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FIXED POINTS FOR VARIANTS OF COMPATIBLE MAPPINGS WITH APPLICATION IN DYNAMIC PROGRAMMING

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Abstract. In this paper, we obtain some common fixed point theorems for pairs of compatible mappings of type (R), type (E) and type (K) satisfying generalized (ψ, ϕ)-weak contraction involving cubic terms of metric functions. We also provide examples for the validity of our results and as applications, we obtain the existence and uniqueness of common solutions of certain functional equations arising in dynamic programming.

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1. Introduction

The importance of fixed point theorem lies in finding solutions of many problems of applied sciences, engineering and economics. The popular mathematician in the area of fixed point theory was Banach [3], who established a famous theorem popularly known as Banach contraction principle. Thereafter, researchers formulated and established many contractive conditions to modify Banach contraction principle. It is worth mentioning that the most appreciable work in this direction was due to Jungck [16], when he used the notion of commutative mappings. See [2, 8, 9, 24, 25] for various contractions and their applications.

Jungck's common fixed result is the simplest fixed point result for a pair of mappings. This result is too strong and it is natural to seek for weaker assumptions. Further, Jungck [17] weakened the notion of commutative mappings and weak commutative mappings, given by Sessa [34], to compatible mappings. In 1993, Jungck, Murthy and Cho [18] generalized the notion of compatible mappings to compatible mappings of type (A). The process of generalizing the concept of compatible mappings still going on. Pathak and Khan [29], Pathak *et al.* [27, 28], Rohen and Singh [32], Singh and Singh [35] and Jha *et al.* [14] weakened this concept of compatible mappings to compatible mappings of type (B), type (P), type (C), type (R), type (E) and type (K) respectively. See [7, 21, 30, 33, 38] for more informations on compatible mappings and applications.

Another direction of generalization of Banach contraction principle concerns with the use of control function. In 1969, Boyd and Wong [5] generalized Banach contraction principle by introducing ϕ contraction condition of the form $d(Tu, Tv) \leq \phi(d(u, v))$ for all $u, v \in E$, where T is a self mapping of a complete metric space E and $\phi : [0, \infty) \to [0, \infty)$ is an upper semi continuous function from right such that $0 \leq \phi(t) < t$ for all $t > 0$. The function ϕ appeared in the ϕ contraction is known as control function. The idea of control function was given by Khan *et al.* [20] as follows: an increasing continuous function $\phi : [0, \infty) \to [0, \infty)$ vanishing only at zero is known as control function. Further, Alber and Guerre-Delabriere [1] generalized ϕ contraction to $\phi-$weak contraction in Hilbert spaces, which was further extended by Rhoades [31] in the setting of complete metric space as follows: A self-mapping T defined on a complete metric space is said to be a ϕ-weak contraction if for each $u, v \in E$, there exists a continuous non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ vanishing only at zero such that $d(Tu, Tv) \leq d(u, v) - \phi(d(u, v))$.

In 2009, Zhang and Song [39] proved the following fixed point theorem for pair of mappings satisfying generalized $\phi-$weak contraction.

Theorem 1.1. ([39]) *Let (E, d) be a metric space and $\phi : [0, \infty) \to [0, \infty)$ be a lower semi-continuous function with $\phi(t) > 0$, for all $t > 0$ and $\phi(0) = 0$. Let (S, T) be a pair of self-mappings defined on E such that*

$$d(Su, Tv) \leq M(u, v) - \phi(M(u, v)) \quad \text{for all } u, v \in E,$$

where $M(u, v) = \max\{d(u, v), d(u, Su), d(v, Tv), \frac{d(u,Tv)+d(v,Su)}{2}\}$. Then S and T have a unique common fixed point in E.

In 2013, Murthy and Prasad [23] proved a fixed point theorem for a mapping satisfying a weak contraction involving cubic terms of metric function.

Theorem 1.2. ([23]) *Let T be a self-mapping on a complete metric space (E, d) and $\phi : [0, \infty) \to [0, \infty)$ be a continuous function with $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) > 0$ for each $t > 0$ such that*

$$\begin{aligned}[1+pd(u, v)]d^2(Tu, Tv) \leq & p \max\Big\{\frac{1}{2}[d^2(u, Tu)d(v, Tv) + d(u, Tu)d^2(v, Tv)], \\ & d(u, Tu)d(u, Tv)d(v, Tu), d(u, Tv)d(v, Tu)d(v, Tv)\Big\} \\ & + m(u, v) - \phi(m(u, v)),\end{aligned}$$

where

$$\begin{aligned}m(u, v) = \max\Big\{& d^2(u, v), d(u, Tu)d(v, Tv), d(u, Tv)d(v, Tu), \\ & \frac{1}{2}[d(u, Tu)d(u, Tv) + d(v, Tu)d(v, Tv)]\Big\},\end{aligned}$$

and $p \geq 0$ is a real number. Then T has a unique fixed point in E.

Jain *et al.* [11, 12, 13], Jain and Kumar [10], Jung *et al.* [15], Kumar and Kumar [22] had extended and generalized Theorem 1.2 for various types of commuting and minimal commuting mappings in metric spaces. In 2022, Kavita and Kumar [19] generalized the results of Jain *et al.* [10, 11, 13] and Murthy and Prasad [23] by introducing a generalized $(\psi, \phi)-$weak contraction involving cubic terms of metric functions.

Main purpose of this paper is to establish the existence of fixed point for pairs of compatible mappings of type (R), type (E) and type (K) using newly introduced generalized (ψ, ϕ)-weak contraction along with continuity and reciprocal continuity that improves Theorem 1.2 and the results of Jain *et al.* [11, 12, 13] and Jung *et al.* [15] and many results cited in the literature.

2. Preliminaries

Now, we recall some definitions and results which will be needed in the sequel.

Definition 2.1. Let (E, d) be a metric space. Two mappings $S, T : E \to E$ are said to be

(i) compatible [17] if

$$\lim_{n\to\infty} d(STu_n, TSu_n) = 0,$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$;

(ii) compatible of type (A) [18] if

$$\lim_{n\to\infty} d(SSu_n, TSu_n) = 0 \text{ and } \lim_{n\to\infty} d(TTu_n, STu_n) = 0,$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$;

(iii) compatible of type (P) [29] if

$$\lim_{n\to\infty} d(SSu_n, TTu_n) = 0,$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$;

(iv) compatible of type (B) [29] if

$$\lim_{n\to\infty} d(STu_n, TTu_n) \leq \frac{1}{2}[\lim_{n\to\infty} d(STu_n, Sz) + \lim_{n\to\infty} d(Sz, SSu_n)]$$

and

$$\lim_{n\to\infty} d(TSu_n, SSu_n) \leq \frac{1}{2}[\lim_{n\to\infty} d(TSu_n, Tz) + \lim_{n\to\infty} d(Tz, TTu_n)],$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$;

(v) compatible of type (C) [28] if

$$\lim_{n\to\infty} d(STu_n, TTu_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(STu_n, Sz) + \lim_{n\to\infty} d(Sz, SSu_n) + \lim_{n\to\infty} d(Sz, TTu_n)]$$

and

$$\lim_{n\to\infty} d(TSu_n, SSu_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(TSu_n, Tz) + \lim_{n\to\infty} d(Tz, TTu_n) + \lim_{n\to\infty} d(Tz, SSu_n)],$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$;

(vi) compatible of type (R) [32] if

$$\lim_{n\to\infty} d(STu_n, TSu_n) = 0$$

and

$$\lim_{n\to\infty} d(SSu_n, TTu_n) = 0,$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$;

(vii) compatible of type (E) [35] if

$$\lim_{n\to\infty} SSu_n = \lim_{n\to\infty} STu_n = Tz$$

and

$$\lim_{n\to\infty} TTu_n = \lim_{n\to\infty} TSu_n = Sz,$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$;

(viii) compatible of type (K) [14] if

$$\lim_{n\to\infty} SSu_n = Tz \text{ and } \lim_{n\to\infty} TTu_n = Sz,$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$.

Remark 2.2. Notion of compatible mappings of type (R) is a combination of the notion of compatible mappings and compatible mappings of type (P), but it is stronger than compatible mappings and compatible mappings of type (P) (see [32, Examples 1 and 2]).

Remark 2.3. If $Sz = Tz$, then compatible of type (E) implies compatible, compatible of type (A), type (B), type (C) and type (P), however the converse may not be true (see [36, Example 2.4]).

Remark 2.4. If $Sz \neq Tz$, then 'compatible of type (E)' is neither compatible nor compatible of type (A), type (C), type (P) (see [36, Example 2.3]).

In 1999, Pant [26] introduced the notion of reciprocally continuity as follows.

Definition 2.5. ([26]) A pair (S, T) of self mappings of a metric space (E, d) is said to be reciprocally continuous, if $\lim_{n\to\infty} STu_n = Sz$ and $\lim_{n\to\infty} TSu_n = Tz$, whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$.

Remark 2.6. It is clear that a pair of continuous self mappings is reciprocally continuous, but the converse may not be true (see [26]).

Remark 2.7. Compatibility and reciprocal continuous are independent of each other (see [37]).

In 2011, Singh and Singh [36] split the concept of compatible mappings of type (E) to the concept of $S-$compatible mappings of type (E) and T-compatible mappings of type (E) and further, split the notion of reciprocally continuity to the notion of $S-$reciprocally continuous and T-reciprocally continuous.

Definition 2.8. ([36]) Let (E, d) be a metric space and $S, T : E \to E$ be two mappings. If $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$, then the pair (S, T) is said to be

(1) $S-$compatible of type (E) if $\lim_{n\to\infty} SSu_n = \lim_{n\to\infty} STu_n = Tz$;
(2) $T-$compatible type (E) if $\lim_{n\to\infty} TTu_n = \lim_{n\to\infty} TSu_n = Sz$;
(3) $S-$ reciprocally continuous if $\lim_{n\to\infty} STu_n = Sz$;
(4) $T-$ reciprocally continuous, if $\lim_{n\to\infty} TSu_n = Tz$.

Remark 2.9. Compatible of type (E) implies both S-compatible of type (E) and T-compatible of type (E), however the converse may not true, see the example given below.

Example 2.10. Let $E = [0, 5]$ and d be a usual metric. Let $S, T : E \to E$ be two mappings defined as $Su = 5, Tu = 1$, for $u \in [0, \frac{5}{2}] - \{\frac{5}{4}\}$, $Su = 0, Tu = 5$, for $u = \frac{5}{4}$ and $Su = \frac{5-u}{2}, Tu = \frac{u}{2}$, for $u \in (\frac{5}{2}, 5]$. Clearly, S and T are not continuous at $u = \frac{5}{2}, \frac{5}{4}$. Assume that $u_n \to \frac{5}{2}$, $u_n > \frac{5}{2}$, for all n. Then $Su_n = \frac{5-u_n}{2} \to \frac{5}{4} = t$ and $Tu_n = \frac{u_n}{2} \to \frac{5}{4} = t$. Therefore, we have $SSu_n = S(\frac{5-u_n}{2}) = 5 \to 5$, $STu_n = S(\frac{u_n}{2}) = 5 \to 5, Tt = 5$ and $TTu_n = T(\frac{u_n}{2}) = 1 \to 1$, $TSu_n = T(\frac{5-u_n}{2}) = 1 \to 1$, $St = 0$. Thus, the pair (S, T) is S-compatible of type (E), but not compatible of type (E).

Remark 2.11. The reciprocal continuity of the pair (S, T) implies both S-reciprocal continuity and T-reciprocal continuity, however, the converse may not be true, see example given below.

Example 2.12. Let $E = [0, 5]$ and d be a usual metric. Let $S, T : E \to E$ be two mappings defined as $Su = 5, Tu = 0$ for $u \in [0, \frac{5}{2})$ and $Su = 5-u, Tu = u$, for $u \in [\frac{5}{2}, 5]$. Let $\{u_n\}$ be a sequence in E such that $u_n \to \frac{5}{2}$, $u > \frac{5}{2}$, for all n. Then $Su_n = 5 - u_n \to \frac{5}{2}$, $Tu_n = u_n \to \frac{5}{2} = t$, $STu_n = S(u_n) = 5 - u_n \to \frac{5}{2}$, $St = \frac{5}{2}$ and $TSu_n = T(5-u_n) = 0 \to 0$, $Tt = \frac{5}{2}$. It follows that $\lim_{n\to\infty} STu_n = \frac{5}{2} = St$ and $\lim_{n\to\infty} TSu_n = 0 \neq Tt = \frac{5}{2}$. Therefore, the pair (S,T) is S-reciprocally continuous, but it is neither $T-$reciprocally continuous nor reciprocally continuous.

Now, we present propositions which are useful for our work.

Proposition 2.13. ([32]) *Let (E,d) be a metric space and $S,T : E \to E$ be mappings such that the pair (S,T) is compatible of type (R).*

(i) *If $Sz = Tz$, then $STz = SSz = TTz = TSz$ for some $z \in E$.*
(ii) *If $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$, then*
  (a) *$\lim_{n\to\infty} TSu_n = Sz$, when S is continuous at z,*
  (b) *$\lim_{n\to\infty} STu_n = Tz$, when T is continuous at z,*
  (c) *$STz = TSz$ and $Sz = Tz$, when S and T are continuous at z.*

Proposition 2.14. ([36]) *Let S and T be two self-mappings defined on a metric space (E,d) and $\{u_n\}$ be a sequence in E such that*

$$\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z,$$

for some $z \in E$. Assume that one of the following conditions is satisfied:

(i) *the pair (S,T) is S-compatible of type (E) and S-reciprocally continuous,*
(ii) *the pair (S,T) is T-compatible of type (E) and T-reciprocally continuous.*

Then we have

(a) *$Sz = Tz$,*
(b) *if there exists $t \in E$ such that $St = Tt = z$, then $STt = TSt$.*

3. Main results

In this section, we study compatible mappings of type (R), type (E) and type (K) to establish the existence and uniqueness of fixed point for pairs of compatible mappings of type (R), type (E) and type (K) by using the control function $\psi \in \Psi$, where Ψ is a collection of all nondecreasing, upper semi continuous (in each coordinate variables) functions $\psi : [0,\infty)^4 \to [0,\infty)$ such that $\max\{\psi(t,t,0,0), \psi(0,0,0,t), \psi(0,0,t,0), \psi(t,t,t,t)\} \leq t$, for each $t > 0$.

Let Φ be a collection of all continuous functions $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) > 0$ for each $t > 0$ and $\phi(0) = 0$.

Let (E, d) be a metric space and f, g, S and T be self-mappings defined on E satisfying the following conditions:

(C_1) $S(E) \subset g(E)$ and $T(E) \subset f(E)$,
(C_2) for all $u, v \in E$, there exist a function $\psi \in \Psi$, a function $\phi \in \Phi$ and a real number $p > 0$ such that

$$\begin{aligned}[1+&pd(fu, gv)]d^2(Su, Tv)\\ &\leq p\psi\Bigg(d^2(fu, Su)d(gv, Tv), d(fu, Su)d^2(gv, Tv),\\ &\qquad d(fu, Su)d(fu, Tv)d(gv, Su), d(fu, Tv)d(gv, Su)d(gv, Tv)\Bigg)\\ &\quad + m(fu, gv) - \phi(m(fu, gv)),\end{aligned}$$

where

$$\begin{aligned}m(fu, gv) = \max\Big\{&d^2(fu, gv), d(fu, Su)d(gv, Tv), d(fu, Tv)d(gv, Su),\\ &\frac{1}{2}[d(fu, Su)d(fu, Tv) + d(gv, Su)d(gv, Tv)]\Big\}.\end{aligned}$$

Let $u_0 \in E$ be an arbitrary point. Using containment condition (C_1), one can find points $u_1, u_2 \in E$ such that $Su_0 = gu_1 = v_0$ and $Tu_1 = fu_2 = v_1$. Continuing in this manner, one can construct sequences such that

$$v_{2n} = Su_{2n} = gu_{2n+1},\ v_{2n+1} = Tu_{2n+1} = fu_{2n+2} \tag{3.1}$$

for all $n = 0, 1, 2, 3, \cdots$.

First, we establish the existence of fixed point for pairs of compatible mappings of type (R).

Theorem 3.1. *Let (E, d) be a complete metric space. Let (f, S) and (g, T) be pairs of compatible mappings of type (R) defined on E satisfying the conditions (C_1) and (C_2). Also, if one of S, T, f and g is continuous, then f, g, S and T have a unique common fixed point in E.*

Proof. Following proof of [19, Theorem 2.2], the sequence $\{v_n\}$ defined by the equation (3.1) is a Cauchy sequence in E. Since (E, d) is a complete metric space, $v_n \to w \in E$, as $n \to \infty$. Consequently, the subsequences $\{Su_{2n}\}$, $\{fu_{2n}\}$, $\{Tu_{2n+1}\}$, and $\{gu_{2n+1}\}$ also converge to the same point w.

Suppose that f is continuous. Then $\{ffu_{2n}\}$ and $\{fSu_{2n}\}$ converge to fw as $n \to \infty$. Since f and S are compatible mappings of type (R) on E, it follows from Proposition 2.13(ii) that $\{Sfu_{2n}\}$ converges to fw as $n \to \infty$.

Now, we claim that $fw = w$. For this, letting $u = fu_{2n}$, $v = u_{2n+1}$ and letting $n \to \infty$ in (C_2), we get

$$[1 + pd(fw, w)]d^2(fw, w) \le p\psi(0, 0, 0, 0) + m(fw, w) - \phi(m(fw, w)),$$

where

$$m(fw, w) = \max\Big\{d^2(fw, w), d(fw, fw)d(w, w), d(fw, w)d(w, fw), \frac{1}{2}[d(fw, fw)d(fw, w) + d(w, fw)d(w, w)]\Big\} = d^2(fw, w).$$

By using the value of $m(fw, w)$ with the property of ϕ and ψ, the above inequality reduces to $pd^3(fw, w) + \phi(d^2(fw, w)) \le 0$, which is possible only if $d(fw, w) = 0$, that is, $fw = w$.

Next, we show that $Sw = w$. Taking $u = w$ and $v = u_{2n+1}$ in (C_2) and letting $n \to \infty$, we get

$$[1 + pd(fw, w)]d^2(Sw, w) \le p\psi(0, 0, 0, 0) + m(w, w) - \phi\big(m(fw, w)\big),$$

where

$$m(fw, w) = \max\big\{d^2(fw, w), d(fw, Sw)d(w, w), d(fw, w)d(w, Sw), \frac{1}{2}\big[d(fw, Sw)d(fw, w) + d(w, Sw)d(w, w)\big]\big\} = 0.$$

After simplification, we get $d^2(Sw, w) = 0$. This implies that $Sw = w$. Therefore, $fw = w = Sw$, that is, w is a common fixed point of f and S.

Since $S(E) \subset g(E)$, there exists a point $u^* \in E$ such that $w = Sw = gu^*$. Now, we show that $Tu^* = w$. Letting $u = w$ and $v = u^*$ in (C_2), we have

$$[1 + pd(w, gu^*)]d^2(w, Tu^*) \le p\psi(0, 0, 0, 0) + m(fw, gu^*) - \phi\big(m(fw, gu^*)\big),$$

where

$$\begin{aligned} &m(fw, gu^*) \\ &\quad = \max\big\{d^2(fw, gu^*), d(fw, Sw)d(gu^*, Tu^*), d(fw, Tu^*)d(gu^*, Sw), \\ &\qquad \frac{1}{2}\big[d(fw, Sw)d(fw, Tu^*) + d(gu^*, Sw)d(gu^*, Tu^*)\big]\big\} = 0. \end{aligned}$$

Simplifying the above inequality, we conclude that $Tu^* = w$.

Since (g, T) is a pair of compatible mappings of type (R) in E and $gu^* = w = Tu^*$, by Proposition 2.13(i), we have $gTu^* = Tgu^*$ and hence $gw = gTu^* = Tgu^* = Tw$. Next, we prove that w is a fixed point of g, i.e., $gw = w$. For this, taking $u = v = w$ in (C_2), we have

$$[1 + pd(w, gw)]d^2(w, Tw) \le p\psi(0, 0, 0, 0) + m(w, gw) - \phi\big(m(w, gw)\big),$$

where

$$m(w, gw) = \max\big\{d^2(w, gw), 0, d(w, Tw)d(gw, w), 0\big\} = d^2(w, gw).$$

Solving the above inequality, we get $d(w, gw) = 0$, which implies that $gw = w$. Thus $w = gw = Tw = fw = Sw$, that is, w is a common fixed point of S, T, f and g.

In a similar way, we can also complete the proof, when the mapping g is considered to be continuous.

Assume that S is continuous. Then $\{SSu_{2n}\}$ and $\{Sfu_{2n}\}$ converge to Sw as $n \to \infty$. Since the pair (f, S) is compatible of type (R) on E, it follows from Proposition 2.13(ii) that $\{fSu_{2n}\}$ converges to Sw as $n \to \infty$. Now, we claim that $w = Sw$. Taking $u = Su_{2n}$, $v = u_{2n+1}$ and letting $n \to \infty$ in (C_2) and using the definitions of ϕ and ψ, we have

$$[1 + pd(Sw, w)]d^2(Sw, w) \leq p\psi(0, 0, 0, 0) + m(Sw, w) - \phi(m(Sw, w)),$$

where

$$m(Sw, w) = \max\Big\{d^2(Sw, w), d(Sw, w)d(w, w), d(Sw, w)d(w, Sw), \frac{1}{2}[d(Sw, Sw)d(Sw, w) + d(w, Sw)d(w, w)]\Big\} = d^2(Sw, w).$$

Simplifying the above inequality, we get $d^2(Sw, w) = 0$, that is, $Sw = w$.

Since $S(E) \subset g(E)$, for w, there exists a point $u' \in E$ such that $w = Sw = gu'$. We claim that $w = Tu'$. Letting $u = Su_{2n}$ and $v = u'$ in (C_2) and taking the limit as $n \to \infty$, we get

$$[1 + pd(w, w)]d^2(w, Tu') \leq p\psi(0, 0, 0, 0) + 0 - \phi(0), \text{ i.e., } d^2(w, Tu') \leq 0,$$

which is possible only when $d(w, Tu') = 0$. This implies that $Tu' = w$. Since the pair (g, T) is compatible type (R) on E and $gu' = Tu' = w$, by Proposition 2.13(i), we have $gTu' = Tgu'$. Hence $gw = gTu' = Tgu' = Tw$. Putting $u = u_{2n}$, $v = w$ and taking $n \to \infty$ in (C_2), we get

$$[1 + pd(w, Tw)]d^2(w, Tw) \leq p\psi(0, 0, 0, 0) + m(w, Tw) - \phi(m(w, Tw)),$$

where

$$m(w, Tw) = \max\Big\{d^2(w, Tw), 0, d(w, Tw)d(Tw, w), 0\Big\} = d^2(w, Tw).$$

Solving the above inequality, we obtain $w = Tw$.

Since $T(E) \subset f(E)$, for w, there exists a point $v' \in E$ such that $w = Tw = fv'$. Next, we claim that $w = Sv'$. Putting $u = v'$ and $v = w$ in (C_2), we get

$$[1 + pd(fv', gw)]d^2(Sv', Tw) \leq p\psi(0, 0, 0, 0) + m(fv', gw) - \phi\big(m(fv', gw)\big),$$

where

$$m(fv', gw) = \max\Bigg\{d^2(fv', gw), d(fv', Sv')d(gw, Tw), d(fv', Tw)d(gw, Sv'), \frac{1}{2}\big[d(fv', Sv')d(fv', Tw) + d(gw, Sv')d(gw, Tw)\big]\Bigg\} = 0.$$

After simplification, we conclude that $d(Sv', w) = 0$, which gives that $Sv' = w$. Since the pair (S, f) is compatible of type (R) on E and $Sv' = fv' = w$, by Proposition 2.13(i), we have $fSv' = Sfv'$, that is, $fw = fSv' = Sfv' = Sw$. Hence $w = fw = Sw = Bw = Tw$, that is, w is a common fixed point of S, T, f and g. Similarly, we can complete the proof when T is continuous.

Next, we have to show the uniqueness of fixed point. Suppose $w_1 \neq w_2$ are two common fixed points of S, T, f and g. Taking $u = w_1$ and $v = w_2$ in (C_2), we get

$$\begin{aligned}&[1 + pd(fw_1, gw_2)]d^2(Sw_1, Tw_2)\\ &\quad \leq p\psi(0,0,0,0) + m(fw_1, gw_2) - \phi\big(m(fw_1, gw_2)\big),\end{aligned}$$

where

$$\begin{aligned}&m(fw_1, gw_2)\\ &= \max\Bigg\{d^2(fw_1, gw_2), d(fw_1, Sw_1)d(gw_2, Tw_2), d(fw_1, Tw_2)d(gw_2, Sw_1),\\ &\qquad \frac{1}{2}\big[d(fw_1, Sw_1)d(fw_1, Tw_2) + d(gw_2, Sw_1)d(gw_2, Tw_2)\big]\Bigg\}\\ &= d^2(w_1, w_2).\end{aligned}$$

Solving the above inequality, we get $pd^3(w_1, w_2) + \phi(d^2(w_1, w_2)) \leq 0$, which is a contradiction to the definition of ϕ and p. Hence our assumption is wrong and $w_1 = w_2$, that is, fixed point is unique. This completes the proof. ☐

Next, we prove fixed point theorem for compatible mappings of type (E) along with split reciprocal continuity as follows.

Theorem 3.2. *Self-mappings f, g, S and T of a complete metric space (E, d) satisfying conditions (C_1) and (C_2) have a unique common fixed point in E, if the pairs (f, S) and (g, T) satisfy either of the following conditions:*

(a) *(f, S) is f-compatible of type (E) and f-reciprocally continuous, (g, T) is g-compatible of type (E) and $g-$reciprocally continuous.*
(b) *(f, S) is S-compatible of type (E) and S-reciprocally continuous, (g, T) is T-compatible of type (E) and T-reciprocally continuous.*

Proof. Following proof of [19, Theorem 2.2], the sequence $\{v_n\}$ defined by (3.1), is a Cauchy sequence in E. Since (E, d) is a complete metric space, $\{v_n\}$ converges to a point $w \in E$. Consequently, the subsequences $\{Su_{2n}\}$, $\{fu_{2n}\}$, $\{Tu_{2n+1}\}$, and $\{gu_{2n+1}\}$ also converge to the same point w.

Suppose that pair (f, S) is $f-$compatible of type (E) and f-reciprocally continuous. Then by Proposition 2.14, $fw = Sw$. Since $S(E) \subset g(E)$, there exists a point $u^* \in E$ such that $Sw = gu^*$, that is, $fw = Sw = gu^*$. Now, we claim that $Tu^* = gu^*$. For this, taking $u = w$, $v = u^*$ in (C_2), we get

$$[1 + pd(fw, gu^*)]d^2(Sw, Tu^*) \leq p\psi(0,0,0,0) + m(fw, gu^*) - \phi(m(fw, gu^*)),$$

where

$$m(fw, gu^*) = \max\Big\{d^2(fw, gu^*), d(fw, Sw)d(gu^*, Tu^*), d(fw, Tu^*)d(gu^*, Sw), \frac{1}{2}[d(fw, Sw)d(fw, Tu^*) + d(gu^*, Sw)d(gu^*, Tw)]\Big\} = 0.$$

Using the value of $m(fw, gu^*)$ along with the property of ϕ and ψ, the above inequality reduces to $d^2(Sw, Tu^*) \leq 0$. This is true only if $d(Sw, Tu^*) = 0$, which further gives $Tu^* = Sw = gu^*$.

Since the pair (g, T) is $g-$compatible of type (E) and $g-$reciprocally continuous and $gu^* = Tu^*$, by Proposition 2.14, $gw = gTu^* = Tgu^* = Tw$. We claim that w is a fixed point of f, that is, $fw = w$. Letting $u = w$ and $v = u_{2n+1}$ in (C_2) and taking the limit as $n \to \infty$, we have

$$[1 + pd(fw, w)]d^2(Sw, w) \leq p\psi(0,0,0,0) + m(fw, w) - \phi(m(fw, w)),$$

where

$$m(fw, w) = \max\Big\{d^2(fw, w), d(fw, Sw)d(w, w), d(fw, w)d(w, Sw), \frac{1}{2}[d(fw, Sw)d(fw, w) + d(w, Sw)d(w, w)]\Big\} = d^2(fw, w).$$

Solving the above inequality, we get $pd^3(fw, w) + \phi(d^2(fw, w)) \leq 0$. This is true only if $d(fw, w) = 0$, which implies that $fw = w$. Therefore, we have $w = fw = Sw$.

Now, we prove that w is a fixed point of g. Putting $u = v = w$ in (C_2), we get

$$[1 + pd(w, gw)]d^2(w, Tw) \leq p\psi(0,0,0,0) + m(w, gw) - \phi(m(w, gw)),$$

where

$$m(w, gw) = \max\Big\{d^2(w, gw), d(w, w)d(gw, Tw), d(w, Tw)d(gw, w), \frac{1}{2}[d(w, w)d(w, Tw) + d(gw, w)d(gw, Tw)]\Big\} = d^2(w, gw).$$

Simplifying, we get $d(w, gw) = 0$, which implies that $w = gw$. Thus, $w = fw = Sw = gw = Tw$, that is, w is a common fixed point of f, g, S and T.

Similarly, one can complete the proof when the pairs (f, S) and (g, T) satisfy the condition (b). Uniqueness follows easily. This completes the proof. □

Now, we prove fixed point theorem for reciprocally continuous pairs of compatible mappings of type (K).

Theorem 3.3. *Let f, g, S and T be self-mappings defined on a complete metric space (E, d) satisfying the conditions (C_1) and (C_2). Then S, T, f and g have a unique common fixed point in E, provided that (f, S) and (g, T) are the pairs of reciprocally continuous and compatible mappings of type (K).*

Proof. Following proof of [19, Theorem 2.2], the sequence $\{v_n\}$, defined by (3.1), is a Cauchy sequence in E. Since (E, d) is a complete metric space, $v_n \to w \in E$. Consequently, the subsequences $\{Su_{2n}\}$, $\{fu_{2n}\}$, $\{Tu_{2n+1}\}$, and $\{gu_{2n+1}\}$ also converge to the same point w.

Since the mappings f and S are compatible of type (K), $ffu_{2n} \to Sw$, $SSu_{2n} \to fw$ as $n \to \infty$. Also reciprocal continuity of the pair (f, S) implies that $fSu_{2n} \to fw$ and $Sfu_{2n} \to Sw$.

Similarly, compatibility of type (K) along with reciprocal continuity of the pair (g, T) implies that $ggu_{2n} \to Tw$, $TTu_{2n} \to gw$, $gTu_{2n} \to gw$ and $Tgu_{2n} \to Tw$ as $n \to \infty$.

Now, we claim that $gw = fw$. Taking $u = Su_{2n}$, $v = Tu_{2n+1}$ and letting $n \to \infty$ in (C_2), we get

$$[1 + pd(fw, gw)]d^2(fw, gw) \leq p\psi(0, 0, 0, 0) + m(fw, gw) - \phi(m(fw, gw)),$$

where

$$\begin{aligned} m(fw, gw) = \max\Big\{ & d^2(fw, gw), d(fw, fw)d(gw, gw), d(fw, gw)d(gw, fw), \\ & \frac{1}{2}[d(fw, fw)d(fw, gw) + d(gw, fw)d(gw, gw)]\Big\} \\ = {} & d^2(fw, gw). \end{aligned}$$

Solving the above inequality, we get $d(fw, gw) = 0$, which implies that $fw = gw$.

Next, we prove that $gw = Sw$. Letting $u = w$ and $v = Tu_{2n+1}$ and taking the limit as $n \to \infty$ in (C_2), we get

$$\begin{aligned} & [1 + pd(fw, gw)]d^2(Sw, gw) \\ & \quad \leq p\psi\Big(d^2(fw, Sw)d(gw, gw), d(fw, Sw)d^2(gw, gw), \\ & \qquad d(fw, Sw)d(fw, gw)d(gw, Sw), d(fw, gw)d(gw, Sw)d(gw, gw)\Big) \\ & \qquad + m(fw, gw) - \phi\big(m(fw, gw)\big), \end{aligned}$$

where

$$m(fw,gw) = \max\Big\{d^2(fw,gw), d(fw,Sw)d(gw,gw), d(fw,gw)d(gw,Sw), \frac{1}{2}\big[d(fw,Sw)d(fw,gw) + d(gw,Sw)d(gw,gw)\big]\Big\} = 0.$$

Simplifying the above inequality, we get $d(Sw,gw) = 0$, hat is, $Sw = gw$. So $fw = gw = Sw$.

Next, we claim that $Sw = Tw$. Putting $u = v = w$ in (C_2), we have

$$\begin{aligned}
&[1 + pd(fw,gw)]d^2(Sw,Tw) \\
&\quad \leq p\psi\Big(d^2(fw,Sw)d(gw,Tw), d(fw,Sw)d^2(gw,Tw), \\
&\qquad d(fw,Sw)d(fw,Tw)d(gw,Sw), d(fw,Tw)d(gw,Sw)d(gw,Tw)\Big) \\
&\qquad + m(fw,gw) - \phi\big(m(fw,gw)\big),
\end{aligned}$$

where

$$\begin{aligned}
m(fw,gw) &= \max\Big\{d^2(fw,gw), d(fw,Sw)d(gw,Tw), d(fw,Tw)d(gw,Sw), \\
&\qquad \frac{1}{2}\big[d(fw,Sw)d(fw,Tw) + d(gw,Sw)d(gw,Tw)\big]\Big\} \\
&= 0.
\end{aligned}$$

That is, $[1 + 0]d^2(Sw,Tw) \leq p\psi(0,0,0,0) + 0 - \phi(0)$, i.e., $d^2(Sw,Tw) \leq 0$, which is true for $Sw = Tw$. Hence $gw = Tw = fw = Sw$, i.e., w is a coincidence point of S,T,f and g. It remains to prove that w is a common fixed point of S,T,f and g. Letting $u = u_{2n}$ and $v = w$ in (C_2) and letting $n \to \infty$, we get

$$[1 + pd(w,gw)]d^2(w,Tw) \leq p\psi(0,0,0,0) + m(w,gw) - \phi(m(w,gw)),$$

where $m(w,gw) = \max\{d^2(w,gw), 0, d(w,Tw)d(gw,w), 0\} = d^2(w,gw)$.

After simplification, we get $d(w,Tw) = 0$, i.e., $w = Tw$. Hence $fw = gw = Sw = Tw = w$. Therefore, w is a common fixed point of f,g,S and T.

The uniqueness can be proved easily. This completes the proof. □

Remark 3.4. Putting $f = g = I$(Identity mapping of E) and $S = T$ in Theorems 3.1, 3.2 and 3.3, we get generalized versions of Theorem 1.2.

Remark 3.5. Define ψ in Theorems 3.1 and 3.3 as follows: $\psi(t_1, t_2, t_3, t_4) = \max\{\frac{1}{2}[t_1+t_2], t_3, t_4\}$. Then Theorems 3.1 and 3.3 reduce to the results of Jain *et al.* [12] for compatible mappings of type (R) and type (K), respectively.

Remark 3.6. Theorem 3.2 improve the results of Jain *et al.* [12] for compatible mappings of type (E) by using the control function ψ and replacing the continuity of mappings with the concept of split reciprocal continuity of mappings.

Letting $f = g = A$ and $S = T = B$ in Theorems 3.1, 3.2 and 3.3, one can deduce the following corollaries.

Corollary 3.7. *Let (E, d) be a complete metric space. Suppose that $A, B : E \to E$ are two mappings satisfying the following conditions:*

(C_1*) $A(E) \subset B(E)$,

(C_2*) *for all $u, v \in E$ there exist a real number $p > 0$, a function $\psi \in \Psi$ and a function $\phi \in \Phi$ such that*

$$\begin{aligned}[1 + pd(Bu, Bv)]d^2(Au, Av) &\le p\,\psi\Big(d^2(Bu, Au)d(Bv, Av), d(Bu, Au)d^2(Bv, Av),\\ &\qquad d(Bu, Au)d(Bu, Av)d(Bv, Au), d(Bu, Av)d(Bv, Au)d(Bv, Av)\Big)\\ &\quad + m(Bu, Bv) - \phi(m(Bu, Bv)),\end{aligned}$$

where

$$\begin{aligned}m(Bu, Bv) &= \max\Big\{d^2(Bu, Bv), d(Bu, Au)d(Bv, Av), d(Bu, Av)d(Bv, Au),\\ &\qquad \frac{1}{2}[d(Bu, Au)d(Bu, Av) + d(Bv, Au)d(Bv, Av)]\Big\},\end{aligned}$$

(C_3*) *either B or A is continuous.*

If the pair (B, A) is compatible of type (R), then there exists a unique point $w \in E$ such that $Bw = w = Aw$.

Corollary 3.8. *Let A and B be self mappings of a complete metric space (E, d) satisfying the conditions (C_1*) and (C_2*) of Corollary 3.7. If the pair (A, B) satisfies either of the following conditions:*

(a) *(A, B) is A-compatible of type (E) and $A-$reciprocally continuous;*

(b) *(A, B) is B-compatible of type (E) and $B-$reciprocally continuous.*

Then A and B have a unique common fixed point in E.

Corollary 3.9. *Let (E, d) be a complete metric space. Suppose that $A, B : E \to E$ are two mappings satisfying the conditions (C_1*) and (C_2*). If (A, B) is compatible of type (K) as well as reciprocally continuous, then A and B have a unique common fixed point in E.*

Now, we present examples in support of Theorems 3.1 and 3.2.

Example 3.10. Let $E = [0, 5]$ and d be a usual metric. Define $f, g, S, T : E \to E$ as $Tu = Su = \frac{5+u}{2}$, $gu = fu = \frac{5}{2} + u$, $0 \leq u < \frac{5}{2}$, $Tu = Su = \frac{5}{2}$, $\frac{5}{2} \leq u \leq 5$, $fu = gu = \frac{5}{2}$, $u = \frac{5}{2}$ and $fu = gu = \frac{24}{5}$, $\frac{5}{2} < u \leq 5$. Clearly, $S(E) = [\frac{5}{2}, \frac{15}{4}) = T(E)$ and $f(E) = g(E) = [\frac{5}{2}, 5)$. The mappings are not continuous at $u = \frac{5}{2}$. Let $\{u_n\}$ be a sequence in E such that $u_n \to 0$, $u_n > 0$, for all n. Then Su_n, $fu_n \to \frac{5}{2} = t$ and $SSu_n = S(\frac{5+u_n}{2}) \to \frac{5}{2}$, $Sfu_n = S(\frac{5}{2} + u_n) \to \frac{5}{2}$, $ffu_n = f(\frac{5}{2} + u_n) \to \frac{24}{5}$ and $fSu_n = f(\frac{5+u_n}{2}) \to \frac{24}{5}$. Also, we have $ft = \frac{5}{2} = St$. Thus $SSu_n, Sfu_n \to \frac{5}{2} = ft = f\frac{5}{2}$ and $Sfu_n \to \frac{5}{2} = St = S\frac{1}{2}$. Therefore, the pair (f, S) is S-compatible of type (E) and S-reciprocally continuous and the pair (g, T) is T-compatible of type (E) and T-reciprocally continuous. In particular, if we take $\psi(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}$, where $t_i \geq 0, i = 1, 2, 3, 4$, $\phi(t) = \frac{3}{2}t, t \geq 0$ and $\frac{3}{2} < p$, then it satisfies all the conditions of Theorem 3.2 and $\frac{5}{2}$ is the unique common fixed point of f, g, S and T.

Example 3.11. Let $E = [2, 20]$ and d be a usual metric. Let $f, g, S, T : E \to E$ be defined as $fu = 2$, $2 \leq u \leq 10$, $fu = u - 8$, $10 < u \leq 20, Su = 2$, $2 \leq u \leq 20$, $gu = 2, Tu = 2, u = 2$, $gu = 6, Tu = 3$, $2 < u \leq 20$. Here $T(E) = \{2, 3\} \subset [2, 12] = f(E)$ and $S(E) = \{2\} \subset \{2, 6\} = g(E)$. For the sequence $\{u_n\}$, where $u_n = 2$, for each n, pairs (f, S) and (g, T) are compatible of type (R). If we define a function $\phi : [0, \infty) \to [0, \infty)$ as $\phi(t) = \frac{3}{2}t$, for each $t \geq 0$ and define a function $\psi : [0, \infty)^4 \to [0, \infty)$ as $\psi(w_1, w_2, w_3, w_4) = \max\{w_1, w_2, w_3, w_4\}$, $w_i \geq 0, i = 1, 2, 3, 4$ and take a real number $p \geq \frac{3}{2}$, then all the conditions of Theorem 3.1 are satisfied and 2 is the unique common fixed point of f, g, S and T.

4. Fixed point for weak integral contraction

In 2001, Branciari [6] generalized Banach contraction principle by introducing an integral type contraction. Similarly, we analyze our results for mappings satisfying a generalized (ψ, ϕ)-weak integral type contraction.

Theorem 4.1. *Let f, g, S and T be four self-mappings of a complete metric space (E, d) satisfying the conditions (C_1) and*

(C_4) *for* $u, v \in E$, *there exist functions* $\phi \in \Phi$, $\psi \in \Psi$ *and a positive real number* p *such that*

$$\int_{o}^{M(u,v)} \zeta(t)\,dt \le \int_{o}^{N(u,v)} \zeta(t)\,dt,$$

where $M(u,v) = [1 + pd(fu, gv)]d^2(Su, Tv)$, *and*

$$N(u,v) = p\psi\Big(d^2(fu, Su)d(gv, Tv), d(fu, Su)d^2(gv, Tv), d(fu, Su)d(fu, Tv)d(gv, Su), d(fu, Tv)d(gv, Su)d(gv, Tv)\Big) + m(fu, gv) - \phi(m(fu, gv)).$$

Here

$$m(fu, gv) = \max\Big\{d^2(fu, gv), d(fu, Su)d(gv, Tv), d(fu, Tv)d(gv, Su), \frac{1}{2}[d(fu, Su)d(fu, Tv) + d(gv, Su)d(gv, Tv)]\Big\},$$

and $\zeta : [0, \infty) \to [0, \infty)$ *is a Lebesgue integrable function which is summable on each compact subset of* $[0, \infty)$ *such that for each* $\epsilon > 0$, $\int_{o}^{\epsilon} \zeta(t)\,dt > 0$.

If the pairs (f, S) *and* (g, T) *are compatible mappings of type* (R) *and one of* f, g, S *and* T *is continuous, then* f, g, S *and* T *have a unique common fixed point.*

Theorem 4.2. *Let* f, g, S *and* T *be four self mappings of a complete metric space* (E, d) *satisfying the conditions* (C_1) *and* (C_4). *If the pairs* (f, S) *and* (g, T) *satisfy either of the following conditions:*

(a) (f, S) *is* f*-compatible of type* (E) *and* f*-reciprocally continuous,* (g, T) *is* g*-compatible of type* (E) *and* g*-reciprocally continuous;*

(b) (f, S) *is* S*-compatible of type* (E) *and* S*-reciprocally continuous,* (g, T) *is* T*-compatible of type* (E) *and* T*-reciprocally continuous.*

Then f, g, S *and* T *have a unique common fixed point.*

Theorem 4.3. *Let* f, g, S *and* T *be four self-mappings of a complete metric space* (E, d) *satisfying the conditions* (C_1) *and* (C_4). *If pairs* (f, S) *and* (g, T) *are compatible mappings of type* (K) *and reciprocally continuous, then* f, g, S *and* T *have a unique common fixed point.*

Remark 4.4. Taking $\zeta(t) = c$ (some non-zero constant) in Theorems 4.1, 4.2 and 4.3, these theorems reduce to Theorems 3.1, 3.2 and 3.3, respectively.

5. Some applications to dynamic programming

Let U, V denote Banach spaces, and $\hat{S} \subset U$, $D \subset V$ be state space and decision space, respectively. Let $\mathbb{R}$ denotes the set of all real numbers and $B(\hat{S}) = \{h : \hat{S} \to \mathbb{R}, h \text{ is bounded}\}$. Basic form of functional equation given by Bellman and Lee [4] is as follows:

$$g(u) = \operatorname*{opt}_{v} G(u, v, g(\tau(u, v))),$$

where $u \in \hat{S}$, $v \in D$, τ is the transformation process, $g(u)$ is the optimal return with initial state u and the opt denotes max or min.

In this section, we shall discuss the application of our results in finding a common solution of the following functional equations that are arising in dynamic programming :

$$f_i(u) = \sup_{v \in D} F_i(u, v, f_i(\tau(u, v))), u \in S, \tag{5.1}$$

$$g_i(u) = \sup_{v \in D} G_i(u, v, g_i(\tau(u, v))), u \in S, \tag{5.2}$$

where $\tau : \hat{S} \times D \to \hat{S}$ and $F_i, G_i : \hat{S} \times D \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$. Define the mappings P_i, Q_i, $i = 1, 2$ as follows

$$\begin{aligned} P_i h(u) &= \sup_{v \in D} F_i(u, v, h(\tau(u, v))), \\ Q_i k(u) &= \sup_{v \in D} G_i(u, v, k(\tau(u, v))), \end{aligned} \tag{5.3}$$

for all $u \in \hat{S}$, $h, k \in B(\hat{S})$, $i = 1, 2$.

Theorem 5.1. *Suppose that the following conditions hold:*

(D_1) F_1, F_2, G_1 *and* G_2 *are bounded,*

(D_2) *for all* $u, t \in \hat{S}, v \in D$, $h, k \in B(\hat{S})$,

$$\begin{aligned} \big|F_1(u, v, h(t)) - F_2(u, v, k(t))\big| \le M^{-1}\Big(& p\,\psi\big(d^2(Q_1h, P_1h)d(Q_2k, P_2k), \\ & d(Q_1h, P_1h)d^2(Q_2k, P_2k), \\ & d(Q_1h, P_1h)d(Q_1h, P_2k)d(Q_2k, P_1h) \\ & d(Q_1h, P_2k)d(Q_2k, P_1h)d(Q_2k, P_2k)\big) \\ & + m(Q_1h, Q_2k) - \phi(m(Q_1h, Q_2k))\Big), \end{aligned}$$

where

$$m(Q_1h, Q_2k) = \max\Big\{d^2(Q_1h, Q_2k), d(Q_1h, P_1h)d(Q_2k, P_2k), d(Q_1h, P_2k)d(Q_2k, P_1h)\frac{1}{2}[d(Q_1h, P_1h)d(Q_1h, P_2k) + d(Q_2k, P_1h)d(Q_2k, P_2k)]\Big\},$$

$M = [1 + pd(Q_1h, Q_2k)]d(P_1h, P_2k)$,$P_1h \neq P_2k$, $\phi \in \Phi$, $\psi \in \Psi$, *p is a positive real number and the mappings* P_1, P_2, Q_1, Q_2 *are defined as in* (5.3),

(D_3) *for any sequence* $\{k_n\} \subset B(\hat{S})$ *and* $k \in B(\hat{S})$ *such that*

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |k_n(u) - k(u)| = 0,$$

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |Q_ik_n(u) - Q_ik(u)| = 0 \ \text{or} \ \lim_{n\to\infty} \sup_{u\in\hat{S}} |P_ik_n(u) - P_ik(u)| = 0,$$

hold for $i = 1$ *or* $i = 2$,

(D_4) *for any* $h \in B(\hat{S})$, *there exist* $k_1, k_2 \in B(\hat{S})$ *such that*

$$P_1h(u) = Q_2k_1(u), P_2h(u) = Q_1k_2(u), u \in \hat{S},$$

(D_5) *for any sequence* $\{k_n\}$ *of* $B(\hat{S})$, *if there exists* $h \in B(\hat{S})$ *such that*

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |Q_ik_n(u) - h(u)| = \lim_{n\to\infty} \sup_{u\in\hat{S}} |P_ik_n(u) - h(u)| = 0,$$

then

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |Q_iP_ik_n(u) - P_iQ_ik_n(u)| = 0, i = 1, 2,$$

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |Q_iQ_ik_n(u) - P_iP_ik_n(u)| = 0, i = 1, 2.$$

Then the system of functional equations (5.1) *and* (5.2) *has a unique common solution in* $B(\hat{S})$.

Proof. Let $d(h, k) = \sup\{|h(u)-k(u)| : u \in \hat{S}\}$ for any $h, k \in B(\hat{S})$. Obviously, $(B(\hat{S}), d)$ is a complete metric space. By conditions $(D_1) - (D_5)$, P_i, Q_i are self-mappings of $B(\hat{S})$. One of P_i, Q_i is continuous for $i = 1, 2$, $P_1(B(\hat{S})) \subset Q_2(B(\hat{S}))$ and $P_2(B(\hat{S})) \subset Q_1(B(\hat{S}))$ and the pairs of mappings (P_1, Q_1) and (P_2, Q_2) are compatible of type (R). For $\eta > 0$, $u \in \hat{S}$ and $k_1, k_2 \in B(\hat{S})$, there exist $v_1, v_2 \in D$ such that

$$P_ik_i(u) < F_i(u, v_i, k_i(u_i)) + \eta, \tag{5.4}$$

where $u_i = \tau(u, v_i), i = 1, 2$. Also, we have

$$P_1 k_1(u) \geq F_1(u, v_2, k_1(u_2)), \tag{5.5}$$
$$P_2 k_2(u) \geq F_2(u, v_1, k_2(u_1)). \tag{5.6}$$

From (5.4), (5.6) and (D_2), we have

$$\begin{aligned} P_1 k_1(u) - P_2 k_2(u) &< F_1(u, v_1, k_1(u_1)) - F_2(u, v_1, k_2(u_1)) + \eta \\ &\leq M^{-1}\Big(p\,\psi\big(d^2(Q_1 h, P_1 h) d(Q_2 k, P_2 k), \\ &\quad d(Q_1 h, P_1 h) d^2(Q_2 k, P_2 k), \\ &\quad d(Q_1 h, P_1 h) d(Q_1 h, P_2 k) d(Q_2 k, P_1 h) \\ &\quad d(Q_1 h, P_2 k) d(Q_2 k, P_1 h) d(Q_2 k, P_2 k)\big) \\ &\quad + m(Q_1 h, Q_2 k) - \phi(m(Q_1 h, Q_2 k))\Big) + \eta. \end{aligned} \tag{5.7}$$

From (5.4), (5.5) and (D_2), we have

$$\begin{aligned} P_1 k_1(u) - P_2 k_2(u) &> F_1(u, v_2, k_1(u_2)) - F_2(u, v_2, k_2(u_2)) - \eta \\ &\geq -M^{-1}\Big(p\,\psi\big(d^2(Q_1 h, P_1 h) d(Q_2 k, P_2 k), \\ &\quad d(Q_1 h, P_1 h) d^2(Q_2 k, P_2 k), \\ &\quad d(Q_1 h, P_1 h) d(Q_1 h, P_2 k) d(Q_2 k, P_1 h) \\ &\quad d(Q_1 h, P_2 k) d(Q_2 k, P_1 h) d(Q_2 k, P_2 k)\big) \\ &\quad + m(Q_1 h, Q_2 k) - \phi(m(Q_1 h, Q_2 k))\Big) - \eta. \end{aligned} \tag{5.8}$$

From (5.7) and (5.8), we obtain

$$\begin{aligned} |P_1 k_1(u) - P_2 k_2(u)| &\leq M^{-1}\Big(p\,\psi\big(d^2(Q_1 h, P_1 h) d(Q_2 k, P_2 k), \\ &\quad d(Q_1 h, P_1 h) d^2(Q_2 k, P_2 k), \\ &\quad d(Q_1 h, P_1 h) d(Q_1 h, P_2 k) d(Q_2 k, P_1 h) \\ &\quad d(Q_1 h, P_2 k) d(Q_2 k, P_1 h) d(Q_2 k, P_2 k)\big) \\ &\quad + m(Q_1 h, Q_2 k) - \phi(m(Q_1 h, Q_2 k))\Big) + \eta. \end{aligned} \tag{5.9}$$

Since $\eta > 0$ is arbitrary and (5.9) is true for all $u \in \hat{S}$, taking supremum, we get

$$
\begin{aligned}
[1 + pd(Q_1k_1, Q_2k_2)]d^2(P_1k_1, P_2k_2) \le\ & p\,\psi\big(d^2(Q_1h, P_1h)d(Q_2k, P_2k),\\
& d(Q_1h, P_1h)d^2(Q_2k, P_2k),\\
& d(Q_1h, P_1h)d(Q_1h, P_2k)d(Q_2k, P_1h)\\
& d(Q_1h, P_2k)d(Q_2k, P_1h)d(Q_2k, P_2k)\big)\\
& + m(Q_1h, Q_2k) - \phi(m(Q_1h, Q_2k)).
\end{aligned}
$$

Therefore, by Theorem 3.1, where P_1, P_2, Q_1, Q_2 correspond to the mappings S, T, f, g, respectively, P_1, P_2, Q_1 and Q_2 have a unique common fixed point $k^* \in B(\hat{S})$, that is, $k^*(u)$ is a unique common solution of the system of functional equations (5.1) and (5.2). □

Theorem 5.2. *Suppose that the conditions* (D_1), (D_2) *and* (D_4) *of Theorem 5.1 are satisfied. Then the system of functional equations* (5.1) *and* (5.2) *has a unique common solution in* $B(\hat{S})$ *provided that either of the following conditions is satisfied:*

(D_6) *for any sequence* $\{k_n\}$ *of* $B(\hat{S})$, *if there exists* $k \in B(\hat{S})$ *such that*

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |Q_ik_n(u) - k(u)| = \lim_{n\to\infty} \sup_{u\in\hat{S}} |P_ik_n(u) - k(u)| = 0,$$

then, for $i = 1, 2$,

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |Q_iP_ik_n(u) - Q_ik(u)| = 0,$$

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |Q_iQ_ik_n(u) - P_ik(u)| = \lim_{n\to\infty} \sup_{u\in\hat{S}} |Q_iP_ik_n(u) - P_ik| = 0,$$

(D_7) *for any sequence* $\{k_n\}$ *of* $B(\hat{S})$, *if there exists* $k \in B(\hat{S})$ *such that*

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |Q_ik_n(u) - k(u)| = \lim_{n\to\infty} \sup_{u\in\hat{S}} |P_ik_n(u) - k(u)| = 0,$$

then, for $i = 1, 2$,

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |P_iQ_ik_n(u) - P_ik(u)| = 0,$$

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |P_iP_ik_n(u) - Q_ik(u)| = \lim_{n\to\infty} \sup_{u\in\hat{S}} |P_iQ_ik_n(u) - Q_ik| = 0.$$

Proof. From (D_6), the pair (Q_i, P_i) is Q_i-compatible of type (E) and Q_i-reciprocally continuous for $i = 1, 2$ and from (D_7), the pair (Q_i, P_i) is P_i-compatible of type (E) and P_i-reciprocally continuous for $i = 1, 2$. Following the proof of Theorem 5.1, we conclude that all the conditions of Theorem 3.2 are satisfied. Hence the system of functional equations (5.1) and (5.2) has a unique common solution in $B(\hat{S})$. □

Theorem 5.3. *Suppose that the conditions* (D_1), (D_2) *and* (D_4) *of Theorem* 5.1 *and the following conditions are satisfied:*

(D_8) *for any sequence* $\{k_n\}$ *of* $B(\hat{S})$, *if there exists* $k \in B(\hat{S})$ *such that*

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |Q_i k_n(u) - k(u)| = \lim_{n\to\infty} \sup_{u\in\hat{S}} |P_i k_n(u) - k(u)| = 0,$$

then, for $i = 1, 2$,

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |Q_i P_i k_n(u) - Q_i k(u)| = 0, \lim_{n\to\infty} \sup_{u\in\hat{S}} |P_i Q_i k_n(u) - P_i k(u)| = 0,$$

(D_9) *for any sequence* $\{k_n\}$ *of* $B(\hat{S})$, *if there exists* $k \in B(\hat{S})$ *such that*

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |Q_i k_n(u) - k(u)| = \lim_{n\to\infty} \sup_{u\in\hat{S}} |P_i k_n(u) - k(u)| = 0,$$

then, $i = 1, 2$,

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |Q_i Q_i k_n(u) - P_i k(u)| = 0, \lim_{n\to\infty} \sup_{u\in\hat{S}} |P_i P_i k_n(u) - Q_i k(u)| = 0.$$

Then the system of functional equations (5.1) *and* (5.2) *has a unique common solution in* $B(\hat{S})$.

Proof. By following the proof of Theorem 5.1, all the conditions of Theorem 3.3 are satisfied and by applying Theorem 3.3, the system of functional equations (5.1) and (5.2) has a unique common solution in $B(\hat{S})$. □

6. Conclusion and future works

In this paper, we studied pairs of compatible mappings of type (R), type (E) and type (K) to obtain some common fixed point theorems for pairs of compatible mappings of type (R), type (E) and type (K) satisfying generalized (ψ, ϕ)-weak contraction involving cubic terms of metric functions. We provided useful examples and application to dynamical programming for the validity of our results. In next study, we will try to apply the results to some set-valued or interval-valued mappings to obtain similar results.

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