Nonlinear Functional Analysis and Applications
Vol. 30, No. 3 (2025), pp. 665-680
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2025.30.03.02
http://nfaa.kyungnam.ac.kr/journal-nfaa

AN INERTIAL NEW ITERATION PROCESS IN HILBERT SPACES

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Abstract. In this paper, we prove some convergence theorems for generalized (α, β)-nonexpansive mapping in Hilbert space via new inertial iteration process. Our results generalized and extended the previous result of Iner PNSP (introduced by Samir et al. [11]). Also, we provide an example that numerically compares our new inertial iteration process with existing iteration processes Iner PNSP.

[0]Received July 18, 2024. Revised November 25, 2024. Accepted January 27, 2025.
[0]2020 Mathematics Subject Classification: 47H09, 47H10.
[0]Keywords: Generalized (α, β)-nonexpansive mapping, inertial iteration process, strong and weak convergence.
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1. Introduction

In the last half century, several mathematicians have studied approximation methods for fixed point problems and various iterations (see e.g., Mann iteration, Ishikawa iteration, Noor iteration, S-iteration, normal S-iteration, Picard normal S-iteration, etc.) for several classes of nonlinear mappings (see e.g., contraction, nonexpansive, generalized classes of nonexpansive mappings, etc.) to solve the mathematical problems such as convex optimization problems and variational inequalities problems (See reference [1]-[13], [25, 29]).

The inertial iteration method is used to compute fixed points for nonexpansive mappings from the algorithms as mentioned earlier. We find that the subsequent iteration of algorithms depends only on the previous iterate, but the inertial method's defining property is that the following iteration depends on multiple previous iterates.

In 2008, Mainge [20] studied the convergence of the inertial Mann algorithm by combining the Mann algorithm and the inertial extrapolation:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = w_n + \beta_n(Sw_n - w_n) \end{cases} \tag{1.1}$$

for $n \geq 1$, and study for speeding up the convergence of the given algorithm, he proved that the sequence x_n converges weakly to a fixed point of the mapping under certain assumptions and also applied the method to convex feasibility problems, fixed point problems, and monotone inclusions.

In 2018, Dong et al. [13] introduced a modified inertial Mann algorithm and an inertial CQ algorithm by unifying the accelerated Mann algorithm with the inertial extrapolation as follows:

Let $T : H \to H$ be nonexpansive mapping such that $Fix(T) \neq \phi$, choose $\mu \in (0, 1)$, $\lambda > 0$, $x_0, x_1 \in H$ arbitrarily and set $d_0 = \frac{(T(x_0)x_1)}{\lambda}$, compute d_{n+1} and x_{n+1} as follows:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ d_{n+1} = \frac{1}{\lambda}(T(w_n) - w_n) + \beta_n d_n, \\ y_n = w_n + \lambda d_{n+1}, \\ x_{n+1} = \mu\gamma_n w_n + (1-\mu)\gamma_n y_n \end{cases} \tag{1.2}$$

for $n \geq 1$ under some condition they proved that the sequence $\{x_n\}$ generated by the algorithm converges weakly to a fixed point of T, they also study an inertial CQ algorithm by combining the CQ-algorithm and the inertial extrapolation defined as follows:

Let H be a Hilbert space and $T : H \to H$ be a nonexpansive mapping such that $Fix(T) \neq \phi$. Let $\{\alpha_n\} \subset [\alpha_1, \alpha_2]$ where $\alpha_1 \in (-\infty, 0]$ and $\alpha_2 \in [0, \infty)$ and $\{\beta_n\} \subset [\beta_1, 1]$ where $\beta_1 \in (0, 1)$ set $x_0, x_1 \in H$ arbitrarily. Define the iterative sequence process

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = (1 - \beta_n)x_n + \beta_n T w_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|w_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases}$$

They showed that the sequence $\{x_n\}$ converges in norm $P_{Fix(T)}x_0$. In this study, they also performed numerical experiments to illustrate that the modified inertial Mann algorithm and inertial CQ algorithm significantly reduce the running time compared with some previous methods without the inertial extrapolation.

In 2019, Phon-on et al. [25] focus on a combination of modified S-iteration process and the inertial extrapolation to obtain a new method which accelerates the approximation of a fixed point of a nonexpansive mapping in a Banach space defined as follows:

Let H be a Banach space and $S_1, S_2 : H \to H$ be nonexpansive such that $F = Fix(S_1) \cap Fix(S_2) \neq \phi$. Define

$$\begin{cases} w_n = x_n + \gamma_n(x_n - x_{n-1}), \\ y_n = (1 - \beta_n)w_n + \beta_n S_1 w_n, \\ x_{n+1} = (1 - \alpha_n)S_1 w_n + \alpha_n S_2 y_n \end{cases}$$

for $n \geq 1$, where $\{\gamma_a\}, \{\beta_n\}$ and $\{\alpha_n\}$ satisfies the following conditions

- (D_1) $\sum_{n=1}^{\infty} \gamma_n < \infty, \gamma_n \subset [0, \gamma], 0 \leq \gamma < 1$, and $\{\alpha_n\}, \{\beta_n\} \subset [\delta, 1 - \delta]$ for $\delta \in (0, 0.5)$.
- (D_2) $\{(S_i(w_n - w_n)\}$ is bounded for $i = 1, 2$.
- (D_3) $\{(S_i(w_n - y)\}$ is bounded for $i = 1, 2$, and $y \in F$.

He proved that under some assumptions, the weak and strong convergent iteration process found the common fixed point and checked speeding up the convergence of the algorithm. He proved the convergence theorems of a sequence generated by our new method for finding a common fixed point of nonexpansive mappings in a Banach space. He also presents numerical examples to illustrate that the acceleration of our algorithm is effective.

In 2021, Samir et al. [11] introduced an inertial Picard normal S-iteration process (InerPNSP), by combining the inertial extrapolation and Picard normal S-iteration process and study convergence analysis for finding fixed points of the nonexpansive mapping defined as let $T : H \to H$ be nonexpansive mapping in a Banach space such that $Fix(T) \neq \phi$,

$$\begin{cases} w_n = x_n + \gamma_n(x_n - x_{n-1}), \\ z_n = (1-\beta_n)w_n + \beta_n T(w_n), \\ y_n = (1-\alpha_n)T(z_n) + \alpha_n T(z_n), \\ x_{n+1} = T(y_n), \end{cases}$$

where α_n, β_n, γ_n satisfy:

- (A_1) $\sum_{n=1}^{\infty} \gamma_n \subset [0, \gamma], 0 \leq \gamma < 1, \{\alpha_n\}, \{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 0.5)$;
- (A_2) $\{T(w_n) - w_n\}$ is bounded;
- (A_3) $\{T(w_n) - p\}$ is bounded.

In the fields of pure and applied mathematics as well as numerous other scientific disciplines, fixed point theory is extremely important (see [15, 16, 19, 27, 28] and the references therein). Finding the fixed points of nonlinear operators is one of the most important issues in operator theory (see [9, 10, 11]. Numerous fixed point problems can be used to simulate issues from a variety of fields, including image reconstruction and signal processing [6], variational inequalities [22], and convex feasibility problems [4].

2. Preliminaries

In this section, we summarized some notations, basic definition and fruitfully lemmas, which play significant role in the convergence of analysis of our algorithm.

We assume that X is a Hilbert space with the norm $\|.\|$, now we embrace the following notations.

1. The strong convergence of sequence $\{x_n\}$ to x is denoted by $x_n \to x$.
2. The weak convergence of sequence $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$.
3. The set of fixed point of a mapping T is denoted by $Fix(T) = \{x \in X : Tx = x\}$.

Definition 2.1. ([3],[17],[18],[20],[21],[26]) Let H be a nonempty subset of X and $T : H \to H$ be mapping. Then, it is said to be

(i) nonexpansive (NE), if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in H,$$

(ii) quasi-nonexpansive mapping (QNE), if
$$\|Tx - p\| \leq \|x - p\| \text{ for all } x \in H, \text{ and } p \in Fix(T).$$

(iii) mean nonexpansive (MNE), if there exists $\alpha, \beta \geq 0$ such that
$$\|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Ty\| \text{ for all } x, y \in H,$$

(iv) satisfy condition C (SC), if
$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in H,$$

(v) satisfy condition $(C\lambda)$, if
$$\lambda\|Tx - x\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in H,$$

(vi) generalized mean nonexpansive (GMNE), if there exists $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta < 1$ such that for all $x, y \in H$,
$$\lambda\|Tx - x\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Tx\|,$$

(vii) α-nonexpansive$(\alpha - NE)$, if there exists $\alpha < 1$ such that $x, y \in H$,
$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2.$$

It is worth mentioning that(NE) are continuous on their domains, but (MNE), (GMNE) mappings satisfying condition (C), Condition $(C\lambda)$ need not be continuous. Due to this fact, these mappings are more fascinating and applicable than nonexpansive mappings. Now a natural question arises.

Question 2.2. Does a class of mapping exist, that contains (MNE), (GMNE), mappings satisfying condition (C), condition $(C\lambda)$, (α-NE) mappings, and other nonexpansive type mappings in existence in the literature?

The affirmative answer to Question 2.2 as given by Akutsah and Narain [2] in 2021 by introducing a new class of mapping, namely, generalized (α, β)-nonexpansive mappings type as follows.

Definition 2.3. Let H be a nonempty subset of a Banach space X. A mapping $T : H \to H$ is said to be generalized (α, β)-nonexpansive $(G(\alpha, \beta) - NE)$ type 1, if there exists $\alpha, \beta, \lambda \in [0, 1)$ with $\alpha \leq \beta, \alpha + \beta < 1$ such that for all $x, y \in H$, with $\lambda\|Tx - x\| \leq \|x - y\|$, then
$$\|Tx - Ty\| \leq \alpha\|y - Tx\| + \beta\|x - Ty\| + (1 - (\alpha + \beta))\|x - y\|. \tag{2.1}$$

From Definition 2.3, it is easy to see that the following statements are true.

(a) $\alpha = \beta = 0$ and $\lambda = \frac{1}{2}$, then $(G(\alpha, \beta) - NE)$ type 1, satisfy condition (C).

(b) $\alpha=\beta=0$ and $\lambda\in[0,1)$, then $(G(\alpha,\beta)-NE)$ type 1, satisfy condition $(C\lambda)$.

(c) Every (NE), (MNE), condition (C), $(C\lambda)$ are $(G(\alpha,\beta)-NE)$ type 1 (see Proposition 3.4,[2]), but converse is not true (see [2, Example 3.5]).

Proposition 2.4. ([2, Proposition 3.6]) *Let H be a nonempty subset of a Banach space X and $T: H\to H$ be $(G(\alpha,\beta)-NE)$ type 1, $Fix(T)$ is nonempty then T is (QNE).*

Definition 2.5. ([24]) Assume that X is a Banach space and $\{x_n\}$ be a sequence in X such that $x_n\rightharpoonup x$. Then X is said to have Opial's property, if

$$\liminf_{n\to\infty}\|x_n-x\|\le\limsup_{n\to\infty}\|x_n-y\|,$$

for all $y\in X, y\neq x$.

Theorem 2.6. ([2, Theorem 3.15]) *Let H be a nonempty closed subset of a Banach space X with opial property and $T: H\to H$ be a $(G(\alpha,\beta)-NE)$ type 1 mapping, $\lambda=\frac{\gamma}{2},\gamma\in[0,1)$. If $x_n\rightharpoonup x$ and $\lim_{n\to\infty}\|Tx_n-x_n\|=0$, then T has a fixed point, that is, $(I-T)$ is demiclosed at zero, where I is the identity mapping on X.*

Theorem 2.7. ([2, Theorem 3.16]) *Let H be a nonempty compact subset of a Banach space X and $T: H\to H$ be a $(G(\alpha,\beta)-NE)$ type 1 mapping and $\lambda=\frac{\gamma}{2},\gamma\in[0,1)$. Then T has a fixed point in H if and only if T admits on almost fixed point sequence.*

Lemma 2.8. ([5]) *Assume that H is a real Hilbert space. Then following inequality holds:*

$$\|cx+(1-c)y\|^2 \quad\le c\|x\|^2+c\|y\|^2-c(1-c)\|x-y\|^2, \tag{2.2}$$

where $x,y\in H$.

Lemma 2.9. ([4]) *Assume that $\{r_n\}$, $\{d_n\}$ and $\{q_n\}$ are sequences in $[0,\infty)$ such that*

$$r_{n+1}\le r_n+q_n(r_n-r_{n-1})+d_n$$

for all $n\ge1$, $\sum_{n=1}^{\infty}d_n<\infty$ and there is real number q with $q_n<q<1$ for all $n\ge1$. Then we have

(1) $\sum_{n\geq1}[r_n-r_{n-1}]<\infty$, *where $[a]+=\max(a,0)$,*
(2) *there is $r^*\in[0,\infty)$ such that $\lim_{n\to\infty}r_n=r^*$.*

Lemma 2.10. ([23]) *Assume that X is a uniformly convex Banach space and $\{s_n\}$ is a sequence in $[\delta, 1-\delta]$ for $\delta \in (0,1)$. Assume that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $\liminf_{n\to\infty} \|x_n\| \le c$, $\liminf_{n\to\infty} \|y_n\| \le c$ and $\liminf_{n\to\infty} \|s_n x_n + (1-s_n)y_n\| = c$ for some $c \ge 0$. Then*

$$\liminf_{n\to\infty} \|x_n - y_n\| = 0.$$

Lemma 2.11. ([5]) *Assume that H is a nonempty closed convex subset of a Hilbert space H and $T : H \to H$ is a generalized (α, β)-nonexpansive mapping. Assume that $\{x_n\}$ is a sequence in H and $x \in H$ such that $x_n \to x$ as $n \to \infty$. Then $x \in Fix(T)$.*

Lemma 2.12. ([12]) *Assume that X is a Banach space with Opial's property. Assume that $\{x_n\}$ is a sequence in X and $x, y \in H$ such that $\lim_{n\to\infty} \|x_n - x\|$ and $\lim_{n\to\infty} \|x_n - y\|$ exist. If $\{x_{n_i}\}$ and $\{x_{n_j}\}$ are two subsequences of $\{x_n\}$ converge to x and y, respectively. Then $x = y$.*

3. Strong and week convergence theorem

In this section, we introduce the new inertial iteration process and study the convergence analysis for finding fixed points of generalized (α, β)-nonexpansive mapping in the framework of a Hilbert space.

Let H be a nonempty closed convex subset of a Hilbert space X and $T : H \to H$ be a generalized (α, β)-nonexpansive mapping with $Fix(T) \neq \phi$.

Algorithm 3.1. New inertial iteration algorithm:

(1) **Initialization:** Select x_0, x_1 arbitrarily.

(2) **Iteration Step:** Select $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ as iteration parameters in $[0,1]$ and compute $(n+1)^{th}$ iterative term as follows:

$$\begin{cases} w_n = x_n + \lambda_n(x_n - x_{n-1}), \\ z_n = (1-\beta_n)w_n + \beta_n T(w_n), \\ y_n = T(z_n), \\ x_{n+1} = (1-\alpha_n)T(z_n) + \alpha_n T(y_n), \end{cases} \tag{3.1}$$

where α_n, β_n, λ_n satisfy:

(C1) $\sum_{n=1}^{\infty} \lambda_n \subset [0, \lambda]$, $0 \le \lambda < 1$, $\{\alpha_n\}, \{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 0.5)$;

(C2) $\{T(w_n) - w_n\}$ is bounded;

(C3) $\{T(w_n) - p\}$ is bounded.

Theorem 3.2. *Let X be a Hilbert space. Let $p \in F = Fix(T)$. Let the sequence $\{x_n\}$ generated by 3.1 satisfying condition (A1), (A2) and (A3). Then*

(1) $\lim_{n\to\infty} \|x_n - p\|$ *exists.*

(2) $\lim_{n\to\infty} \|x_n - T(x_n) = 0\|$.

Proof. Since T is a generalized (α, β)-nonexpansive mapping, then by triangular inequality

$$\begin{aligned} \|z_n - p\| &= \|(1-\beta_n)w_n + \beta_n T(w_n) - p\| \\ &\leq (1-\beta_n)\|w_n - p\| + \beta_n\|T(w_n) - p\| \\ &\leq (1-\beta_n)\|w_n - p\| + \beta_n\|w_n - p\| \\ &= \|w_n - p\|, \ \forall n \in \mathbb{N}, \end{aligned} \tag{3.2}$$

from (3.1) and (3.2),

$$\begin{aligned} \|y_n - p\| &= \|T(z_n) - p\| \\ &\leq \|z_n - p\| \\ &\leq \|w_n - p\|, \end{aligned} \tag{3.3}$$

from (3.1), (3.2) and (3.3),

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)T(z_n) + \alpha_n T(y_n) - p\| \\ &\leq (1-\alpha_n)\|T(z_n) - p\| + \alpha_n\|T(y_n) - p\| \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|y_n - p\| \\ &\leq \|z_n - p\| \\ &\leq \|w_n - p\|. \end{aligned} \tag{3.4}$$

Now we will prove $\{w_n - p\}$ is bounded. By condition (A1) and (A2),

$$\begin{aligned} \|w_n - p\| &= \|w_n - T(w_n) + T(w_n) - p\| \\ &\leq \|T(w_n) - w_n\| + \|T(w_n) - p\| \\ &\leq K \end{aligned}$$

for some $K \in (0, \infty)$. Thus $\{w_n - p\}$ is bounded and by (3.4), $\{x_n - p\}$ and $\{x_n - x_{n-1}\}$ are bounded. By (2.2),

$$\begin{aligned} \|w_n - p\|^2 &= \|x_n + \lambda_n(x_n - x_{n-1} - p)\|^2 \\ &= \|(1+\lambda_n)(x_n - p) - \lambda_n(x_{n-1} - p)\|^2 \\ &= (1+\lambda_n)\|(x_n - p)\|^2 - \lambda_n\|(x_{n-1} - p)\|^2 \\ &\quad +\lambda_n(1+\lambda_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.5}$$

Using (3.4) and (3.5), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \|w_n-p)\|^2\\ &= \|(1+\lambda_n)(x_n-p)-\lambda_n(x_{n-1}-p)\|^2\\ &= (1+\lambda_n)\|(x_n-p)\|^2-\lambda_n\|(x_{n-1}-p)\|^2\\ &\quad +\lambda_n(1+\lambda_n)\|x_n-x_{n-1}\|^2. \qquad (3.6)\end{aligned}$$

Let $r_n=\|x_n-p\|^2$. Then by (3.6)

$$r_{n+1}\leq r_n+\lambda_n(r_n-r_{n-1})+d_n,$$

where $d_n=\lambda_n(1+\lambda_n)\|x_n-x_{n-1}\|^2$. By condition (A1),

$$\begin{aligned}\sum_{n=1}^{\infty} d_n &= \sum_{n=1}^{\infty}\lambda_n(1+\lambda_n)\|x_n-x_{n-1}\|^2\\ &\leq \sum_{n=1}^{\infty}\lambda(1+\lambda)(2M)^2\\ &< \infty.\end{aligned}$$

From Lemma 2.9, there is $r^*\in[0,\infty)$ such that $\lim_{n\to\infty} r_n=r^*$. Therefore, $\lim_{n\to\infty}\|x_n-p\|^2$ exists and hence $\lim_{n\to\infty}\|x_n-p\|$.

Now, we will prove $\lim_{n\to\infty}\|x_n-T(x_n)\|=0$. Assume that $c=\lim_{n\to\infty}\|x_n-p\|$. Since T is generalized (α,β)-nonexpansive,

$$\begin{aligned}\|x_n-T(x_n)\| &\leq \|x_n-p\|+\|T(x_n)-p\|\\ &\leq \|x_n-p\|+\|x_n-p\|\\ &= 2\|x_n-p\|. \qquad (3.7)\end{aligned}$$

If $c=0$, then by (3.7), $\|x_n-T(x_n)\|\to 0$ as $n\to 0$. Assume that $c>0$. Now $\sum_{n=1}^{\infty} d_n$ implies that $\lim_{n\to\infty} d_n=0$. From (3.5), we have

$$\begin{aligned}\lim_{n\to\infty}\|w_n-p\|^2 &= \lim_{n\to\infty}(1+\lambda_n)\|x_n-p\|^2-\lambda_n\|x_{n-1}-p\|^2\\ &\quad +\lambda_n(1+\lambda_n)\|x_n-x_{n-1}\|^2\\ &= \lim_{n\to\infty}\|x_n-p\|^2\\ &= c^2,\end{aligned}$$

which implies $\lim_{n\to\infty}\|w_n-p\|=c$. Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\|y_n-p\| &\leq \limsup_{n\to\infty}\|w_n-p\|\\ &= c. \qquad (3.8)\end{aligned}$$

Now, we claim that $\liminf_{n\to\infty} \|y_n - p\| \geq c$. Since T is generalized (α, β)-nonexpansive mapping, by (3.1)

$$\begin{aligned} \|x_{n+1} - p\| &= \|T(y_n) - p\|. \\ &\leq \|y_n - p\|. \end{aligned}$$

On taking limit inferior both sides

$$\begin{aligned} \liminf_{n\to\infty} \|x_{n+1} - p\| &\leq \liminf_{n\to\infty} \|y_n - p\| \\ c &\leq \liminf_{n\to\infty} \|y_n - p\|. \end{aligned} \tag{3.9}$$

By (3.8) and (3.9)

$$\lim_{n\to\infty} \|y_n - p\| = c.$$

Now, by (3.2),

$$\begin{aligned} \limsup_{n\to\infty} \|z_n - p\| &\leq \limsup_{n\to\infty} \|w_n - p\| \\ &= c. \end{aligned} \tag{3.10}$$

Since T is generalized (α, β)-nonexpansive mapping, by (3.1)

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)T(z_n) + \alpha_n T(y_n) - p\| \\ &\leq (1-\alpha_n)\|T(z_n) - p\| + \alpha_n\|T(y_n) - p\| \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|y_n - p\| \\ &\leq \|z_n - p\|. \end{aligned}$$

On taking limit inferior both sides

$$\liminf_{n\to\infty} \|x_{n+1} - p\| \leq \liminf_{n\to\infty} \|z_n - p\|,$$

it implies that

$$c \leq \liminf_{n\to\infty} \|z_n - p\|. \tag{3.11}$$

By (3.10) and (3.11),

$$\lim_{n\to\infty} \|z_n - p\| = c.$$

Now, we have

$$\limsup_{n\to\infty} \|T(w_n) - p\| \leq \limsup_{n\to\infty} \|w_n - p\| \leq c,$$

$$\limsup_{n\to\infty} \|(1-\beta_n)(w_n - p) + \beta_n(T(w_n) - p\| \leq \limsup_{n\to\infty} \|z_n - p\| \leq c,$$

by Lemma 2.10,

$$\|T(w_n) - w_n\| = 0. \tag{3.12}$$

Now, we have

$$\limsup_{n\to\infty} \|T(z_n) - p\| \le \limsup_{n\to\infty} \|z_n - p\| \ \le c,$$

$$\limsup_{n\to\infty} \|(1-\alpha_n)(z_n - p) + \alpha_n(T(z_n) - p\| \le \limsup_{n\to\infty} \|z_n - p\| \ \le c,$$

by Lemma 2.10,

$$\|T(z_n) - z_n\| = 0.$$

Now, since $z_n - w_n = \beta_n(T(w_n) - w_n)$, by (3.12),

$$\begin{aligned} 0 &\le \lim_{n\to\infty} \|z_n - w_n\| \\ &= \lim_{n\to\infty} \beta_n \|T(w_n) - w_n\| \\ &\le \lim_{n\to\infty} \|T(w_n) - w_n\| \\ &= 0. \end{aligned} \tag{3.13}$$

Now, since $y_n - z_n = \alpha_n(T(z_n) - z_n)$, by (3.12),

$$\begin{aligned} 0 &\le \lim_{n\to\infty} \|y_n - z_n\| \\ &= \lim_{n\to\infty} \alpha_n \|T(z_n) - z_n\| \\ &\le \lim_{n\to\infty} \|T(z_n) - z_n\| \\ &= 0. \end{aligned} \tag{3.14}$$

Now, by (3.13) and (3.14),

$$\begin{aligned} 0 &\le \lim_{n\to\infty} \|y_n - w_n\| \\ &= \lim_{n\to\infty} \|y_n - z_n| + \lim_{n\to\infty} \|z_n - w_n\|| \\ &= 0. \end{aligned} \tag{3.15}$$

Now since $w_n - x_n = \lambda_n(x_n - x_{n-1})$, by (3.4),

$$\begin{aligned} 0 &\le \lim_{n\to\infty} \|w_n - x_n\| \\ &= \lim_{n\to\infty} \lambda_n \|x_n - x_{n-1}\| \\ &= 0. \end{aligned} \tag{3.16}$$

Now, by (3.12) and (3.16),

$$\begin{aligned} 0 &\le \lim_{n\to\infty} \|Tw_n - x_n\| \\ &\le \lim_{n\to\infty} \|Tw_n - w_n\| + \lim_{n\to\infty} \|w_n - x_n\| \\ &= 0. \end{aligned} \tag{3.17}$$

Using (3.15) and (3.16),

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \|x_n - y_n\| \\ &\leq \lim_{n\to\infty} \|x_n - w_n\| + \lim_{n\to\infty} \|w_n - y_n\| \\ &= 0. \end{aligned} \tag{3.18}$$

Now, since T is a generalized (α, β)-nonexpansive mapping and using (3.13), (3.14), (3.17), (3.18), we have

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \|Tx_n - x_n\| \\ &= \lim_{n\to\infty} \|Tx_n - Ty_n\| + \lim_{n\to\infty} \|Ty_n - Tz_n\| \\ &\quad + \lim_{n\to\infty} \|Tz_n - Tw_n\| + \lim_{n\to\infty} \|Tw_n - x_n\| \\ &= \lim_{n\to\infty} \|x_n - y_n\| + \lim_{n\to\infty} \|y_n - z_n\| \\ &\quad + \lim_{n\to\infty} \|z_n - w_n\| + \lim_{n\to\infty} \|Tw_n - x_n\| \\ &= 0. \end{aligned}$$

Therefore, we have $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. □

Theorem 3.3. *Assume that H is a Hilbert space. Also assume that $T : H \to H$ is a generalized (α, β)-nonexpansive mapping with $F = Fix(T) \neq \phi$. Then the sequence $\{x_n\}$ generated by (3.1) weakly converges to a fixed point of T.*

Proof. Assume that $p \in F$. Then from Theorem 3.2, $\lim_{n\to\infty} \|x_n - p\|$ exists, therefore $\{x_n\}$ is bounded. Assume that $\{x_{n_i}\}$ and $\{x_{n_j}\}$ are two subsequences of the sequence $\{x_n\}$ with weak limits p_1 and p_2, respectively. Again by Theorem 3.2, $\lim_{n\to\infty} \|x_{n_i} - T(x_{n_i})\| = 0$ and $\lim_{n\to\infty} \|x_{n_j} - T(x_{n_j})\| = 0$. Since every Hilbert space has Opial's property (see [24]) and by Lemma 2.11, $T(p_1) = p_1$ and $T(p_2) = p_2$, that is, $p_1, p_2 \in F$. From Theorem 3.2, $\lim_{n\to\infty} \|x_n - p_1\|$ and $\lim_{n\to\infty} \|x_n - p_2\|$ exist and both sequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ weakly converge to p_1 and p_2, respectively. From Lemma 2.12, $p_1 = p_2$. Thus $\{x_n\}$ converges weakly to a fixed point of T. □

4. Numerical experiments

Example 4.1. Define a mapping $T : [0, 1] \to [0, 1]$ as

$$T(x) = \begin{cases} 1 - x \text{ if } 0 \leq x < \frac{1}{8}, \\ \frac{x+7}{8} \text{ if } \frac{1}{8} \leq x \leq 1. \end{cases}$$

Then, it is easy to see that T satisfies condition (C) thus it is a generalized (α, β)-nonexpansive mapping.

Figure 1, and Figure 2 show the convergence behavior of our new iteration process and InerPNSP where $\alpha_n = \frac{1}{\sqrt{n^3 0+40}}, \beta_n = \frac{1}{\sqrt{n^1 0+50}}$ and $\lambda_n = \frac{3}{300n^3}$ for different initial values. we numerically compare our new iteration process with existing iterative processes InerPNSP.

Case I : Taking $x_0 = 0$ and $x_1 = 0.1$.
Case II : Taking $x_0 = 0.125$ and $x_1 = 0.8$.

Figure 1. Convergence graph of ***FSRIP*** and ***FMIP*** for $t_n = \frac{1}{\sqrt{n^3+1}}$
1.PNG

(a) Tabulated iterative values

Iter	ISRIP	S-Iter	Normal S	Ishikawa	Mann
1	0.1	0.1	0.1	0.1	0.1
2	0.980988	0.926309	0.932221	0.545463	0.542229
3	0.988458	0.992079	0.993668	0.844118	0.82715
4	0.996334	0.999065	0.999333	0.965059	0.95123
5	0.999835	0.999886	0.999925	0.994475	0.988729
6	0.999993	0.999986	0.999991	0.999338	0.997723
7	1	0.999998	0.999999	0.999937	0.999581
8	1	1	1	0.999995	0.999928
9	1	1	1	1	0.999988
10–20	1	1	1	1	1

(b) Convergence Graph

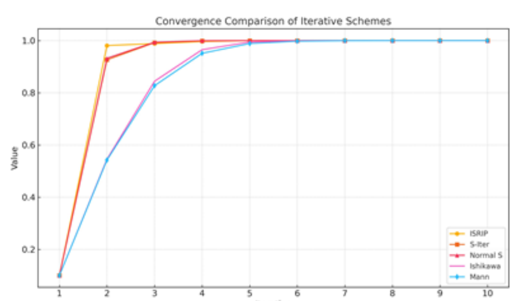

Figure 1: Comparison table and graph for initial values $x_0 = 0$, $x_1 = 0.1$.

Figure 2. Convergence graph of ***FSRIP*** and ***FMIP*** for $t_n = \frac{1}{\sqrt{n^3+1}}$
2.PNG

(a) Tabulated Iterative Values

Iter	ISRIP	S-Iter	Normal S	Ishikawa	Mann
1	0.125	0.125	0.125	0.125	0.125
2	0.8	0.8	0.8	0.8	0.8
3	0.999321	0.982988	0.984783	0.898957	0.896738
4	0.999507	0.998171	0.998578	0.965348	0.961009
5	0.996766	0.999784	0.999850	0.992233	0.988999
6	0.999999	0.999974	0.999983	0.998772	0.997457
7	1	0.999997	0.999998	0.999853	0.999486
8	1	1	1	0.999986	0.999905
9	1	1	1	0.999999	0.999984
10–20	1	1	1	1	1

(b) Convergence Graph

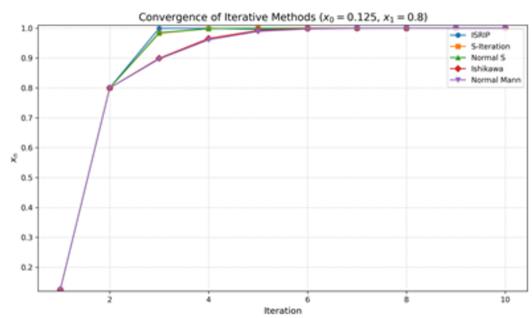

Figure 2: Comparison table and graph for initial values $x_0 = 0.125$, $x_1 = 0.8$.

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