligned} \tag{3.5}
$$

which is a contradiction. Therefore, for every $n \in \mathbb{N}$, we have

$$G\left(x_n, x_{n+1}, x_{n+2}\right) < G\left(x_{n-1}, x_n, x_{n+1}\right)$$

so from (3.4), we have

$$
\begin{aligned}
G(x_n, x_{n+1}, x_{n+2}) &\leq \psi([(\alpha_1 + \alpha_2)G(x_n, x_{n+1}, x_{n+2})^s]^{\frac{1}{s}} \\
&= \psi((\alpha_1 + \alpha_2)^{\frac{1}{s}} G(x_{n-1}, x_n, x_{n+1})) \\
&= \psi(G(x_{n-1}, x_n, x_{n+1})) \\
&= \psi^2(G(x_{n-1}, x_n, x_{n+1})).
\end{aligned} \tag{3.6}
$$

Continuing inductively, we have

$$G(x_n, x_{n+1}, x_{n+2}) < \psi^n(G(x_0, x_1, x_2)). \tag{3.7}$$

Now, since

$$G(x_n, x_n, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+2}) \leq \psi^n(G(x_0, x_1, x_2)),$$

for all n in $\mathbb{N}$ with $x_{n+1} \neq x_{n+2}$, then for any $n, m \in \mathbb{N}$ with $n < m$ and by rectangular inequality, we have

$$\begin{aligned} G(x_n, x_n, x_m) \leq & G(x_n, x_n, x_{n+1}) + G(x_{n+1}, x_{n+1}, x_{n+2}) \\ & + \cdots + G(x_{m-1}, x_{m-1}, x_m) \\ \leq & (\psi^n + \psi^{n+1} + \psi^{n+2} + \cdots + \psi^{m-1}) G(x_0, x_1, x_2) \\ = & \sum_{i=1}^{m-1} \psi^i (G(x_0, x_1, x_2)) \\ \leq & \sum_{i=1}^{\infty} \psi^i (G(x_0, x_1, x_2)). \end{aligned} \tag{3.8}$$

Since ψ is a (c)-comparison function then the series $\sum_{i=1}^{\infty} \psi^i(G(x_0, x_1, x_2))$ is convergent. Hence, denoting by $S_p = \sum_{i=1}^{\infty} \psi^i(G(x_0, x_1, x_2)$, we have

$$G(x_n, x_n, x_m) \leq S_{m-1} - S_{n-1}.$$

Therefore, as $n, m \to \infty$, we see that

$$G(x_n, x_n, x_m) \to 0.$$

Thus $\{x_n\}_{n \in N}$ is G-convergent to z, that is,

$$\lim_{n \to \infty} G(x_n, x_n, z) = 0.$$

We will now show that z is a fixed point of T. By the assumption that T is continuous, we have

$$\begin{aligned} \lim_{n \to \infty} G(z, z, Tz) &= \lim_{n \to \infty} G(x_{n+1}, x_{n+1}, Tz) \\ &= \lim_{n \to \infty} G(Tx_n, Tx_n, Tz) \\ &= \lim_{n \to \infty} G(Tx_n, Tx_n, Tx_n) \\ &= 0. \end{aligned}$$

In order to demonstrate that $Tz = z$, assume $Tz \neq z$ on the contrary.
Then by (3.1) and Definition 2.3, we obtain

$$\begin{aligned} G(z, Tz, T^2 z) &\leq \beta(z, Tz, T^2 z) G(Tz, T^2 z, T^3 z) \\ &= \beta(z, Tz, T^2 z) G(Tz, T^2 z, z) \\ &= \psi(M((z, Tz, T^2 z) \\ &< M(z, Tz, T^2 z), \end{aligned}$$

where

$$\begin{aligned} M(z,Tz,T^2z) &= \Big[\alpha_1\Big(\frac{(G(z,Tz,T^2z)G(Tz,T^2z,T^3z))}{G(z,Tz,T^2z)}\Big)^s + \alpha_2(G(z,Tz,T^2z))^s\Big]^{\frac{1}{s}} \\ &= \alpha_1(G(z,Tz,T^2z))^s + \alpha_2(G(z,Tz,T^2z))^s]^{\frac{1}{s}} \\ &= [(\alpha_1+\alpha_2)(G(z,Tz,T^2z))^s]^{\frac{1}{s}} \\ &= (\alpha_1+\alpha_2)^{\frac{1}{s}}G(z,Tz,T^2z) \\ &\le G(z,Tz,T^2z), \end{aligned} \tag{3.9}$$

which is a contradiction. Hence, $Tz = z$.
Case 2: For $s = 0$, we have

$$\begin{aligned} M(x_{n-1},x_n,Tx_n) &= (G(x_{n-1},Tx_{n-1},T^2x_{n-1}))^{\alpha_1}(G(x_n,Tx_n,T^2x_n)^{\alpha_2} \\ &= (G(x_{n-1},x_n,x_{n+1})^{\alpha_1}(G(x_n,x_{n+1},x_{n+2})^{\alpha_2}. \end{aligned}$$

Now, if $G(x_{n-1},x_n,x_{n+1}) \le G(x_n,x_{n+1},x_{n+2})$, then

$$\begin{aligned} M(x_{n-1},x_n,Tx_n) &= (G(x_{n-1},x_n,x_{n+1}))^{\alpha_1}(G(x_n,x_{n+1},x_{n+2})^{\alpha_2} \\ &= (G(x_n,x_{n+1},x_{n+2}))^{\alpha_1+\alpha_2} \\ &\le G(x_n,x_{n+1},x_{n+2}), \end{aligned}$$

thus from (3.4), we get

$$G(x_n,x_{n+1},x_{n+2}) < G(x_n,x_{n+1},x_{n+2}),$$

which is a contradiction. Therefore,

$$\begin{aligned} G(x_n,x_{n+1},x_{n+2}) &< \psi(G(x_{n-1},x_n,x_{n+1})) \\ &< \psi^2(G(x_{n-1},x_n,x_{n+1})) \\ &< \psi^n(G(x_0,x_1,x_2)). \end{aligned}$$

By similar arguments in the case of $s > 0$, we can show that there exists a G Cauchy sequence $\{x_n\}_{n\in N} \subset (X,G)$, hence $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

To see that z is a fixed point of T, under the assumption that T is continuous, we have

$$\begin{aligned} \lim_{n\to\infty} G(z,z,Tz) &= \lim_{n\to\infty} G(x_{n+1},x_{n+1},Tz) \\ &= \lim_{n\to\infty} G(Tx_n,Tx_n,Tz) \\ &= \lim_{n\to\infty} G(Tx_n,Tx_n,Tx_n) \\ &= 0 \end{aligned}$$

and by the uniqueness of limit, $Tz = z$.

Similarly, if T^3 is continuous, as in case 1, we have that $T^3z = z$. Suppose on the contrary that $Tz \neq z$. Then

$$\begin{aligned} G(z,Tz,T^2z) &\leq \beta(z,Tz,T^2z)G((Tz,T^2z,T^3z). \\ &= \beta(z,Tz,T^2z)G(Tz,T^2z,z) \\ &< M(z,Tz,T^2z) \\ &= G(z,Tz,T^2z) \\ &= G(z,Tz,T^2z), \end{aligned} \tag{3.10}$$

which is a contradiction. Hence, $Tz = z$. □

Theorem 3.3. *If in Theorem* 3.2*, in the case of* $s > 0$*, we suppose supplementary that* (X,G) *is regular with respect to* $\alpha(x,y,Ty) \geq 1$ *for any* $x, y \in Fix(T)$*, then the fixed point of* T *is unique.*

Proof. Let $v, z \in Fix(T)$ be such that $v \neq z$. By replacing this in (3.1) and noting the additional hypothesis, we have

$$\begin{aligned} G(z,v,Tv) &\leq (z,v,Tv)G(Tz,Tv,T^2v) \\ &\leq \psi(M(z,v,Tv)) \\ &< M(z,v,Tv) \\ &= [\alpha_1(\frac{(G(z,Tz,T^2z)G(v,Tv,T^2v))}{G(z,v,Tv)})^s + \alpha_2(G(z,v,Tv))^s]^{\frac{1}{s}} \\ &= (\alpha_2(G(z,v,Tv))^s)^{\frac{1}{s}} \\ &= \alpha_2^{\frac{1}{s}}(G(z,v,Tv)) \\ &\leq G(z,v,Tv), \end{aligned}$$

which is a contradiction. Hence, $v = z$, and so, the fixed point of T is unique. □

Example 3.4. Let $X = [-1,1]$ and $G : X \times X \times X \to \mathbb{R}_+$be defined by $G(x,y,Ty) = |x-y| + |x-Ty| + |y-Ty|$ for all $x, y \in X$. Then (X,G) is a complete G-metric space. Define $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\psi(t) = \frac{t}{2}$ for all $t \geq 0$, $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{5}, & \text{if } x \in \{-1,1\} \\ \frac{1}{5}, & \text{if } x \in (-1,1) \end{cases}$$

for all $x \in X$ and $\beta : X \times X \times X \to \mathbb{R}_+$ by

$$\beta(x,y,Ty) = \begin{cases} 1, & \text{if } x,y \in \{-1\} \cup [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Then obviously, $\psi \in \Psi$, T is triangular β_G-orbital admissible, T is continuous for all $x \in X$ and T^3 continuous for any $x \in Fix(T^3)$. Also, there exist $x_0 = \frac{1}{2} \in X$ such that $\beta((\frac{1}{2}), T(\frac{1}{2}), T^2(\frac{1}{2})) = \beta(\frac{1}{2}, \frac{1}{5}, \frac{1}{5}) \geq 1$. Hence, condition (i)-(iv) of Theorem 3.2 are satisfied. To see that is an admissible hybrid (β_G, ψ)-contraction, notice that $\beta(x, y, Ty) = 0$ for all $x, y \in (-1, 0)$ and $x, y \in (-1, 1)$. Hence, inequality (3.1) holds for all $x, y \in (-1, 1)$.

Now for $x, y \in \{-1, 1\}$, if $x = y = 1$, then $G(Tx, Ty, T^2y) = 0$ for all $q \geq 0$. If $x = y = -1$, letting $\alpha_1 = 1, \alpha_2 = 0$ and $s = 2$, we obtain

$$\begin{aligned}
\beta(x,y,Ty)G(Tx,Ty,T^2y) &= \beta(-1,-1,-\frac{1}{5})G(-\frac{1}{5},-\frac{1}{5},\frac{1}{5}) \\
&= \frac{4}{5} \\
&< \frac{18}{10} \\
&= \frac{1}{2}(\frac{18}{5}) \\
&= \frac{1}{2}(M(-1,-1,-\frac{1}{5})) \\
&= \psi(M(x,y,Ty)).
\end{aligned}$$

Also, for $q = 0$, we take $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{2}$,

$$\begin{aligned}
\beta(x,y,Ty)G(Tx,Ty,T^2y) &= \frac{4}{5} \\
&< \frac{1}{2}(\frac{12}{5}) \\
&= \psi(M(x,y,Ty)).
\end{aligned}$$

If $x \neq y$, then letting $\alpha_1 = 0$, $\alpha_2 = 1$ and $s = 2$, we obtain

$$\begin{aligned}
\beta(x,y,Ty)G(Tx,Ty,T^2y) &= \beta(-1,1,\frac{1}{5})G(-\frac{1}{5},\frac{1}{5},\frac{1}{5}) \\
&= \beta(1,-1,-\frac{1}{5})G(\frac{1}{5},-\frac{1}{5},\frac{1}{5}) \\
&= \frac{4}{5} \\
&< \frac{4}{2} \\
&= \frac{1}{2}(4) = \frac{1}{2}(M(-1,1,\frac{1}{5})) \\
&= \frac{1}{2}(M(1,-1,-\frac{1}{5})) \\
&= \psi(M(x,y,Ty)).
\end{aligned}$$

Also, for $q = 0$, we take $\alpha_1 = 1, \alpha_2 = 0$. Then

$$\begin{aligned}(x,y,Ty)G(Tx,Ty,T^2y) &= \beta(-1,1,\frac{1}{5})G(-\frac{1}{5},\frac{1}{5},\frac{1}{5}) \\ &= \beta(1,-1,-\frac{1}{5})G(\frac{1}{5},-\frac{1}{5},\frac{1}{5}) \\ &= \frac{4}{5} \\ &< \frac{12}{10} \\ &= \frac{1}{2}(\frac{12}{5}) \\ &= \frac{1}{2}(M(-1,1,\frac{1}{5})) \\ &= \frac{1}{2}(M(1,-1,-\frac{1}{5})) \\ &= \psi(M(x,y,Ty)).\end{aligned}$$

Hence, inequality (3.1) is satisfied for all $x, y \in X$. Therefore, T is an admissible hybrid (β_G, ψ)-contraction which satisfies all the assumptions of Theorem 3.2 and $x = \frac{1}{5}$ is the fixed point of T.

Corollary 3.5. *Let (X, G) be a complete G-metric space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$, the mapping $T : X \to X$ satisfies:*

$$G(Tx, Ty, T^2y) \leq kG(x, y, Ty).$$

Then T has a fixed point in X.

4. Ulam-type stability

Ulam stability was introduced by Ulam [21, 22] and is seen as type of data dependence. This notion was further developed by Hyers and other researchers [9]. Karapinar and Fulga [6] investigated general Ulam-type stability in the sence of a fixed point problem in the framework of G-metric space. Suppose that $T : X \to X$ is a self-mapping in a G-metric space (X, G). Then we say that the fixed point problem

$$Tx = x \tag{4.1}$$

has the general Ulam–type stability if and only if there exists an increasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$, continuous at 0, $\mu(0) = 0$ such that for every $\epsilon > 0$ and for each $y' \in X$ which satisfies the inequality

$$(y', Ty', T^2y') \leq \epsilon, \tag{4.2}$$

there exists a solution $z \in X$ of (4.1) such that

$$G(z, y', Ty') \leq \mu(\epsilon). \tag{4.3}$$

For a positive number C, we take $\mu(t) = Ct$ for all $t \geq 0$. Then the fixed point of (4.1) is said to be Ulam–type stable. On a G-metric space (X, G), the fixed point problem (4.1) is said to be well-posed if the following assumptions are satisfied:

(1) T has a unique fixed point $z \in X$;
(2) $G(x_n, z, z) = 0$ for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that

$$\lim_{n\to\infty} G(x_n, Tx_n, T^2x_n) = 0.$$

Theorem 4.1. *Let (X, G) be a complete G-metric space. If in addition to the assumption of Theorem 3.3, we have $\alpha_2 < \frac{1}{K}$ where $K = \max\{1, 2^{q-1}\}$, then the following hold:*

(1) *The fixed point equation (4.1) is Ulam–Hyers stable if $\alpha(u, v, Tv) \geq 1$ for any u, v satisfying (4.2);*
(2) *The fixed point equation (4.1) is well–posed if $\alpha(z, Tx_n, T^2x_n) \geq 1$ for any $\{x_n\}_{n\in N}$ in X such that $\lim_{n\to\infty} G(x_n, Tx_n, T^2x_n) = 0$ and $Fix(T) = \{z\}$.*

Proof. (i) In Theorem 3.3, we have shown that there exists a unique $z \in X$ such that $Tz = z$. Let $y' \in X$ such that for any $\epsilon > 0$, we have

$$G(y', Ty', T^2y') \leq \epsilon.$$

Then obviously, z satisfies (4.2) and so we have $\beta(z, y', Ty') \geq 1$. Hence, by rectangular inequality,

$$\begin{aligned} G(z, y', Ty') &\geq G(z, Ty', T^2y') + G(T^2y', y', Ty') \\ &= G(Tz, Ty', T^2y') + G(y'Ty', T^2y') \\ &\leq \beta(z, y'Ty')G(Tz, Ty', T^2y') + G(y', Ty', T^2y') \\ &\leq \psi(M(z, y', Ty')) + G(y', Ty', T^2y') \\ &< M(z, y', Ty') + G(y', Ty', T^2y') \\ &= \Big[\alpha_1 \frac{(G(z, Tz, T^2z)G(y', Ty', T^2y'))^s}{G(z, y', Ty')} + \alpha_2(G(z, y', Ty'))^s\Big]^{\frac{1}{s}} \\ &\quad + G(y', Ty', T^2y') \end{aligned}$$

$$
\begin{aligned}
&=\Big[\alpha_1\frac{(G(z,z,z)G(y',Ty',T^2y'))^s}{G(z,y',Ty')}+\alpha_2(G(z,y',Ty'))^s\Big]^{\frac{1}{s}}\\
&\quad+G(y',Ty',T^2y')\\
&=(\alpha_2(G(z,y',Ty'))^s)^{\frac{1}{s}}+G(y',Ty',T^2y')\\
&=(\alpha_2(G(z,y',Ty'))^s)^{\frac{1}{s}}+\epsilon.
\end{aligned}
$$

Therefore, we have

$$(G(z,y',Ty'))^s\le K[\alpha_2(G(z,y',Ty')]^s+\epsilon^s,$$

where $K=\max\{1,2^{q-1}\}$. Hence, the above inequality reduces to

$$(G(z,y',Ty'))^s\le\frac{\epsilon^s}{1-K\alpha_2}.$$

Which is equivalent to

$$G\left(z,y',Ty'\right)\le C\epsilon,$$

where $C=\frac{1}{1-K\alpha_2}$ for $s>0,\alpha_2\in[0,1)$ such that $\alpha_2<\frac{1}{K}$.

(ii) Taking into account the supplementary condition and since $\mathrm{Fix}(T)=\{z\}$, then we have

$$
\begin{aligned}
G(z,x_n,Tx_n)&\le G(z,Tx_n,T^2x_n)+G(T^2x_n,x_n,Tx_n)\\
&=G(Tz,Tx_n,T^2x_n)+G(x_n,Tx_n,T^2x_n)\\
&\le\beta(z,x_n,Tx_n)G(Tz,Tx_n,T^2x_nt)+G(x_n,Tx_n,T^2x_n)\\
&\le\psi(M(z,x_n,Tx_n)+G(x_n,Tx_n,T^2x_n).\\
&<M(z,x_n,Tx_n)+G(x_n,Tx_n,T^2x_n)\\
&=\Big[\alpha_1\frac{(G(z,Tz,T^2z)G(x_n,Tx_n,T^2x_n))^s}{G(z,x_n,Tx_n)}+\alpha_2(G(z,x_n,Tx_n))^s\Big]^{\frac{1}{s}}\\
&\quad+G(x_n,Tx_n,T^2x_n)\\
&=\Big[\alpha_1\frac{(G(z,z,z)G(x_n,Tx_n,T^2x_n))^s}{G(z,x_n,Tx_n)}+\alpha_2(G(z,x_n,Tx_n))^s\Big]^{\frac{1}{s}}\\
&\quad+G(x_n,Tx_n,T^2x_n)\\
&=(\alpha_2(G(z,x_n,Tx_n))^s)^{\frac{1}{s}}+G(x_n,Tx_n,T^2x_n).
\end{aligned}
$$

Therefore, we have

$$(G(z,x_n,Tx_n))^s\le K\alpha_2(G(z,x_n,Tx_n))^s+(G(x_n,Tx_n,T^2x_n))^s,$$

where $K = \max\{1, 2^{q-1}\}$. Hence, the above inequality reduces to

$$G(z, x_n, Tx_n) \leq \frac{1}{1 - K\alpha_2} G(x_n, Tx_n, T^2x_n).$$

Letting $n \to \infty$ and keeping in mind Definition 2.3 and

$$\lim_{n\to\infty} G(x_n, Tx_n, T^2x_n) = 0,$$

then we obtain

$$\begin{aligned} Lim_{n\to\infty} G(x_n, z, z) &= \lim_{n\to\infty} G(z, x_n, Tx_n) \\ &\leq \lim_{n\to\infty} G(x_n, Tx_n, T^2x_n) \\ &= 0. \end{aligned}$$

That is, the fixed point equation (4.1) is well–posed. □

5. Applications to solution of integral equations

In this section, Corollary 3.5 is applied to examine the existence criteria for a solution to the following integral equation:

$$u(t) = h(t) + \int_a^b \omega(t, s) f(s, u(s)) ds, \quad t \in [a, b], \tag{5.1}$$

where $h : [a, b] \times \mathbb{R} \to \mathbb{R}$, $\omega : [a, b] \times [a, b] \to \mathbb{R}_+$ and $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ are given continuous function, and the function u is unknown.

Let $X = C([a, b], \mathbb{R})$ be the set of all real–valued continuous functions defined on $[a, b]$. We equip X with the mapping:

$$G(u, v, w) = \max_{a\leq t\leq b} (|u(t) - v(t)| + |u(t) - w(t)| + |v(t) - w(t)|). \tag{5.2}$$

Then, it is clear that (X, G) is a complete G–metric space. Consider the self mapping $T : X \to X$ is defined by

$$T(u(t)) = h(t) + \int_a^b \omega(t, s) f(s, u(s)) ds, \quad t \in [a, b]. \tag{5.3}$$

One can see that u^* is a fixed point of T if and only if u^* is a solution to (5.1).

Now, we study the existing conditions of the integral equation (5.1) under the following hypotheses:

Theorem 5.1. *Assume that the following conditions are satisfied:*

(1) $|f(s, x) - f(s, y)| + |f(s, x) - f(s, y)| + |f(s, y) - f(s, z)|$
$\leq |x - y| + |x - z| + |y - z|, \ \forall\, t \in [a, b], x, y, z \in \mathbb{R}$,

(2) $\max_{t\in[a,b]} \int_a^b |\omega(t, s)| ds = \eta < 1$.

Then, the integral equation (5.1) *has a solution in* X.

Proof. Taking (5.2) into account, we obtain

$$\begin{aligned}
G(Tu,Tv,T^2v) &= \max_{t\in[a,b]}(|\ Tu(t)-Tv(t)|+|Tu(t)-T^2v(t)|+|Tv(t)-T^2v(t)\ |) \\
&= \max_{t\in[a,b]}(|\int_a^b \omega(t,s)(f(s,u(s))-f(s,v(s))ds\ | \\
&+|\int_a^b \omega(t,s)(f(s,u(s))-f(s,Tv(s))ds\ | \\
&+|\int_a^b \omega(t,s)(f(s,v(s))-f(s,Tv(s))ds\ |) \\
&\leq \max_{t\in[a,b]}^{b}\int_a^b |\omega(t,s)|[|\ (f(s,u(s))-f(s,v(s)) \qquad (5.4)\\
&+|\ (f(s,u(s))-f(s,Tv(s))|+|(f(s,v(s))-f(s,Tv(s))\ |]ds \\
&\leq \max_{t\in[a,b]}\int_a^b |\omega(t,s)|[|u(s)-v(s)|+|u(s)-Tv(s)| \\
&+|v(s)-Tv(s)|]ds \\
&\leq (\max_{t\in[a,b]}\int_a^b |\omega(t,s)|ds)\max_{t\in[a,b]}^{b}\int_a^b[|u(s)-v(s)| \\
&+|u(s)-Tv(s)|+|v(s)-Tv(s)|]ds \\
&= \eta G(u,v,Tv).
\end{aligned}$$

Hence, all the conditions of Corollary 3.5 are satisfied. It follows that T has a fixed point u^* in X, which corresponds to a solution to the integral equation (5.1). □

Example 5.2. Let $X = C([0,1]),\mathbb{R})$ and consider

$$u(t) = \frac{t^2}{(3+t)} + \frac{1}{7}\int_0^1 \frac{s^2}{(3+t)}\frac{1}{(5+u(s))}ds, \quad t\in[0,1]. \tag{5.5}$$

To obtain the solution of (5.1), we prove that $u(t)$ is a fixed point of T, that is, $Tu(t) = u(t)$. Notice that the integral equation (5.4) is a special case of (5.1), where

$$h(t) = \frac{t^2}{(5+t)}, \quad \omega(t,s) = \frac{s^2}{(5+t)}, \quad f(s,t) = \frac{1}{2(5+u(s))}.$$

Obviously, the functions $h(t)$, $\omega(t,s)$ and $f(s,t)$ are continuous. Moreover, for all $u, v \in \mathbb{R}$,

$$\begin{aligned}
&| \ f(s,u) - f(s,v)| + |f(s,u) - f(s,Tv)| + |f(s,v) - f(s,Tv) \ | \\
&\quad = \Big|\frac{1}{2(5+u(s))} - \frac{1}{2(5+v(s))}\Big| + \Big|\frac{1}{2(5+u(s))} - \frac{1}{2(5+Tv(s))}\Big| \\
&\qquad + \Big|\frac{1}{2(5+v(s))} - \frac{1}{2(5+Tv(s))}\Big| \\
&\quad = \Big|\frac{v-u}{2(5+u(s))(2(5+v(s))}\Big| + \Big|\frac{Tv-u}{2(5+u(s))2(5+Tv(s))}\Big| \\
&\qquad + \Big|\frac{Tv-v}{2(5+v(s))2(5+Tv(s)}\Big| \\
&\quad \le \frac{1}{2}\Big(|u-v| + |u-Tv| + |v-Tv|\Big) \\
&\quad \le |u-v| + |u-Tv| + |v-Tv|.
\end{aligned}$$

Also, notice that

$$\begin{aligned}
\max_{t\in[0,1]} \int_0^1 |\omega(t,s)|ds &= \max_{t\in[0,1]} \int_0^1 |\frac{s^2}{(3+s)}|ds \\
&\le \frac{1}{9} \\
&< 1.
\end{aligned}$$

Hence, all the condition of Theorem 5.1 are satisfied. Therefore, the integral equation (5.1) has a solution in X.

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