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DISTRIBUTED CONTROL OF AN INCOMPRESSIBLE NAVIER-STOKES EQUATIONS UNDER CONJUGATION CONDITIONS

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Abstract. This paper is concerned with the study of first-order necessary optimality conditions for the non-stationary incompressible fluid flows described by the Navier-Stokes equations subject to conjugation conditions within a bounded domain $\varrho \subset \mathbb{R}^n (2 \leq n \leq 3)$. First, we establish the existence and uniqueness of the weak solution to this system by using the Lax-Milgram lemma then, we prove the existence and uniqueness of distributed control for a quadratic cost functional, subject to the non-stationary incompressible fluid flows governed by Navier-Stokes equations with conjugation and Dirichlet conditions.

1. Introduction

The mathematical analysis of the optimal control of the Navier-Stokes equations with conjugation conditions which describe fluid flow plays a major role in the flow Control, Environmental Engineering and Biomedical applications. In modeling blood flow, controls might dictate how blood flow is manipulated through stents or other medical devices. For Environmental Engineering, conjugation conditions can be used to model how controls in pollution dispersal influence the flow field. Moreover the problem involves finding control inputs that minimize a cost function while satisfying the equations governing fluid motion. When incorporating conjugation conditions, you typically deal with distributed control or constraints that link the control inputs to the state of

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the fluid. In three dimensional form we address the distributed control of the non-stationary and incompressible Navier-Stokes equations as follows

$$\begin{cases} \frac{\partial V_1}{\partial t} - \zeta \Delta V_1 + V_1 \frac{\partial V_1}{\partial \eta_1} + V_2 \frac{\partial V_1}{\partial \eta_2} + V_3 \frac{\partial V_1}{\partial \eta_3} + \frac{\partial q_1^*}{\partial \eta_1} = f_1^*(\eta, t) & \text{in } \varrho_{T^*}, \\ \frac{\partial V_2}{\partial t} - \zeta \Delta V_2 + V_1 \frac{\partial V_2}{\partial \eta_1} + V_2 \frac{\partial V_2}{\partial \eta_2} + V_3 \frac{\partial V_2}{\partial \eta_3} + \frac{\partial q_2^*}{\partial \eta_2} = f_2^*(\eta, t) & \text{in } \varrho_{T^*}, \\ \frac{\partial V_3}{\partial t} - \zeta \Delta V_3 + V_1 \frac{\partial V_3}{\partial \eta_1} + V_2 \frac{\partial V_3}{\partial \eta_2} + V_3 \frac{\partial V_3}{\partial \eta_3} + \frac{\partial q_3^*}{\partial \eta_3} = f_3^*(\eta, t) & \text{in } \varrho_{T^*}, \\ \nabla . V = \frac{\partial V_1}{\partial \eta_1} + \frac{\partial V_2}{\partial \eta_2} + \frac{\partial V_3}{\partial \eta_3} = 0 & \text{in } \varrho_{T^*}, \end{cases} \tag{1.1}$$

where $\varrho_{T^*} = \varrho \times (0, T^*)$ and ϱ is a domain that consists of two open, non-intersecting and bounded, continuous, strictly Lipschitz domains ϱ_1 and ϱ_2 such that

$$\varrho = (\varrho_1 \cup \varrho_2), (\varrho_1 \cap \varrho_2) = \varphi \text{ and } \bar{\varrho} = (\bar{\varrho}_1 \cup \bar{\varrho}_2).$$

On $\Gamma_{T^*} = \Gamma \times (0, T^*)$, $\Gamma = (\partial\varrho_1 \cup \partial\varrho_2) \setminus \gamma$, $(\gamma = \partial\varrho_1 \cap \partial\varrho_2 \neq \varphi)$, the boundary condition of the Dirichlet (no-slip) and the initial condition are given by

$$\begin{cases} V_1(\eta, 0) = V_{1,0}(\eta) \text{ in } \varrho, \\ V_2(\eta, 0) = V_{2,0}(\eta) \text{ in } \varrho, \\ V_3(\eta, 0) = V_{1,0}(\eta) \text{ in } \varrho, \\ V_1(\eta, t) = V_2(\eta, t) = V_3(\eta, t) = 0 \text{ on } \Gamma_{T^*}, \end{cases} \tag{1.2}$$

where $V(\eta, t) = (V_1, V_2, V_3))$ is the velocity of the fluid, q^* represents the scalar pressure, f^* stands for the given external body forces and ζ is a constant. On the section $\gamma_{T^*} = \gamma \times (0, T^*)$ of the domain $\bar{\varrho}$ the conjugation conditions are given by

$$\begin{cases} \left[\frac{\partial V_1}{\partial \mu_A}\right] = \left[\sum_{i,j=1}^{n} \frac{\partial V_1}{\partial \eta_j} \cos(\mu, \eta_i)\right] = 0 \quad \text{on} \quad \gamma_{T^*}, \\ \left[\frac{\partial V_2}{\partial \mu_A}\right] = \left[\sum_{i,j=1}^{n} \frac{\partial V_2}{\partial \eta_j} \cos(\mu, \eta_i)\right] = 0 \quad \text{on} \quad \gamma_{T^*}, \\ \left[\frac{\partial V_3}{\partial \mu_A}\right] = \left[\sum_{i,j=1}^{n} \frac{\partial V_3}{\partial \eta_j} \cos(\mu, \eta_i)\right] = 0 \quad \text{on} \quad \gamma_{T^*}, \end{cases} \tag{1.3}$$

$$\left\{ \begin{array}{l} \left\{ \frac{\partial V_1}{\partial \mu_A} \right\}^{\pm} = r[V_1] \quad \text{on} \quad \gamma_{T^*}, \\ \\ \left\{ \frac{\partial V_2}{\partial \mu_A} \right\}^{\pm} = r[V_2] \quad \text{on} \quad \gamma_{T^*}, \\ \\ \left\{ \frac{\partial V_3}{\partial \mu_A} \right\}^{\pm} = r[V_3] \quad \text{on} \quad \gamma_{T^*}, \end{array} \right. \tag{1.4}$$

where

$$[\xi] = \xi^+ - \xi^-,$$

$$\xi^+ = \{\xi\}^+ = \xi(\eta, t) \quad \text{for} \quad (\eta, t) \in \gamma^+_{T^*} = (\partial\varrho_2 \cap \gamma) \times (0, T^*),$$

$$\xi^- = \{\xi\}^- = \xi(\eta, t) \quad \text{for} \quad (\eta, t) \in \gamma^-_{T^*} = (\partial\varrho_1 \cap \gamma) \times (0, T^*),$$

$\cos(\mu, \eta_i) = i$-th direction cosine of μ, μ being the normal to γ and such normal is directed into the domain ϱ_2, $\frac{\partial V}{\partial \mu_A}$ is directional derivative of V and

$$\left\{ \begin{array}{l} 0 \le r = r(\eta) \le r_1, \quad r \in C(\gamma), \\ r_1 \quad \text{is a positive constant.} \end{array} \right. \tag{1.5}$$

This paper has considered the applied problems of distributed type of the Navier-Stokes system. The optimal control conditions in terms of the states of systems (4.2) and equations (1.2), (1.3), (1.4) have been established. Various optimal control problems governed by classical Navier-Stokes equations has been studied by many authors see ([4], [5], [8], [11], [18], [19], [20]). Hyder and El-Badawy [9] extended this discussion to time-fractional Navier-Stokes equations. Lions [10] discussed the optimal control problems for finite order elliptic, parabolic and hyperbolic operators with finite number of variables. Gali and Serag ([6], [7]) extended this discussion to cooperative systems. Using the theory of Sergienko and Deineka [17], Serag et.al. ([12], [13], [14], [15], [16]) and [2] introduced some control problems for cooperative systems under conjugation conditions. Some applications were introduced in ([1], [3]). In the present work, we study the optimal control of distributed type for the non-stationary incompressible fluid flows of the Navier-Stokes equations under conjugation conditions.

2. Function spaces and some properties

To start, we define the following spaces: $\mathcal{D}(\varrho)$ the space of infinitely differentiable functions with compact support in ϱ, $\mathcal{D}^*(\varrho)$ its dual and the space of divergence free functions is defined by

$$\Lambda = \mathcal{D}_{div}(\varrho_{T^*}) = \{V \in (D(\varrho_{T^*}))^3; \nabla \cdot V = 0\}.$$

For the functional setting of the problem (1.1) we take a Hilbert space $L^2(\varrho)$ and a Sobolev space

$$H_0^1(\varrho) = \{V \in L^2(\varrho) : \tfrac{\partial V}{\partial \eta_i} \in L^2(\varrho), V|_\Gamma = 0, \quad i = 1, 2, 3\}.$$

We define the next spaces

$$H = \{V \in (L^2(\varrho))^3, \nabla \cdot V = 0\},$$

$$V_\theta = \{V \in (H_0^1(\varrho))^3, \nabla \cdot V = 0, \frac{\partial V}{\partial \mu_A}|_\Gamma = 0, \left[\frac{\partial V}{\partial \mu_A}\right]_\gamma = 0\},$$

where H and V_θ are Hilbert spaces and there are the complete closure of $\mathcal{D}_{div}(\varrho_{T^*})$ in $L^2(\varrho)$ and $H_0^1(\varrho)$, respectively. The norm in $L^2(\varrho)$ is $|V|^2 = (V, V)$, where

$$(V, \chi) = \sum_{i=1}^{3} \int_\varrho V_i(\eta)\chi_i(\eta) d\eta$$

and the norm in $H_0^1(\varrho)$ is

$$\|V\|^2 = \left(\frac{\partial V}{\partial \eta_i}, \frac{\partial V}{\partial \eta_i}\right).$$

Since V_θ is dense subset of H and $V_\theta \subset H \subset V_\theta^*$ with continuous and dense embedding, we construct the function space for our problem as follows

$$\mathbb{W}_0(0, T^*) = \left\{V : V \in L^2(0, T^*; V_\theta), \frac{\partial V}{\partial t} \in L^2(0, T^*; V_\theta^*)\right\}.$$

Moreover,

$$L^2(0, T^*; V_\theta) \subset L^2(0, T^*; (L^2(\varrho))^3) \subset L^2(0, T^*; V_\theta^*),$$

so $\mathbb{W}_0(0, T^*)$ is a Hilbert space with the norm equipped by

$$\|V\|^2_{\mathbb{W}_0(0,T^*)} = \left\{\|V\|^2_{L^2(0,T^*;V_\theta)} + \|\frac{\partial V}{\partial t}\|^2_{L^2(0,T^*;V_\theta^*)}\right\}.$$

To prove the existence of the state of the system (1.1)-(1.5) we apply the extended holder inequality which stated that:

Theorem 2.1. ([11]) *Let ϱ be a bounded set in $\mathbb{R}^3$. If $V \in L^{p^*}(\varrho)$, $\chi^* \in L^{q^*}(\varrho)$ and $\bar{\chi} \in L^{r^*}(\varrho)$ with*

$$\frac{1}{p^*} + \frac{1}{q^*} + \frac{1}{r^*} = 1, \qquad 1 \le p^*, q^*, r^* \le \infty,$$

then $V\ \chi^*\bar{\chi} \in L^1(\varrho)$ and

$$\int_\varrho |V(\eta)\chi^*(\eta)\bar{\chi}(\eta)| d\eta \le \|V\|_{L^{p^*}} \|\chi^*\|_{L^{q^*}} \|\bar{\chi}\|_{L^{r^*}}. \tag{2.1}$$

And also apply the following two theorems.

Theorem 2.2. ([19], Sobolev imbedding theorem) *Let $\varrho \subset \mathbb{R}^n$ and let $1 < p^* < \infty$ and $m \geq 0$. Then the following imbedding exist and are continuous:*

(i) *for* $m^*p^* < n^*$: $W^{m^*,p^*}(\varrho) \subseteq L^{q^*}(\varrho)$ *if* $1 \leq q^* \leq \frac{n^*p^*}{n^*-m^*p^*}$,

(ii) *for* $m^*p^* = n^*$: $W^{m^*,p^*}(\varrho) \subseteq L^{q^*}(\varrho)$ *if* $1 \leq q^* < \infty$,

(iii) *if* $\varrho \subset \mathbb{R}^2$, *then* $H^1(\varrho) = W^{1,2}(\varrho) \subseteq L^{q^*}(\varrho)$, $\forall\, 1 \leq q^* < \infty$,

(iv) *if* $\varrho \subset \mathbb{R}^3$, *then* $H^1(\varrho) \subseteq L^6(\varrho)$.

Theorem 2.3. ([11], Banach Contraction Principle) *Let T^* be an operator defined on the Banach space X. Assume that T is a contraction, i.e.*

$$\|T^*_x - T^*_y\|\| \leq \alpha_1 \|x^* - y^*\|, \quad \forall\, x^*, y^* \in X, \ \ 0 \leq \alpha_1 < 1.$$

Then there exists a unique element $x^* \in X$ such that $Tx^* = x^*$.

3. Weak solution

To begin the notation of weak solutions for the non-stationary Navier-Stokes equations (1.1).

Let V_θ and H be spaces defined above and a forcing term $f^* \in L^2(0, T^*; V^*_\theta)$ be given. A weak formulation consists in multiplying the momentum equation and the continuity equation with an arbitrary test function $\chi \in \Lambda$ and then integrate over ϱ_{T^*}. Finally, integration by parts is applied to reduce the derivative order of some terms. The variational form of the momentum equation reads directly

$$\begin{aligned} &\int_0^{T^*} (\frac{\partial V}{\partial t}, \chi)_{(L^2(\varrho))^3} dt + \int_0^{T^*} ((V.\nabla)V, \chi)_{(L^2(\varrho))^3} dt \\ &+ \zeta \int_0^{T^*} (\nabla V, \nabla \chi)_{(L^2(\varrho))^3} dt + \int_0^{T^*} (\nabla q, \chi)_{(L^2(\varrho))^3} dt \\ &= \int_0^{T^*} (f^*, \chi)_{(L^2(\varrho))^3} dt, \qquad \forall\, \chi \in \Lambda. \end{aligned} \tag{3.1}$$

Applying integration by parts, the weak form of the pressure term is given by

$$\int_0^{T^*} \int_\varrho \nabla q^* \chi d\eta \ dt = \int_0^{T*} (q^*, \frac{\partial \chi}{\partial n})_{(L^2(\Gamma))^3} - \int_0^{T^*} (q^*, \nabla \cdot \chi) = 0.$$

Using the similar procedure on the continuity equation yields

$$\int_0^{T^*} (\nabla \cdot V(t), \chi(t))_{(L^2(\varrho))^3} dt = 0.$$

Definition 3.1. We call $V \in L^2(0, T^*; V_\theta)$ with $f^* \in L^2(0, T^*; V_\theta^*)$ and $V_{i,0} \in H$ be given, a weak solution to the no-slip boundary-value problem for the non-stationary Navier-Stokes system in (1.1)-(1.4) if the variational equation.

$$(\frac{\partial V}{\partial t}, \chi) + a^*(V, \chi) + b^*(V, V, \chi) = F(\chi), \quad \forall \chi \in V_\theta \tag{3.2}$$

is satisfied.

To specify the problem setting, we introduce a nonlinear operator $B : L^2(0, T^*; V_\theta) \to L^2(0, T^*; V_\theta^*)$ by

$$(B(V, V), \bar{\chi}) = \int_0^{T^*} ((B(V))(t), \bar{\chi})_{(L^2(\varrho))^3} dt = \int_0^{T^*} b^*(V(t), V(t), \bar{\chi}(t)) dt,$$

where $b^* : V_\theta \times V_\theta \times V_\theta \longrightarrow \mathbb{R}$ is the trilinear form defined by:

$$b^*(V, \chi, \bar{\chi}) = \int_0^{T^*} ((V \cdot \nabla)\chi, \bar{\chi})_{(L^2(\varrho))^3}\ dt = \int_0^{T^*} \int_\varrho \sum_{i,j=1}^{3} V_i \frac{\partial \chi_j}{\partial \eta_i} \bar{\chi}_j d\eta\ dt, \tag{3.3}$$

$$\begin{aligned}((V \cdot\ \nabla)\chi, \bar{\chi}) = \Big(& V_1 \frac{\partial \chi_1}{\partial \eta_1} \bar{\chi}_1 + V_2 \frac{\partial \chi_1}{\partial \eta_2} \bar{\chi}_1 + V_3 \frac{\partial \chi_1}{\partial \eta_3} \bar{\chi}_1, \\ & V_1 \frac{\partial \chi_2}{\partial \eta_1} \bar{\chi}_2 + V_2 \frac{\partial \chi_2}{\partial \eta_2} \bar{\chi}_2 + V_3 \frac{\partial \chi_2}{\partial \eta_3} \bar{\chi}_2, \\ & V_1 \frac{\partial \chi_3}{\partial \eta_1} \bar{\chi}_3 + V_2 \frac{\partial \chi_3}{\partial \eta_2} \bar{\chi}_3 + V_3 \frac{\partial \chi_3}{\partial \eta_3} \bar{\chi}_3 \Big).\end{aligned}$$

Lemma 3.2. ([11]) *For $n^* = 3$, the trilinear form $b^*(V, \chi, \bar{\chi})$ defined in* (3.3) *is continuous and satisfies the following properties*

(i) $|b^*(V, \chi, \bar{\chi})| \leq C_1 \|V\|_{V_\theta} \|\chi\|_{V_\theta} \|\bar{\chi}\|_{V_\theta}$,
(ii) $b^*(V, \chi, \bar{\chi}) = -b^*(V, \bar{\chi}, \chi)$,
(iii) $b^*(V, \bar{\chi}, \chi) = 0$.

Proof. To prove the continuity of the form $b^*(V, \chi, \bar{\chi})$, let

$$V_i \in L^{\frac{2n*}{n*-2}}(\varrho), \quad \frac{\partial \chi_j}{\partial \eta_i} \in L^2(\varrho), \quad \bar{\chi}_i \in L^{n*}(\varrho).$$

By the extended Holder inequality, equation (3.3) becomes

$$\begin{aligned}|b^*(V, \chi, \bar{\chi})| &= \Big| \int_0^{T^*} \int_\varrho \sum_{i,j=1}^{3} V_i \frac{\partial \chi_j}{\partial \eta_i} \bar{\chi}_j d\eta\ dt \Big| \\ &\leq \|V_i\|_{L^6(\varrho)} \|\frac{\partial \chi_j}{\partial \eta_i}\|_{L^2(\varrho)} \|\bar{\chi}_i\|_{L^3(\varrho)}.\end{aligned} \tag{3.4}$$

From the Sobolev imbedding theorem

$$\|V\|_{L^{\frac{2n*}{n*-2}}(\varrho)} \leq C\|V\|_{H_0^1(\varrho)},$$

we obtain

$$|b^*(V,\chi,\bar{\chi})| \leq C_1\|V_i\|_{V_\theta}\|\chi_{V_\theta}\|\bar{\chi}_i\|_{V_\theta}. \tag{3.5}$$

Thus, $b^*(V,\chi,\bar{\chi})$ is continuous on $V_\theta \times V_\theta \times V_\theta$.

Moreover,we prove the above properties as follows: Applying the Green's formula to equation (3.3), we obtain

$$\begin{aligned} b^*(V,\chi,\bar{\chi}) = &-\int_0^{T^*}\int_\varrho \sum_{i,j=1}^{3} \frac{\partial V_i}{\partial \eta_i}\chi_j\bar{\chi}_j \quad d\eta\ dt \\ &-\int_0^{T^*}\int_\varrho \sum_{i,j=1}^{3} V_i\chi_j \ \frac{\partial \bar{\chi}_j}{\partial \eta_i} \quad d\eta\ dt \\ &+\int_0^{T^*}\int_\Gamma \sum_{i,j=1}^{3} V_i\chi_j\bar{\chi}_j \cdot n d\Gamma dt. \end{aligned} \tag{3.6}$$

Therefore,

$$b^*(V,\chi,\bar{\chi}) = -b^*(V,\bar{\chi},\chi).$$

If we replace $\bar{\chi}$ by χ in (3.6), we have

$$\begin{aligned} b^*(V,\chi,\chi) = &-\int_0^{T^*}\int_\varrho \sum_{i,j=1}^{3} \frac{\partial V_i}{\partial \eta_i}\chi_j\chi_j \quad d\eta\ dt \\ &-\int_0^{T^*}\int_\varrho \sum_{i,j=1}^{3} V_i\chi_j \frac{\partial \chi_j}{\partial \eta_i} \quad d\eta\ dt \\ &+\int_0^{T^*}\int_\Gamma \sum_{i,j=1}^{3} V_i\chi_j\chi_j \cdot n d\Gamma dt. \end{aligned}$$

Thus, we obtain

$$b^*(V,\chi,\chi) = -b^*(V,\chi,\chi)$$

and

$$b^*(V,\chi,\chi) = 0.$$

□

On the other hand, we consider the bilinear form $a^*: V_\theta \times V_\theta \to \mathbb{R}$, for each $t \in (0, T^*)$,

$$\begin{aligned} a^*(V,\chi) &= (AV,\chi) \\ &= \int_0^{T^*} (AV(t),\chi)_{(L^2(\varrho))^3} dt \\ &= \int_0^{T^*} (\nabla V(t), \nabla\chi(t))_{(L^2(\varrho))^3} dt \\ &= (-\zeta\Delta V(\eta,t), \chi(\eta,t)), \quad \forall\, \chi \in \Lambda, \end{aligned}$$

where the linear operator $A: L^2(0,T^*;V_\theta) \hookrightarrow L^2(0,T^*;V_\theta^*)$ is defined by

$$AV = -\zeta\Delta V.$$

Lemma 3.3. *The form defined in* (3.7) *is symmetric, continuous and coercive over $V_\theta \times V_\theta$.*

Now, we define $F^*: V_\theta \longrightarrow \mathbb{R}$ by

$$F^*(\chi) = (f^*,\chi), \quad \forall\, \chi \in V_\theta,$$

where the functional be linear and continuous and

$$\|F^*\|_{V_\theta^*} = \sup_{\chi \in V_\theta} \frac{|(f^*,\chi)|}{\|\chi\|}.$$

Theorem 3.4. *For every $f^* \in L^2(0,T^*;V_\theta^*)$ and $V_{i,0} \in H$ is given the formula* (3.2) *has a unique solution $V \in \mathbb{W}_0(0,T^*)$.*

Proof. For each $V \in V_\theta$ rewriting (3.2) to be

$$\left(\frac{\partial\chi}{\partial t},\bar{\chi}\right) + a^*(\chi,\bar{\chi}) + b^*(V;\chi,\bar{\chi}) = F^*(\bar{\chi}), \qquad \forall\, \chi,\ \bar{\chi} \in V_\theta, \tag{3.7}$$

and consider the bilinear form $\pi_V(\chi,\bar{\chi})$ as following:

$$\pi_V(\chi,\bar{\chi}) = a^*(\chi,\bar{\chi}) + b^*(V;\chi,\bar{\chi}), \quad \forall\, \chi,\ \bar{\chi} \in V_\theta. \tag{3.8}$$

For each $V \in V_\theta$ and χ is a unique solution to (3.8), we define $\mathcal{B}_0: V_\theta \longrightarrow V_\theta$, by $\mathcal{B}_0(V) = \chi$.

Based on coerciveness of the form $a^*(\chi,\bar{\chi})$ and the skew-symmetric property of the form $b^*(V;\chi,\bar{\chi})$, we deduce the continuity and the coerciveness of $\pi_V(\chi,\bar{\chi})$, that is,

$$\pi_V(\chi,\chi) \geq C_2\|\chi\|^2_{V_\theta}, \quad C_2 > 0. \tag{3.9}$$

Then from Lax-Milgram lemma, the mapping $\mathcal{B}_0(V) = \chi$ is well-defined and the fixed point of $\mathcal{B}_0$ is a unique solution of (3.2). If we replace $\bar{\chi}$ by χ in equation (3.7), we have

$$\|\chi\|_{V_\theta} + \|\frac{\partial \chi}{\partial t}\|_{V_\theta^*} \leq C_3\|F^*\|_{V_\theta^*}. \tag{3.10}$$

Now, let $K^* = \{\chi \in V_\theta : \|\chi\|_{V_\theta} + \|\frac{\partial \chi}{\partial t}, \chi)_{V_\theta^*}\| \leq C_3\|F^*\|_{V_\theta^*}\}$. Then K^* is a bounded, closed and convex subset of V_θ, where $\mathcal{B}_0 : K^* \longrightarrow K^*$, furthermore, $\mathcal{B}_0$ is also a contraction mapping. In fact, if $\mathcal{B}_0(V_1) = \chi_1$, $\mathcal{B}_0(V_2) = \chi_2$, we have, for all $V_1, V_2 \in K^*$,

$$(\frac{\partial \chi_1}{\partial t}, \bar{\chi}) + a^*(\chi_1, \bar{\chi}) + b^*(V_1; \chi_1, \bar{\chi}) = F^*(\bar{\chi}) \ , \ \ \forall \chi_1, \bar{\chi} \in V_\theta,$$
$$(\frac{\partial \chi_2}{\partial t}, \bar{\chi}) + a^*(\chi_2, \bar{\chi}) + b^*(V_2; \chi_2, \bar{\chi}) = F^*(\bar{\chi}) \ , \ \ \forall \chi_2, \bar{\chi} \in V_\theta.$$

Hence, we obtain

$$(\frac{\partial}{\partial t}(\chi_1 - \chi_2), \bar{\chi}) + a^*(\chi_1 - \chi_2, \bar{\chi}) + b^*(V_1; \chi_1, \bar{\chi}) - b^*(V_2; \chi_2, \bar{\chi}) = 0,$$

by replacing $\bar{\chi}$ by $\chi_1 - \chi_2$, we have

$$\begin{aligned}&(\frac{\partial}{\partial t}(\chi_1 - \chi_2), \chi_1 - \chi_2) + a^*(\chi_1 - \chi_2, \chi_1 - \chi_2)\\ &+ b^*(V_1; \chi_1, \chi_1 - \chi_2) - b^*(V_2; \chi_2, \chi_1 - \chi_2) = 0.\end{aligned} \tag{3.11}$$

Take again

$$\pi_{V_2}(\chi_1 - \chi_2, \chi_1 - \chi_2) = a^*(\chi_1 - \chi_2, \chi_1 - \chi_2) + b^*(V_2; \chi_1 - \chi_2, \chi_1 - \chi_2),$$

then (3.11) transformed to

$$(\frac{\partial}{\partial t}(\chi_1 - \chi_2), \chi_1 - \chi_2) + \pi_{V_2}(\chi_1 - \chi_2, \chi_1 - \chi_2) = b^*(V_2 - V_1; \chi_1, \chi_1 - \chi_2),$$

and hence, by using (3.9) and Lemma 3.1, we obtain

$$C_2\|\chi_1 - \chi_2\|^2 + \|\frac{\partial}{\partial t}(\chi_1 - \chi_2)\| \leq C_1\|V_2 - V_1\|\|\chi_1\|\|\chi_1 - \chi_2\|$$

or

$$\mathbb{C}_2\|\chi_1 - \chi_2\|^2 \leq C_1\|V_2 - V_1\|\|\chi_1\|\|\chi_1 - \chi_2\|.$$

Since $\chi_1 \in K$, we have

$$\|\chi_1 - \chi_2\| \leq \frac{C_1}{C_2}\|V_2 - V_1\| \left(C_3\|F^*\| - \|\frac{\partial \chi}{\partial t}\| \right),$$

if we take

$$\frac{C_1}{C_2}\left(C_3\|f^*\| - \|\frac{\partial \chi}{\partial t}\| \right) < 1,$$

then the mapping $\mathcal{B}_0$ is a contraction mapping and the mapping $\mathcal{B}_0(V) = \chi$ has a unique fixed point. □

4. Formulation of the control problem

First, the adjoint state of our problem is established and the set of the first-order necessary optimality conditions is obtained. This problem leads us to formulate the minimization of the cost functional

$$\begin{aligned} J(v) =& \|V_1(u^*) - z^*_{1g}\|^2_{L^2(\varrho_{T^*})} + \|V_2(u^*) - z^*_{2g}\|^2_{L^2(\varrho_{T^*})} \\ &+ \|V_3(u^*) - z^*_{3g}\|^2_{L^2(\varrho_{T^*})} + (N^*v, v)_{(L^2(\varrho_{T^*}))^3}, \quad \forall\, v \in \mathcal{U}^*_{ad}. \end{aligned} \tag{4.1}$$

Subject to equation

$$\begin{cases} \frac{\partial V_1(u^*)}{\partial t} - \zeta \Delta V_1(u^*) + V_1(u^*)\frac{\partial V_1(u^*)}{\partial \eta_1} + V_2(u^*)\frac{\partial V_1(u^*)}{\partial \eta_2} + V_3(u^*)\frac{\partial V_1(u^*)}{\partial \eta_3} + \frac{\partial q^*_1}{\partial \eta_1} \\ \quad = f^*_1(\eta, t) + u^*_1 \quad \text{in } \ \varrho_{T^*}, \\ \frac{\partial V_2(u^*)}{\partial t} - \zeta \Delta V_2(u^*) + V_1(u^*)\frac{\partial V_2(u^*)}{\partial \eta_1} + V_2(u^*)\frac{\partial V_2(u^*)}{\partial \eta_2} + V_3(u^*)\frac{\partial V_2(u^*)}{\partial \eta_3} + \frac{\partial q^*_2}{\partial \eta_2} \\ \quad = f^*_2(\eta, t) + u^*_2 \quad \text{in } \ \varrho_{T^*}, \\ \frac{\partial V_3(u^*)}{\partial t} - \zeta \Delta V_3(u^*) + V_1(u^*)\frac{\partial V_3(u^*)}{\partial \eta_1} + V_2(u^*)\frac{\partial V_3(u^*)}{\partial \eta_2} + V_3(u^*)\frac{\partial V_3(u^*)}{\partial \eta_3} + \frac{\partial q^*_3}{\partial \eta_3} \\ \quad = f^*_3(\eta, t) + u^*_3 \quad \text{in } \ \varrho_{T^*}, \\ \nabla . V(u^*) = \frac{\partial V_1(u^*)}{\partial \eta_1} + \frac{\partial V_2(u^*)}{\partial \eta_2} + \frac{\partial V_3(u^*)}{\partial \eta_3} = 0 \quad \text{in } \ \varrho_{T^*}, \end{cases} \tag{4.2}$$

and equations (1.2), (1.3), (1.4), where the forcing term $f^* + u^*$ is the control.

Consider $\mathcal{U}^* = (L^2(\varrho_{T^*}))^3$ as a control Hilbert space.

For a control $u^* = (u^*_1, u^*_2, u^*_3) \in U^*$, the state $V(u^*) \in L^2(0, T^*; V_\theta)$ of the system is given by (4.2). Specify the observation as

$$Z^*(u^*) = \sum_{i=1}^{3}(z^*_i(u^*)) = C_3 V(u^*) = \sum_{i,j=1}^{3}(V_i(u^*)),$$

where $C_3 \in \pounds(\mathbb{W}_0(0, T^*), (L^2(\varrho_{T^*}))^3)$. For given $z^*_g \in (L^2(\varrho_{T^*}))^3$, the cost functional is got by (4.1), where $N^* \in \mathcal{L}((L^2(\varrho_{T^*}))^3, (L^2(\varrho_{T^*})^3))$ is positive definite Hermitian operator satisfies the following condition:

$$(N^* u^*, u^*) \geq C_0 \|u^*\|^2_{(L^2(\varrho_{T^*}))^3}, \quad C_0 > 0. \tag{4.3}$$

The control problem then is to find:

$$\begin{cases} u^* = (u^*_1, u^*_2, u^*_3) \in U^*_{ad} \quad \text{such that} \\ J(u^*) = \inf J(v), \qquad \forall\, v \in {U^*}_{ad}, \end{cases} \tag{4.4}$$

where U^*_{ad} is closed convex subset of $(L^2(\varrho_{T^*}))^3$.

On the other hand, the weak formulation (4.2) can be written in the following form (see[18])

$$\begin{cases} \frac{dV}{dt} + AV + B(V) = f^* + u^* \quad \text{in } \varrho_{T^*}, \\ V(\eta, t) \in V_\theta, \quad \forall\, t \in (0, T^*), \\ V(\eta, 0) = V_0 \quad \text{in } \varrho_{T^*}. \end{cases} \tag{4.5}$$

T^* is a fixed (but arbitrary) strictly positive real number and the operators A, B are defined above.

5. Linearized equations

We introduce some results regarding linearized equations. For each $V \in \mathbb{W}_0(0, T^*)$ define the derivative $B^{'}(V)$ of the nonlinear operator B by

$$\begin{aligned} \int_0^{T^*} (B^{'}(\bar{V}(t))V(t), \bar{\chi}(t))dt = \int_0^{T^*} &(b^*(\bar{V}(t), V(t), \bar{\chi}(t)) \\ &+ b^*(V(t), \bar{V}(t), \bar{\chi}(t)))dt, \ \ \forall\, \bar{V}, \bar{\chi} \in \mathbb{W}_0(0, T^*) \end{aligned} \tag{5.1}$$

and it's adjoint operator $[B^{'}(V)]^* : L^2(0, T^*, V_\theta) \to L^2(0, T^*, V_\theta{}^*)$ by

$$(B^{'}(\bar{V}) \cdot V, \bar{\chi}) = (V, (B^{'}(\bar{V}))^* \bar{\chi}),$$

that is,

$$((B^{'}(\bar{V}))^* V, \bar{\chi}) = \int_0^{T^*} (b^*(\bar{V}, \bar{\chi}, V) + b^*(\bar{\chi}, \bar{V}, V))dt, \qquad \forall\, \bar{\chi} \in \mathbb{W}_0(0, T^*). \tag{5.2}$$

In fact, to prove (5.2) we use (5.1) as follows

$$((B^{'}(\bar{V})) \cdot V, \bar{\chi}) = \int_0^{T^*} \int_\varrho \sum_{i,j=1}^{3} (\bar{V}_i \frac{\partial V_j}{\partial \eta_i} \bar{\chi}_j + V_i \frac{\partial \bar{V}_j}{\partial \eta_i} \bar{\chi}_j) d\eta \ dt,$$

by integration by parts and (1.1)

$$\begin{aligned} ((B^{'}(\bar{V}))^* V, \bar{\chi}) = &- \int_0^{T^*} \int_\varrho \sum_{i,j=1}^{3} \frac{\partial \bar{V}_i}{\partial \eta_i} V_j \bar{\chi}_j \ d\eta \ dt - \int_0^{T^*} \int_\varrho \sum_{i,j=1}^{3} \bar{V}_i V_j \frac{\partial \bar{\chi}_j}{\partial \eta_i} \ d\eta \ dt \\ &- \int_0^{T^*} \int_\varrho \sum_{i,j=1}^{3} V_i \bar{V}_j \frac{\partial \bar{\chi}_j}{\partial \eta_i} \ d\eta \ dt \\ &- \int_0^{T^*} \int_\varrho \sum_{i,j=1}^{3} \frac{\partial V_i}{\partial \eta_i} \bar{V}_j \bar{\chi}_j d\eta \ dt + \int_0^{T^*} \int_\Gamma \sum_{i,j=1}^{3} V_i \chi_j \bar{\chi}_j \cdot n d\Gamma dt. \end{aligned}$$

Lemma 5.1. *The mapping $u^* \mapsto V(u^*)$, from $L^2(0,T^*;H)$ into $L^2(0,T^*;V_\theta)$, has a Gâteaux derivatives $\frac{\partial V(u^*)}{\partial u^*}.h$ in any direction $h \in L^2(0,T^*;H)$. Moreover, $\frac{\partial V(u^*)}{\partial u^*}.h = w(h)$ is a solution of the linearized problem:*

$$\begin{cases} \frac{dw}{dt} + Aw + B^{'}(V(u^*))(w) = h \quad in \;\; \varrho_{T^*}, \\ w(\eta,t) \in V_\theta, \quad \forall\, t \in (0,T^*), \\ w(0) = 0 \quad in \;\; \varrho. \end{cases} \tag{5.3}$$

Lemma 5.2. *Let $h_1 \in L^2(0,T^*;H)$ and let $w(h_1)$ be the solution of the system (5.1). Then for every $h_2 \in L^2(0,T^*;H)$, we have*

$$\int_0^{T^*}\int_\varrho (h_2 \cdot w(h_1))(\eta,t)d\eta dt = \int_0^{T^*}\int_\varrho (h_1 \cdot \psi^*(h_2))(\eta,t)d\eta dt,$$

where $\psi^*(h_2)$ is a solution of the adjoint linearized problem

$$\begin{cases} -\frac{d\psi^*}{dt} + \zeta A(\psi^*) + (B^{'}(V(u^*)))^*\psi^* = h_2 \quad in \;\; \varrho_T^*, \\ \psi^*(\eta,t) \in V_\theta, \quad \forall\, t \in (0,T^*), \\ \psi^*(T^*) = 0 \quad in \;\; \varrho. \end{cases} \tag{5.4}$$

Proof.

$$\begin{aligned} \int_0^{T^*}\int_\varrho h_2 \cdot w(h_1)d\eta dt &= \int_0^{T^*}\int_\varrho [-\frac{d\psi^*}{dt} + \zeta A(\psi^*) + (B^{'}(V(u^*)))^*\psi^*]wd\eta dt \\ &+ \int_\varrho (\int_0^{T^*}(-\frac{d\psi^*}{dt}\cdot w)dt)d\eta + \int_0^{T^*}(\int_\varrho(\zeta A(\psi^*)\cdot w)d\eta)dt \\ &+ \int_0^{T^*}(\int_\varrho((B^{'}(V(u^*)))^*\psi^*\cdot w)d\eta)dt, \end{aligned}$$

using the definition of $(B^{'}(V(u^*)))^*$ and the property of the self-adjoint of the operator A and the adjoint of the operator B defined above we have

$$
\begin{aligned}
\int_0^{T^*}\int_\varrho h_2\cdot w(h_1)d\eta dt &= \int_\varrho \{[-\psi^* w]_0^{T^*} + \int_0^{T^*}(\psi^*\frac{dw}{dt})\}d\eta \\
&\quad + \zeta\int_0^{T^*}(\int_\varrho \psi^*\cdot Awd\eta)dt \\
&\quad + \int_0^{T^*}(\int_\varrho \psi^*\cdot B^{'}(V(u^*))wd\eta)dt \\
&= \int_0^{T^*}\int_\varrho \psi^*[\frac{dw}{dt}+\zeta Aw + B^{'}(V(u^*))w]d\eta dt \\
&= \int_0^{T^*}\int_\varrho \psi^*\cdot h_1 d\eta dt.
\end{aligned}
$$

□

6. First-order necessary optimality conditions

Theorem 6.1. *If the cost functional is given by* (4.1) *and* (4.3) *is satisfied, then the necessary conditions for the existence of the optimal control* $u^*\in {U^*}_{ad}$ *is the following equation*

$$
\begin{cases}
-\frac{dp^*(u^*)}{dt} + Ap^*(u^*) + [B^{'}(V(u^*))]^* p(u^*) = V(u^*) - z_g^* \quad in \ \ \varrho_T^*, \\
p^*(\eta, T^*; u^*) = 0 \quad in \ \ \bar{\varrho}, \\
p^*(\eta,t) = 0 \quad on \ \ \Gamma_T^*, \\
\left[\frac{\partial p^*(u^*)}{\partial \mu_A^*}\right] = 0 \quad on \ \ \gamma_T^*, \\
\left\{\frac{\partial p^*(u^*)}{\partial \mu_A^*}\right\}^{\pm} = r[p^*(u^*)] \quad on \ \ \gamma_T^*,
\end{cases} \tag{6.1}
$$

$$
\begin{aligned}
&\int_0^{T^*}\int_\varrho (p_1^*(u^*)+N^*u_1^*)(v_1-u_1^*)d\eta dt + \int_0^{T^*}\int_\varrho (p_2^*(u^*)+N^*u_2^*)(v_2-u_2^*)d\eta dt \\
&\quad + \int_0^{T^*}\int_\varrho (p_3^*(u^*)+N^*u_3^*)(v_3-u_3^*)d\eta dt \geq 0, \quad \forall\, v\in {U^*}_{ad}, u^*\in {U^*}_{ad},
\end{aligned} \tag{6.2}
$$

where $p^*(u^*)\in L^2(0,T^*;V_\theta{}^*)$ *is the adjoint state.*

Proof. The control $u^*\in {U^*}_{ad}$ is optimal if and only if ([10])

$$
J^{'}(u^*).(v-u^*)\geq 0, \quad \forall\, v\in {U^*}_{ad},
$$

which is equivalent to

$$\begin{aligned}&\|V_1(u^*)-z^*_{1g}\|^2_{L^2(\varrho^*_T)}+\|V_2(u^*)-z^*_{2g}\|^2_{L^2(\varrho^*_T)}\\&+\|V_3(u^*)-z^*_{3g}\|^2_{L^2(\varrho^*_T)}+(N^*u^*,u^*)_{(L^2(\varrho^*_T))^3}\geq 0,\quad \forall\, v\in U^*_{ad}.\end{aligned}\tag{6.3}$$

Multiplying both sides of the first equation of (6.1) by $(V(v)-V(u^*))$ and integrating over ϱ^*_T, we deduce

$$\begin{aligned}&\int_0^{T^*}\int_\varrho (V(u^*)-z^*_g)(V(v)-V(u^*))d\eta dt\\&=\int_0^{T^*}\int_\varrho(-\frac{dp^*(u^*)}{dt}+Ap^*+(B^{'}V(u^*))^*p^*(u^*))(V(v)-V(u^*)))d\eta dt.\end{aligned}\tag{6.4}$$

Applying the integration by parts and using (5.3), we obtain

$$\int_0^{T^*}\int_\varrho (V(u^*)-z^*_g)(V(v)-V(u^*))d\eta dt=\int_0^{T^*}\int_\varrho p^*(u^*)(v-u^*)d\eta dt$$

and hence (6.3) is equivalent to

$$\begin{aligned}&\int_0^{T^*}\int_\varrho (p^*_1(u^*)(v_1-u^*_1)+(N^*u^*_1,v_1-u^*_1))d\eta dt\\&+\int_0^{T^*}\int_\varrho (p^*_2(u^*)(v_2-u^*_2)+(N^*u^*_2,v_2-u^*_2))d\eta dt\\&+\int_0^{T^*}\int_\varrho (p^*_3(u^*)(v_3-u^*_3)+(N^*u_3,v_3-u^*_3))d\eta dt\geq 0,\end{aligned}$$

which is reduced to (6.2). □

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