Nonlinear Functional Analysis and Applications
Vol. 30, No. 3 (2025), pp. 721-740
ISSN: 1229-1595(print), 2466-0973(online)
https://doi.org/10.22771/nfaa.2025.30.03.05
http://nfaa.kyungnam.ac.kr/journal-nfaa

HYPERABSOLUTE VALUES IN HYPERFIELDS : CONNECTIONS TO HYPERVALUATION

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Abstract. In this paper, we attempt an exposition of the connection between absolute value and hyperstructure theory. We define the hyperabsolute value of a hyperfield and investigate its properties. Moreover, we introduce the notion of a hypermetric space and use it to find a necessary and sufficient condition for two hyperabsolute values of a hyperfield to be equivalent. Finally, we explore the relationship between hyperabsolute values of a hyperfield and its hypervaluations.

1. Introduction

Hyperstructure theory originated in 1934 when Marty [12] introduced the concept of a hypergroup as a natural generalization of a group, based on the notion of a hyperoperation. A hypergroup is an algebraic structure similar to a group, but with a key difference: the composition of two elements results in

[0]Received December 8, 2024. Revised April 7, 2025. Accepted April 15, 2025.
[0]2020 Mathematics Subject Classification: 16Y99, 20N20.
[0]Keywords: Krasner hyperfield, hyperabsolute value, hypermetric space, hypervaluation.
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a nonempty set rather than a single element. Marty analyzed the properties of hypergroups, demonstrated their applications to groups, and illustrated their utility in the study of algebraic functions and relational fractions. For further applications of this theory, see [2, 4].

Since its inception, various definitions and concepts have been introduced, such as hyperrings, hyperfields, and many others [1, 3, 19, 20]. A well-known type of hyperring is the Krasner hyperring [8], which is essentially a ring with modified axioms where addition is defined as a hyperoperation (i.e., $a + b$ is a set). This concept has been extensively studied by various authors. Some fundamental notions of hyperstructure and hyperring theory can be found in [5, 13].

The theory of valuations was initiated in 1912 by the Hungarian mathematician Kurschak (see [10]). He introduced the concept of a valuation on a field as a real-valued function defined on the set of nonzero elements of the field, which satisfies certain properties. The concept of the valuation of the hyperfields was introduced by Krasner [7] in 1957. Following that, Mittas [14, 15, 16, 17, 18] studied in depth the notion of valuation and hypervaluation of hyperfields. Katarzyna Kuhlmann et.al [7] studied the notion of compatibility between valuations and orderings in real hyperfields and other recent studies on hyperfields and valued fields can be found in [11]. Valuations provide a way to measure the "size" of elements in a field, playing a crucial role in number theory and algebraic geometry. They considered a totally ordered canonical hypergroup and defined a hypervaluation of a hyperfield as a mapping onto this hypergroup. This generalization of classical valuation theory offers new perspectives in the study of hyperfields and their applications.

On the other hand, the notion of absolute value (as a measure) was explored in the 17th century in France by Jean-Robert Argand (Although it was known before for real and complex numbers). It is defined as a real valued function on the field (F) elements satisfying certain properties. For all $x, y \in F$,

(1) $|x| \geq 0$;
(2) $|x| = 0$ if and only if $x = \underline{0}$;
(3) $|xy| = |x||y|$;
(4) $|x + y| \leq |x| + |y|$.

In this paper, we extend the notion of absolute value of a field to the hyperabsolute value of a hyperfield. The remainder of this paper is organized as follows: In Section 2, we present some definitions related to hyperstructures. In Section 3, we define hyperabsolute values of hyperfields, prove their properties and present some examples. In Section 4, we use the concept of hyperabsolute value of a hyperfield to define hypermetric space. We also present new notions like convergent sequences, Cauchy sequences and investigate their properties. Moreover, we introduce the concept of equivalent hyperabsolute values and find a necessary and sufficient condition for two hyperabsolute values of a hyperfield to be equivalent. Finally, in Section 5, we find a relationship between hypervaluation on a hyperfield and the hyperabsolute value of it.

Throughout this paper, $\mathbb{R}$ is the set of real numbers, K is a hyperfield, $\underline{0}$ is the additive identity of K, $|\cdot|$ is the standard absolute value of real (complex) numbers and $/\cdot/$ is hyperabsolute value of K.

2. Preliminaries

In this section, we present some definitions related to hyperstructures that are used throughout this paper.

Let H be a nonempty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *binary hyperoperation* on H, where $\mathcal{P}^*(H)$ is the family of all nonempty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. In this definition, if A and B are two nonempty subsets of H and $x \in H$, then we define $A \circ B = \bigcup_{a\in A, b\in B} a \circ b,\ x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$.

A hypergroupoid $(H, \circ)$ is called: a *semihypergroup* if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$; a *quasihypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$ (this condition is called the reproduction axiom); a *hypergroup* if it is a semihypergroup and a quasihypergroup.

A *Krasner hyperring* is an algebraic structure $(\mathbb{R}, +, \cdot)$ which satisfies the following axiom:

(1) $(\mathbb{R}, +)$ is a commutative hypergroup;
(2) there exists $0 \in \mathbb{R}$ such that $0 + x = \{x\}$ for all $x \in \mathbb{R}$;
(3) for every $x \in \mathbb{R}$ there exists unique $x' \in \mathbb{R}$ such that $0 \in x + x'$; (x' is denoted by $-x$);
(4) $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$;
(5) $(\mathbb{R}, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$;
(6) the multiplication "$\cdot$" is distributive with respect to the hyperoperation "$+$".

Note that every ring is a Krasner hyperring. A subhyperring A of a Krasner hyperring $(\mathbb{R}, +, \cdot)$ is a *hyperideal* of $\mathbb{R}$ if $r \cdot a \in A$ $(a \cdot r \in A)$ for all $a \in A, r \in \mathbb{R}$. A commutative Krasner hyperring $(\mathbb{R}, +, \cdot)$ with identity element "1" is a *hyperfield* if $(\mathbb{R} \setminus \{0\}, \cdot)$ is a group. Different examples of finite and infinite hyperfield were constructed. First, we present two examples of finite hyperfields.

Example 2.1. Let $F_2 = \{0, 1\}$ and define $(F_2, +)$ and $(F_2, \cdot)$ by the following tables:

$+$	0	1
0	0	1
1	1	F_2

$\cdot$	0	1
0	0	0
1	0	1

$(F_2, +, \cdot)$ is a hyperfield.

Example 2.2. Let $S = \{0, 1, 2\}$ and define $(S, +)$ and $(S, \cdot)$ by the following tables:

$+$	0	1	2
0	0	1	2
1	1	1	S
2	2	S	2

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $(S, +, \cdot)$ is a hyperfield.

We present the following three examples of infinite hyperfields from [4, 21].

Example 2.3. (**Triangle hyperfield**) Let $\mathbb{V}$ be the set of non-negative real numbers with the following hyperoperations:

$$a \oplus b = \{c \in \mathbb{V} : |a - b| \leq c \leq a + b\}$$

and

$$a \odot b = ab.$$

Then $(\mathbb{V}, \oplus, \odot)$ is a hyperfield. Here, the additive identity $\underline{0} = 0$ and $-a = a$ for all $a \in \mathbb{V}$.

Example 2.4. (**Tropical hyperfield**) Let $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ with the following hyperoperations:

$$a \oplus b = \begin{cases} \max\{a, b\}, & \text{if } a \neq b; \\ \{c \in \mathbb{T} : c \leq a\}, & \text{if } a = b \end{cases}$$

and

$$a \odot b = a + b.$$

Then $(\mathbb{T}, \oplus, \odot)$ is a hyperfield. Here, the additive identity $\underline{0} = -\infty$ and $-a = a$ for all $a \in \mathbb{T}$. Moreover, the multiplicative identity is 0.

Example 2.5. (**Complex tropical hyperfield**) Let $\mathbb{C}$ be the set of complex numbers with the following hyperoperations:

$$a \oplus b = \begin{cases} \{a\}, & \text{if } |a| \geq |b|; \\ \{b\}, & \text{if } |a| \leq |b|; \\ \{c \in \mathbb{C} : c \in S_{ab}\}, & \text{if } a+b \neq 0 \text{ and } |a| = |b|; \\ \{c \in \mathbb{C} : |c| \leq |a|\}, & \text{if } a+b = 0. \end{cases}$$

$$a \odot b = ab,$$

Here, S_{ab} is the shortest arc connecting a to b on the circle with $|a|$ as absolute value. Then $(\mathbb{T}, \oplus, \odot)$ is a hyperfield. Here, the additive identity $\underline{0} = 0$ and the additive inverse of a complex number a is $-a$.

3. Hyperabsolute values of hyperfields

Inspired by the definition of absolute values of a field, we define hyperabsolute values of hyperfields, prove their properties and present some examples.

Definition 3.1. Let K be a hyperfield and $\mathbb{R}_{\geq 0}$ be the set of non-negative real numbers. A *hyperabsolute value* of K is a function

$$/\cdot/ : K \longrightarrow \mathbb{R}_{\geq 0}$$

satisfying the following conditions for all $x, y \in K$:

(1) $/x/ = 0$ if and only if $x = 0$;
(2) $/xy/ = /x//y/$;
(3) $\sup /z/_{z \in x+y} \leq /x/ + /y/$. (Triangle inequality)

Definition 3.2. Let K be a hyperfield. A hyperabsolute value of K is called *non-Archimedean* if for all $x, y \in K$, the following condition is satisfied:

$$sup/z/_{z \in x+y} \leq \max\{/x/, /y/\}.$$

Otherwise, it is called *Archimedean.*

Example 3.3. Let K be any hyperfield. Define the *trivial hyperabsolute value* of K as follows:

$$/x/ = \begin{cases} 0, & \text{if } x = \underline{0}; \\ 1, & \text{otherwise.} \end{cases}$$

One can easily see that $/\cdot/$ defines a non-Archimedean hyperabsolute value.

Next, we present examples of non-trivial hyperabsolute values.

Example 3.4. Let $(\mathbb{V}, \oplus, \odot)$ be the *Triangle hyperfield*, and define $/\cdot/ : \mathbb{V} \to \mathbb{V}$ by

$$/x/ = x, \quad \forall\, x \in \mathbb{V}.$$

We claim that $/\cdot/$ is an *Archimedean hyperabsolute value* on $\mathbb{V}$.

We verify the three conditions of Definition 3.1:

(1) **(Non-negativity and definiteness)** For all $x \in \mathbb{V}$, we have $/x/ = x \geq 0$, and $/x/ = 0$ if and only if $x = 0$.

(2) **(Multiplicativity)** For all $x, y \in \mathbb{V}$, we have

$$/x \odot y/ = x \cdot y = /x/ \cdot /y/.$$

(3) **(Triangle inequality for hyperaddition)** Let $z \in x \oplus y$. Then by the definition of hyperaddition in the Triangle hyperfield,

$$|x - y| \leq z \leq x + y.$$

Since $/z/ = z$, and $/x/ + /y/ = x + y$, we get

$$/z/ = z \leq x + y = /x/ + /y/.$$

To verify the *Archimedean property*, note that in the Triangle hyperfield, $3 \in 1 \oplus 2$, and

$$/3/ = 3 \nleq \max\{/1/, /2/\} = \max\{1, 2\} = 2.$$

Hence, the Archimedean property holds. Therefore, $/\cdot/$ is an Archimedean hyperabsolute value on $\mathbb{V}$.

Example 3.5. Let $(\mathbb{T}, \oplus, \odot)$ be the *Tropical hyperfield*, and define $/\cdot/ : \mathbb{T} \to \mathbb{R}_{\geq 0}$ as follows:

$$/x/ = \begin{cases} 0, & \text{if } x = \underline{0} = -\infty; \\ e^x, & \text{otherwise.} \end{cases}$$

We prove that $/\cdot/$ is a *non-Archimedean hyperabsolute value* on $\mathbb{T}$.
(Non-negativity and definiteness): Clearly, for all $x \in \mathbb{T}$, we have $/x/ \geq 0$, and $/x/ = 0$ if and only if $x = -\infty$.
(Multiplicativity): Let $x, y \in \mathbb{T}$. Then,

$$/x \odot y/ = \begin{cases} e^{x+y} = e^x \cdot e^y, & \text{if } x \neq -\infty, y \neq -\infty; \\ 0, & \text{otherwise} \end{cases} = /x/ \cdot /y/.$$

(Triangle inequality for hyperaddition): Let $z \in x \oplus y$. We consider the following cases:

- Case 1: $x \neq y$. Then $z = \max\{x, y\}$. Without loss of generality, suppose $x > y$. Then

$$/z/ = e^x > e^y, \quad \text{so } /z/ = e^x \leq e^x + e^y = /x/ + /y/.$$

- Case 2: $x = y = -\infty$. Then $z = -\infty$, so

$$/z/ = 0 = /x/ = /y/, \quad \text{and } /z/ \leq /x/ + /y/ = 0 + 0 = 0.$$

- Case 3: $x = y \neq -\infty$. Then $z \leq x = y$, and

$$/z/ = e^z \leq e^x = /x/ \leq /x/ + /x/ = /x/ + /y/.$$

Finally, it is easy to see that $/ \cdot /$ is *non-Archimedean*, as for any $x, y \in \mathbb{T}$, we always have:

$$/x \oplus y/ \subseteq \{z \in \mathbb{T} : /z/ \leq \max\{/x/, /y/\}\},$$

which violates the Archimedean inequality in general. Therefore, $/ \cdot /$ is a non-Archimedean hyperabsolute value on $\mathbb{T}$.

Example 3.6. Let $(\mathbb{C}, \oplus, \odot)$ be the complex tropical hyperfield and define $/ \cdot /$ of $\mathbb{C}$ as follows: For all $x \in \mathbb{C}$, $/x/ = |x|$. We prove that $/ \cdot /$ is a non-Archimedean hyperabsolute value of $\mathbb{C}$.

It is clear that Conditions (1) and (2) of Definition 3.1 is satisfied. We prove Condition (3). Let $x, y \in \mathbb{C}$ and $z \in x \oplus y$. We consider the following three cases:

- Case $|x| \neq |y|$. We get that $/z/ = \max\{|x|, |y|\} = \max\{/x/, /y/\}$.
- Case $x = -y$. We get that $/z/ = |z| \leq |x| = /x/ \leq \max\{/x/, /y/\}$.
- Case $|x| = |y|$ and $x + y \neq 0$. We get that $/z/ = |z| = |x| = /x/ \leq \max\{/x/, /y/\}$.

Proposition 3.7. (**Generalized Triangle inequality**) *Let K be a hyperfield, n be a positive integer greater than 1, $/ \cdot /$ be a hyperabsolute value of K and $x_i \in K$ for all $i = 1, 2, \ldots, n$. Then*

$$\sup /z/_{z \in x_1 + x_2 + \ldots + x_n} \leq /x_1/ + /x_2/ + \ldots + /x_n/.$$

Proof. We prove by induction on the value of n. For $n = 2$, the proof follows from Definition 3.1, Condition (3). Suppose that $\sup /z/_{z \in x_1 + x_2 + \ldots + x_{n-1}} \leq /x_1/ + /x_2/ + \ldots + /x_{n-1}/$ and let $t \in x_1 + x_2 + \ldots + x_{n-1} + x_n$. Then there exists $z \in x_1 + x_2 + \ldots + x_{n-1}$ such that $t \in z + x_{n-1}$. Definition 3.1, Condition (3) asserts that $/z/ \leq /t/ + /x_{n-1}/$. Thus, $/z/ \leq /x_1/ + /x_2/ + \ldots + /x_{n-1}/ + /x_n/$. □

Proposition 3.8. *Let K be a hyperfield, $/ \cdot /$ be a non-Archimedean hyperabsolute value of K and $x_i \in K$ for all $i = 1, 2, \ldots, n$. Then*

$$\sup /z/_{z \in x_1 + x_2 + \ldots + x_n} \leq \max\{/x_1/, /x_2/, \ldots, /x_n/\}.$$

Proof. The proof is straightforward. □

Proposition 3.9. *Let K be a hyperfield, $/\cdot/$ be a hyperabsolute value of K and $x, y \in K$. Then the following are true:*

(1) $/1/ = 1$;
(2) $/x^{-1}/ = \frac{1}{/x/}$;
(3) $/-1/ = 1$;
(4) $/-x/ = /x/$;
(5) *If there exists* $d \in \mathbb{N}$ *such that* $x^d = 1$ *then* $/x/ = 1$;
(6) *If* $/x/ < \epsilon$ *for all* $\epsilon > 0$ *then* $x = \underline{0}$;
(7) *If* $x \in \underbrace{1 + 1 + \ldots 1}_{n-times}$ *then* $/x/ \leq n$.

Proof. (1) Having $1 \odot 1 = 1$ implies that $/1/ = /1 \odot 1/ = /1//1/$. Since $/1/ > 0$, it follows that $/1/ = 1$.
(2) The proof follows from having $1 = /1/ = /x \odot x^{-1}/ = /x//x^{-1}/$ and $/x/ \neq 0$.
(3) Having $(-1)^{-1} = -1$ implies that $/-1/ = \frac{1}{/-1/}$. And since $-1 \neq \underline{0}$ (otherwise, $1 = \underline{0}$), it follows that $/-1/ = 1$.
(4) The proof follows from 3. and having $/-x/ = /(-1)(x)/ = /-1//x/ = /x/$.
(5) We have that $1 = /1/ = /x^d/ = /x/^d$. Thus, $/x/ = 1$.
(6) Suppose, for contradiction, that $x \neq \underline{0}$, then the equation is true for $\epsilon = \frac{/x/}{2}$. The latter implies that $/x/ \leq \frac{/x/}{2}$.
(7) The proof follows from Generalized Triangle Inequality and from (1). □

Corollary 3.10. *Let K be a hyperfield, $/\cdot/$ be a non-Archimedean hyperabsolute value of K and $x \in K$. If* $x \in \underbrace{1 + 1 + \ldots 1}_{n-times}$, *then* $/x/ \leq 1$.

Proof. Since $/\cdot/$ is a non-Archimedean hyperabsolute value of K and $x \in \underbrace{1 + 1 + \ldots 1}_{n-\text{times}}$, it follows, by Proposition 3.8 and Proposition 3.9, that $/x/ \leq \max\{/1/, \ldots, /1/\} = /1/ = 1$. □

Proposition 3.11. *Let K be a hyperfield, $/\cdot/$ be a non-Archimedean hyperabsolute value of K and $/x/ \neq /y/ \in K$. Then $/z/ = \max\{/x/, /y/\}$ for all $z \in x + y$.*

Proof. Without loss of generality, let $/x/ > /y/$. For all $z \in x + y$, we have that $/z/ \leq \max\{/x/, /y/\} = /x/$. Since $z \in x + y$, it follows that $x \in z - y$. And hence, $/x/ \leq \max\{/z/, /-y/\} = \max\{/z/, /y/\}$. Having $/y/ < /x/$ implies that $/x/ \leq /z/$. □

Proposition 3.12. *Let K be a hyperfield, $/\cdot/$ be a non-Archimedean hyperabsolute value of K and $x_i \in K$ such that $/x_i \neq /x_j/$ for all $i \neq j = 1, 2, \ldots, n$. Then for all $z \in x_1 + x_2 + \ldots + x_n$,*

$$/z/ = \max\{/x_1/, /x_2/, \ldots, /x_n/\}.$$

Proof. We prove by induction on n. For $n = 2$, our statement is true by Proposition 3.11. Assume that

$$/t/ = \max\{/x_1/, /x_2/, \ldots, /x_{n-1}/\}$$

for all $t \in x_1 + x_2 + \ldots + x_{n-1}$. Let $z \in x_1 + x_2 + \ldots + x_n$. Then there exists $t \in x_1 + x_2 + \ldots + x_{n-1}$ such that $z \in t + x_n$. Since $/t/ = \max\{/x_1/, /x_2/, \ldots, /x_{n-1}/\}$ and $/x/_n \neq /x_i/$ for all $i = 1, 2, \ldots, n-1$, it follows (by Proposition 3.11) that $/z/ = \max\{/t/, /x_n/\} = \max\{/x_1/, /x_2/, \ldots, /x_n/\}$. □

Proposition 3.13. *Let K be a finite hyperfield and $/\cdot/$ be a hyperabsolute value of K. Then $/\cdot/$ is the trivial hyperabsolute value of K.*

Proof. Let $x \in K \setminus \{\underline{0}\}$ and let the cardinality of K be $r \in \mathbb{N}$. Since $x^r = 1$, it follows by Proposition 3.9, Condition (5) that $/x/ = 1$. □

Proposition 3.14. *Let K be a hyperfield, $x, y, k \in K, k \neq \underline{0}$ and $/\cdot/$ be a hyperabsolute value of K. Then* $\inf /kx - ky/ = /k/ \inf /x - y/$.

Proof. Let $z \in x - y$, $t \in kx - ky$ such that $\inf /kx - ky/ = /t/$ and $\inf /x - y/ = /z/$. Having $z \in x - y$ implies that $kz \in kx - ky$ and hence, $/kz/ \geq /t/ = \inf /kx - ky/$. Having $t \in kx - ky$ implies that $k^{-1}t \in x - y$ and hence, $\frac{/t/}{/k/} = /k^{-1}t/ \geq \inf /x - y/ = /z/$. Therefore, $/t/ = /kz/$. □

Proposition 3.15. *Let K be a hyperfield, $/\cdot/$ be a hyperabsolute value of K and $x, y \in K$. Then*

$$\inf /z/_{z \in x-y} \geq |/x/ - /y/|.$$

Proof. Let $z \in x - y$. By using the definition of a hyperfield, we get that $x \in z + y$ and $y \in x - z$. The triangle inequality implies that $/x/ \leq /z/ + /y/$ and that $/y/ \leq /x/ + /-z/ = /x/ + /z/$. We get that $/x/ - /y/ \leq /z/$ and that $/y/ - /x/ \leq /z/$. Thus, $/z/ \geq |/x/ - /y/|$. □

Proposition 3.16. *Let K be any hyperfield and $/\cdot/$ be a hyperabsolute value of K. Define $/\cdot/_n$ for $n \in \mathbb{N}$ as follows:*

$$/x/_n = /x/^{\frac{1}{n}} \textit{ for all } x \in K.$$

Then $/\cdot/_n$ is a hyperabsolute value of K.

Proof. Let $x, y \in K$. We check the conditions of Definition 3.1 for $/\cdot/_n$.

- $/x/_n = 0 \Leftrightarrow /x/^{\frac{1}{n}} = 0 \Leftrightarrow /x/ = 0$. The latter is equivalent to $x = \underline{0}$.
- $/xy/_n = /xy/^{\frac{1}{n}} = (/x//y/)^{\frac{1}{n}} = /x/^{\frac{1}{n}}/y/^{\frac{1}{n}} = /x/_n/y/_n$.
- Let $z \in x + y$. Then $/z/ \leq /x/ + /y/$. The latter implies that

$$/z/_n = /z/^{\frac{1}{n}} \leq (/x/ + /y/)^{\frac{1}{n}} \leq /x/^{\frac{1}{n}} + /y/^{\frac{1}{n}} = /x/_n + /y/_n. \qquad \square$$

Proposition 3.17. *Let K be any hyperfield and $/ \cdot /$ be a non-Archimedean hyperabsolute value of K. Define $/ \cdot /^{(t)}$ for $t \in \mathbb{R}_{\geq 0}$ as follows:*

$$/x/^{(t)} = /x/^t \text{ for all } x \in K.$$

Then $/ \cdot /^{(t)}$ is a non-Archimedean hyperabsolute value of K.

Proof. Let $x, y \in K$.

- $/x/^{(t)} = 0 \Leftrightarrow /x/^t = 0 \Leftrightarrow /x/ = 0$. The latter is equivalent to $x = \underline{0}$.
- $/xy/^{(t)} = /xy/^t = (/x//y/)^t = /x/^t/y/^t = /x/^{(t)}/y/^{(t)}$.
- Let $z \in x + y$. Then $/z/ \leq \max\{/x/, /y/\}$. The latter implies that $/z/^t \leq \max\{/x/^t, /y/^t\}$. Thus, $/z/^{(t)} \leq \max\{/x/^{(t)}, /y/^{(t)}\}$. $\square$

4. Hypermetric spaces and equivalent hyperabsolute values

In this section, we use the concept of hyperabsolute value of a hyperfield, that is defined in Section 3, to define hypermetric space. Also, we introduce the concept of equivalent hyperabsolute values and find a necessary and sufficient condition for two hyperabsolute values of a hyperfield to be equivalent.

4.1. Hypermetric spaces.

Definition 4.1. Let X be a nonempty set and a mapping $d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$. Then (X, d) is called *a pseudo metric space* if for all $x, y, z \in K$, the following conditions are satisfied.

(1) $d(x, x) = 0$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$.

Example 4.2. Let K be any hyperfield and define $d : K \times K \longrightarrow \mathbb{R}_{\geq 0}$ as follows:

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{otherwise.} \end{cases}$$

Then (K, d) is a hypermetric space. Moreover, this hypermetric is induced by the trivial hyperabsolute value of K, that is, $d(x, y) = \inf /x - y/$.

Definition 4.3. Let (K, d) be a hypermetric space. We define the *open balls*, $B_r(x)$ in K as follows: For $x \in K, r > 0$,

$$B_r(x) = \{y \in K : d(x, y) < r\}.$$

Open subsets in K are defined to be union of open balls in K.

Example 4.4. Let $(\mathbb{V}, \oplus, \odot)$ be the triangle hyperfield and define d on $\mathbb{V}$ as follows:

$$d(x, y) = \inf /x \oplus y/ \text{ for all } x, y \in \mathbb{V}.$$

Then $(\mathbb{V}, d)$ is a hypermetric space. We show that the conditions of hypermetric space are satisfied.

- We have $d(x, x) = \inf /x \oplus x/ = \inf /c/_{0 \leq c \leq 2x} = 0$.
- Let $d(x, y) = \inf /x \oplus y/ = 0$. We get that $0 = d(x, y) = |x - y|$. Thus, $x = y$.
- $d(x, y) = \inf /x \oplus y/ = /y \oplus x/ = d(y, x)$.
- $d(x, z) = \inf /x \oplus z/ = |x - z| \leq |x - y| + |y - z| = \inf /x \oplus y/ + \inf /y \oplus z/ = d(x, y) + d(y, z)$.

Moreover, $B_1(0) = \{x \in \mathbb{V} : d(x, 0) = \inf /x \oplus \underline{0}/ = /x/ = x < 1\} = [0, 1[$ and $B_1(1) = \{x \in \mathbb{V} : d(x, 1) = \inf /x \oplus 1/ = |x - 1| < 1\} =]0, 2[$. One can conclude that $[0, 2[= B_1(0) \cup B_1(1)$ is an open subset of $\mathbb{V}$.

Example 4.5. Let $(\mathbb{T}, \oplus, \odot)$ be the Tropical hyperfield and define d on $\mathbb{T}$ as follows:

$$d(x, y) = \inf /x \oplus y/ \text{ for all } x, y \in \mathbb{T}.$$

Then $(\mathbb{T}, d)$ is a hypermetric space. We show that the conditions of hypermetric space are satisfied.

- We have $d(x, x) = \inf /x \oplus x/ = / - \infty/ = 0$.
- Let $d(x, y) = \inf /x \oplus y/ = 0$ and suppose that $x \neq y$. We get that $0 = \max\{e^x, e^y\}$. Then, $x = y = -\infty$, a contradiction.
- $d(x, y) = \inf /x \oplus y/ = /y \oplus x/ = d(y, x)$.
- $d(x, z) = \inf /x \oplus z/$. If $x = z$ or $x = y$ or $y = z$, we are done. If $x \neq z$, $x \neq y$ and $y \neq z$, then $d(x, z) = \max\{e^x, e^z\} \leq \max\{e^x, e^y\} + \max\{e^y, e^z\} = d(x, y) + d(y, z)$.

For $a \in \mathbb{T}$, $B_1(a) = \{y \in \mathbb{T}\colon d(a, y) < 1\} = \{y \in \mathbb{T} : \inf\{/t/ \colon t \in a - y\} < 1\}$. Clearly, $a \in B_1(a)$. Note that, for any $y \in \mathbb{T}$, $y = -y$. Now, for $y \neq a$, $a - y = a \oplus (-y) = a \oplus y = \max\{a, y\}$, and so, $d(a, y) = \inf\{/t/ : t \in a \oplus (-y)\} = \max\{e^a, e^y\}$. For $0 \leq a$, this gives $1 \leq e^0 \leq d(a, y)$ and hence, $B_1(a) = \{a\}$. However, for $a, y < 0$, $d(a, y) < e^0$ and so $B_1(a) = [-\infty, 0[$.

Proposition 4.6. *Let K be a hyperfield and $/ \cdot /$ be the trivial hyperabsolute value of K. Then all subsets of K are open.*

Proof. It suffices to show that singletons are open subsets in K. Let $x \in K$, we consider the open ball $B_1(x) = \{y \in K : \inf /y - x/ < 1\}$. Since $/ \cdot /$ is the trivial hyperabsolute value of K and $\inf /y - x/ \neq 1$, it follows that $\inf /y - x/ = 0$. Thus, $x = y$. Therefore, we get that $B_1(x) = \{x\}$ is open in K. □

Corollary 4.7. *Let K be a finite hyperfield. Then the power set topology is the only topology induced by the hyperabsolute value of K.*

Proof. The proof is obvious from Propositions 3.13 and 4.6. □

The next theorem shows that a hyperabsolute value of K induces a pseudometric space in the following manner, $d(x, y) = \inf /x - y/$.

Theorem 4.8. *Let $/\cdot/$ be a hyperabsolute values of K and define $d : K \times K \longrightarrow \mathbb{R}_{\geq 0}$ as $d(x, y) = \inf /x - y/$ for all $x, y \in K$. Then (K, d) is a pseudo metric space.*

Proof. Let $x, y, z \in K$. Since $\underline{0} \in x - x$ for all $x \in K$, it follows that $d(x, x) = \inf /x - x/ = 0$. Proposition 3.14 implies that $d(y, x) = \inf /y - x/ = \inf /x - y/ = d(x, y)$. We need to show that the triangle inequality is satisfied, that is, $\inf /x - z/ \leq \inf /x - y/ + \inf /y - z/$. Let $\alpha \in x - y, \beta \in y - z$. Then $-y \in \alpha - x$ and $y \in \beta + z$. We get that $\underline{0} \in -y + y \subseteq \alpha - x + \beta + z$. The latter implies that $\underline{0} \in \alpha + \beta + t$ for some $t \in z - x$. We get now that $-t \in \alpha + \beta$. Applying the definition of hyperabsolute values, we get that $/ - t/ = /t/ \leq /\alpha/ + /\beta/$. Since $t \in z - x$, it follows that $\inf /x - z/ = \inf /z - x/ \leq /t/$. Thus $\inf /x - z/ \leq /\alpha/ + /\beta/$ for all $\alpha \in x - y, \beta \in y - z$. Therefore, $\inf /x - z/ \leq \inf /x - y/ + \inf /y - z/$. □

Condition (*): For a hyperabsolute function $/ \cdot /$ defined on a hyperfield K, suppose $inf/x - y/ = 0$ implies $x = y$ holds for all x, y, then $/ \cdot /$ induces a metric on K.

Remark 4.9. Example 4.2, 4.4 and 4.5 form pseudo metric spaces with respect to the map defined in Theorem 4.8.

The hyperabsolute function defined on a hyperfield K together with the Condition (*) becomes a metric on K. Condition (*) holds good for Examples 6,7 and 8.

Definition 4.10. Let K be a hyperfield, $/ \cdot /$ be a hyperabsolute value of K and $\{x_n\}$ be a sequence in K. Then $\{x_n\}$ is said to converge to $x \in K$ ($x_n \longrightarrow x$) if for every $\epsilon > 0$, there exists a natural number N such that $\inf /x_n - x/ < \epsilon$ for all $n \geq N$.

Example 4.11. Let K be a hyperfield, $/\cdot/$ be a hyperabsolute value of K and $x \in K$. The constant sequence $\{x\}$ is convergent. This is because $d(x,x) = \inf /x - x/ = 0 < \epsilon$.

Example 4.12. Let K be a hyperfield, $/\cdot/$ be a hyperabsolute value of K and $x \in K$ with the property that $/x/ < 1$. Then the sequence $\{x^n\}$ is convergent to $\underline{0}$. This is because $/x^n/ = /x/^n$.

Proposition 4.13. *Let K be a hyperfield and $/\cdot/$ be a hyperabsolute value of K. Then every convergent sequence in K has a unique limit.*

Proof. Let x and y be two limits for $\{x_n\}$ and let $\epsilon > 0$. Then there exists a natural number N such that $\inf /x_n - x/ < \epsilon/2$ and $\inf /x_n - y/ < \epsilon/2$ for all $n \geq N$. Since $\inf /x - y/ \leq \inf /x_n - x/ + \inf /x_n - y/ < \epsilon$, it follows by Proposition 3.9 (7), that $x = y$. □

Proposition 4.14. *Let K be a hyperfield, $k \in K$ and $/\cdot/$ be a hyperabsolute value of K. Then the followings are true:*

(1) *If $x_n \longrightarrow x$ then $/x_n/ \longrightarrow /x/$;*
(2) *$x_n \longrightarrow \underline{0}$ if and only if $/x_n/ \longrightarrow 0$;*
(3) *If $x_n \longrightarrow x$ then $kx_n \longrightarrow kx$;*
(4) *If $x_n \longrightarrow \underline{0}$ and $y_n \longrightarrow \underline{0}$ then $z_n \longrightarrow \underline{0}$ for all $z_n \in x_n + y_n$.*

Proof.

(1) Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\inf /x_n - x/ < \epsilon$ for all $n \geq N$. Applying Proposition 3.15, we get that $|/x_n/ - /x/| \leq \inf /x_n - x/ < \epsilon$. Thus, $/x_n/ \longrightarrow /x/$.
(2) If $x_n \longrightarrow \underline{0}$ then by (1), we get that $/x_n/ \longrightarrow 0$. If $/x_n/ \longrightarrow 0$ then for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $/x_n/ = |/x_n/| < \epsilon$ for all $n \geq N$.
(3) The proof follows from having $\inf /kx_n - kx/ = /k/ \inf /x_n - x/$ (Proposition 3.14).
(4) Let $\epsilon > 0$ and $z_n \in x_n + y_n$. Then there exist $N_1, N_2 > 0$ such that $/x_n/ = \inf /x_n - 0/ < \frac{\epsilon}{2}$, $/y_n/ = \inf /y_n - 0/ < \frac{\epsilon}{2}$ for all $n \geq N = \max\{N_1, N_2\}$. Applying the triangle inequality, we get that $/z_n/ < \epsilon$ for all $n \geq N$. □

Definition 4.15. Let K be a hyperfield and $/\cdot/$ be a hyperabsolute value of K. A sequence $\{x_n\}$ in K is said to be bounded if there exists $M > 0$ such that $/x_n/ \leq M$ for all $n \in \mathbb{N}$.

Proposition 4.16. *Let K be a hyperfield and $/\cdot/$ be a hyperabsolute value of K. Then every convergent sequence in K is bounded.*

Proof. Let $\{x_n\}$ be a sequence in K such that $x_n \longrightarrow x \in K$. Applying the definition of convergent sequences, we get that: there exists $N > 0$ such that $\inf /x_n - x/ < 1$ for all $n \geq N$. Since $x_n \in x_n - x + x$, it follows, by the triangle inequality, that $/x_n/ \leq \inf /x_n - x/ + /x/ = 1 + /x/$. Let $M = \max\{/x_1/, \ldots, /x_{N-1}/, 1 + /x/\}$. It is easy to see that $/x_n/ \leq M$ for all $n \in \mathbb{N}$. □

Definition 4.17. Let K be a hyperfield, $/ \cdot /$ be a hyperabsolute value of K and (x_n) be a sequence in K. Then $\{x_n\}$ is said to be a *Cauchy sequence* if for every $\epsilon > 0$, there exists a natural number N such that $\inf /x_n - x_m/ < \epsilon$ for all $n, m \geq N$.

Proposition 4.18. *Let K be a hyperfield and $/ \cdot /$ be a hyperabsolute value of K. Then every convergent sequence in K is Cauchy.*

Proof. Let $\{x_n\}$ be a convergent sequence in K with $x \in K$ as a convergent limit and $\epsilon > 0$. Then there exists $N > 0$ such that $\inf /x_n - x/ < \frac{\epsilon}{2}, \inf /x_n - x_m/ < \frac{\epsilon}{2}$ for all $m, n \geq N$ implies that $\inf /x_n - x_m/ < \epsilon$. □

The converse of Proposition 4.16 is not true. We illustrate it with the following example.

Example 4.19. Let $K = F_2 = \{0, 1\}$ with the trivial hyperabsolute value on K and $x_n = \{0, 1, 0, 1, \ldots, 0, 1, \ldots\}$ be a bounded sequence in K. One can easily see that $\inf /x_{n+1} - x_n/ = 1$ for all $n \in \mathbb{N}$. Thus, $\{x_n\}$ is not Cauchy in K. Therefore, $\{x_n\}$ is not convergent in K.

Proposition 4.20. *Let K be a hyperfield and $/ \cdot /$ be a hyperabsolute value of K. Then every Cauchy sequence in K is bounded.*

Proof. The proof is similar to that of Proposition 4.16. □

Remark 4.21. Let $\{x_n\}$ be a sequence in K and $n_1 < n_2 < n_3 < \ldots$. A subsequence of $\{x_n\}$ is denoted as $\{x_{n_k}\}$.

Proposition 4.22. *Let K be a hyperfield and $/ \cdot /$ be a hyperabsolute value of K. Then a sequence in K is convergent if and only if every subsequence of it is convergent in K.*

Proof. Let $\{x_n\}$ be a sequence in K. Since $\{x_n\}$ is a subsequence of itself, it follows that if all subsequences of $\{x_n\}$ are convergent then $\{x_n\}$ is convergent. Let $\{x_n\}$ be a convergent sequence with convergent limit $x \in K$, $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ and $\epsilon > 0$. Having $x_n \longrightarrow x$ implies that there exists $N > 0$ such that $\inf /x_n - x/ < \epsilon$ for all $n \geq N$. The latter implies that $\inf /x_{n_k} - x/ < \epsilon$ for all $n_k \geq N$. □

4.2. **Equivalent hyperabsolute values.** Throughout we consider a hyperasolute function $/\cdot/$ satisfing Condition (*).

Definition 4.23. Let K be a hyperfield and $/\cdot/_1, /\cdot/_2$ hyperabsolute values of K. Then $/\cdot/_1$ and $/\cdot/_2$ are *equivalent hyperabsolute values* of K if they induce same metric topology in K. In this case we denote by $/\cdot/_1 \sim /\cdot/_2$.

Remark 4.24. The relation $/\cdot/_1 \sim /\cdot/_2$ is an equivalence relation.

Lemma 4.25. *Let K be a hyperfield and $/\cdot/_1, /\cdot/_2$ be hyperabsolute values of K. If there exists a positive number t such that $/x/_2 = /x/_1^t$ for all $x \in K$, then $/\cdot/_1$ and $/\cdot/_2$ are equivalent hyperabsolute values of K.*

Proof. It suffices to show that open balls in $/\cdot/_1$ are open balls in $/\cdot/_2$ and vice versa. Let $x \in K$ and $r > 0$. An open ball in $/\cdot/_1$ is of the form $\{y \in K : \inf /x-y/_1 < r\}$. Let $z \in x-y$ such that $/z/_1 = \inf /x-y/_1$. Then $/z/_2 = /z/_1^t = \inf /x-y/_2 < r^t$. The latter implies that $\{y \in K : \inf /x-y/_1 < r\} = \{y \in K : \inf /x-y/_2 < r^t\}$. In a similar manner, we can show that open balls in $/\cdot/_2$ are open balls in $/\cdot/_1$. □

Lemma 4.26. *Let K be a hyperfield, $x \in K$ and $/\cdot/_1, /\cdot/_2$ be equivalent hyperabsolute values of K. Then $/x/_1 < 1$ if and only if $/x/_2 < 1$.*

Proof. If $/x/_1 < 1$, then the sequence $x^n \longrightarrow \underline{0}$ (Example 4.12). The latter implies that for every open subset of K containing $\underline{0}$, there exist $N > 0$ such that $x^n \in U$ for all $n \geq N$. If $/x/_1 \geq 1$, then $/x^n/ \geq 1$ for all $n \in \mathbb{N}$. The open ball $B_1(\underline{0}) = \{y \in K : /y/_1 < 1\}$ contains $\underline{0}$ and does not contain x^n for all $n \in \mathbb{N}$. We get similar results if we put $/x/_2$ instead of $/x/_1$. We get now that both conditions: $/x/_1 < 1$ and $/x/_2 < 1$ are equivalent to saying that for every open subset U (containing $\underline{0}$) of K, $x^n \in U$ for all but finitely many n. Thus, $/x/_1 < 1$ if and only if $/x/_2 < 1$. □

Corollary 4.27. *Let K be a hyperfield, $x \in K$ and $/\cdot/_1, /\cdot/_2$ be equivalent hyperabsolute values of K. Then*

$$/x/_1 > 1 \quad \textit{if and only if} \quad /x/_2 > 1.$$

Proof. Since $/x/_1 > 1$, it follows that $x \neq 0$ and that x^{-1} exists. Having $/x^{-1}/_1 = \frac{1}{/x/_1} < 1$ is equivalent to saying $/x^{-1}/_2 = \frac{1}{/x/_2} < 1$. The latter is equivalent to saying $/x/_2 > 1$. □

Corollary 4.28. *Let K be a hyperfield, $x \in K$ and $/\cdot/_1, /\cdot/_2$ be equivalent hyperabsolute values of K. Then*

$$/x/_1 = 1 \quad \textit{if and only if} \quad /x/_2 = 1.$$

Proof. $/x/_1 = 1$ is equivalent to $/x/_1 \not< 1$ and $/x/_1 \not> 1$. Using Lemma 4.26 and Corollary 4.27, we get that our statement is equivalent to $/x/_2 \not< 1$ and $/x/_2 \not> 1$. The latter is equivalent to $/x/_2 = 1$. □

Corollary 4.29. *Let K be a hyperfield and $/\cdot/_1$ be the trivial hyperabsolute value of K. If $/\cdot/_2$ is an equivalent hyperabsolute value of K to $/\cdot/_1$ then $/\cdot/_2$ is the trivial hyperabsolute value of K.*

Proof. Let $x \in K \setminus \{\underline{0}\}$. Then $/x/_1 = 1$. Using Corollary 4.28, we get that $/x/_2 = 1$. □

Lemma 4.30. *Let K be a hyperfield and $/\cdot/_1, /\cdot/_2$ be equivalent hyperabsolute values of K. Then there exists a positive number t such that $/x/_2 = /x/_1^t$ for all $x \in K$.*

Proof. Let $x, y \in K \setminus \{0\}$. If $/x/_1 = 1$ then $/x/_2 = 1 = /x/_1^t$ by Corollary 4.28. Without loss of generality, we assume that $/x/_1 > 1$. If $/x/_1 < 1$, we get that $/x^{-1}/_1 > 1$. Let $t = \frac{\log /x/_2}{\log /x/_1} > 0$. We claim that $\frac{\log /x/_2}{\log /x/_1} = \frac{\log /y/_2}{\log /y/_1}$. If $\frac{\log /x/_2}{\log /x/_1} \neq \frac{\log /y/_2}{\log /y/_1}$, then either $\frac{\log /x/_2}{\log /x/_1} < \frac{\log /y/_2}{\log /y/_1}$ or $\frac{\log /x/_2}{\log /x/_1} > \frac{\log /y/_2}{\log /y/_1}$. We consider the case $\frac{\log /x/_2}{\log /x/_1} < \frac{\log /y/_2}{\log /y/_1}$. The other is done in a similar manner. The case $\frac{\log /x/_2}{\log /x/_1} < \frac{\log /y/_2}{\log /y/_1}$ implies that $\frac{\log /x/_2}{\log /y/_2} < \frac{\log /x/_1}{\log /y/_1}$ over the set of real numbers. The density of rational numbers in $\mathbb{R}$ implies that there exists a rational number $\frac{m}{n}$ such that $\frac{\log /x/_2}{\log /y/_2} < \frac{m}{n} < \frac{\log /x/_1}{\log /y/_1}$. Simplifying our inequality, we get:

$$n \log /x/_2 < m \log /y/_2 \quad \text{and} \quad m \log /y/_1 < n \log /x/_1.$$

The latter implies that

$$/x^n y^{-m}/_2 = \frac{/x^n/_2}{/y^m/_2} < 1 \quad \text{and} \quad /x^n y^{-m}/_1 = \frac{/x^n/_1}{/y^m/_1} > 1.$$

The latter contradicts the result of Lemma 4.26. □

Theorem 4.31. *Let K be a hyperfield and $/\cdot/_1, /\cdot/_2$ be hyperabsolute values of K. Then $/\cdot/_1$ and $/\cdot/_2$ are equivalent hyperabsolute values of K if and only if there exists a positive number t such that $/x/_2 = /x/_1^t$ for all $x \in K$.*

Proof. The proof results from Lemmas 4.25 and 4.30. □

5. Relationship between hyperabsolute values and hypervaluation

In [6], hypervaluation of a hyperfield onto a totally ordered canonical hypergroup was defined. In this section, We consider $\mathbb{R}$ as our totally ordered canonical hypergroup.

Definition 5.1. ([6]) Let K be a hyperfield. A *hypervaluation* on K is a map $v : K \longrightarrow \mathbb{R} \cup \{\infty\}$ that satisfies the following conditions for all $x, y \in K$:

(1) $v(x) = \infty$ if and only if $x = \underline{0}$;
(2) $v(xy) = v(x) + v(y)$;
(3) $v(z) \geq \min\{v(x), v(y)\}$ for all $z \in x + y$.

Example 5.2. Let K be any hyperfield and define $v : K \longrightarrow \mathbb{R} \cup \{\infty\}$ as follows:

$$v(x) = \begin{cases} \infty, & \text{if } x = \underline{0}; \\ 0, & \text{otherwise.} \end{cases}$$

Then it is easy to see that v is hypervaluation on K. Such hypervaluation is called *trivial hypervaluation.*

Lemma 5.3. ([6]) *Let H be a totally ordered canonical hypergroup with identity e and $v : K \to H \cup \{\infty\}$ be a hypervaluation on K. Then*

(1) $v(1) = e$*;*
(2) $v(x^{-1}) = -v(x)$.

Proposition 5.4. *Let K be any hyperfield and v be a hypervaluation on K. Then the following are true for all $x \in K$:*

(1) $v(-1) = 0$*;*
(2) $v(x) = v(-x)$*;*
(3) *If there exist $d \in \mathbb{N}$ with the property that $x^d = 1$, then $v(x) = 0$.*

Proof. (1) Since $(-1)(-1) = 1$, it follows that $0 = v(1) = v(-1) + v(-1)$. Thus, $v(-1) = 0$.
(2) Applying (1), we get that $v(-x) = v(-1) + v(x) = v(x)$.
(3) $0 = v(1) = v(x^d) = dv(x)$. □

Starting from a hyperabsolute value of K, we can construct a hypervaluation on K and vice versa.

Theorem 5.5. *Let K be any hyperfield, $s > 0$ and $/\cdot/$ be a non-Archimedean hyperabsolute value of K. Define $v_s : K \longrightarrow \mathbb{R} \cup \{\infty\}$ as follows:*

$$v_s(x) = \begin{cases} \infty, & \text{if } x = \underline{0}; \\ -s \log /x/, & \text{otherwise.} \end{cases}$$

Then v_s is a hypervaluation on K.

Proof. We prove that the conditions of Definition 5.1 are satisfied for v_s. Let $x, y \in K$. $v_s(x) = \infty$ if and only if $x = \underline{0}$ as $-s \log /x/ \neq \infty$.

$$v_s(xy) = \begin{cases} \infty, & \text{if } x = \underline{0} \text{ or } y = \underline{0}; \\ -s \log /xy/, & \text{otherwise.} \end{cases}$$

Since $/\cdot/$ is an absolute value of K, it follows that $-s\log/xy/ = -s\log(/x//y/) = -s\log/x/ - s\log/y/$. It is easy to see now that $v_s(xy) = v_s(x) + v_s(y)$. Let $z \in x+y$. If $z = \underline{0}$, then $v(z) = \infty \geq \min\{v(x), v(y)\}$. If $z \neq \underline{0}$, then $v_s(z) = -s\log/z/$. Having $/z/ \leq \max\{/x/, /y/\}$ implies that $\log/z/ \leq \max\{\log/x/, \log/y/\}$. The latter implies that

$$\begin{aligned} v_s(z) &= -s\log/z/ \\ &\geq -s\max\{\log/x/, \log/y/\} \\ &= \min\{-s\log/x/, -s\log/y/\} \\ &= \min\{v_s(x), v_s(y)\}. \end{aligned}$$

□

Example 5.6. Let $(\mathbb{T}, \oplus, \odot)$ be the Tropical hyperfield and define v on $\mathbb{T}$ as follows: For all $x \in \mathbb{T}$,

$$v(x) = \begin{cases} \infty, & \text{if } x = \underline{0} = -\infty; \\ -x, & \text{otherwise.} \end{cases}$$

Example 3.5 and Theorem 5.5 assert that v is a hypervaluation on $\mathbb{T}$.

Example 5.7. Let $(\mathbb{C}, \oplus, \odot)$ be the complex tropical hyperfield and define v on $\mathbb{C}$ as follows: For all $x \in \mathbb{C}$,

$$v(x) = \begin{cases} \infty, & \text{if } x = 0; \\ -\ln(|x|), & \text{otherwise.} \end{cases}$$

Example 3.6 and Theorem 5.5 assert that v is a hypervaluation on $\mathbb{C}$.

Theorem 5.8. *Let K be any hyperfield, $q > 1$ and v be a hypervaluation on K. Define $/\cdot/_q : K \longrightarrow \mathbb{R}_{\geq 0}$ as follows:*

$$/x/_q = \begin{cases} 0, & \text{if } x = \underline{0}; \\ q^{-v(x)}, & \text{otherwise.} \end{cases}$$

Then $/\cdot/_q$ is a non-Archimedean hyperabsolute value of K.

Proof. Let $x, y \in K$. We show that conditions of Definition 3.1 are satisfied for $/\cdot/_q$. Since $q^{-v(x)} \neq 0$ for all $x \in K$, it follows that $/x/_q = 0$ if and only if $x = \underline{0}$.

$$/xy/_q = \begin{cases} 0, & \text{if } x = \underline{0} \text{ or } y = \underline{0}; \\ q^{-v(xy)}, & \text{otherwise.} \end{cases}$$

Since v is a hypervaluation on K, it follows that $v(xy) = v(x) + v(y)$. Thus, $q^{-v(xy)} = q^{-v(x)-v(y)} = q^{-v(x)}q^{-v(y)}$. We get now that $/xy/_q = /x/_q/y/_q/$.

Let $z \in x+y$. Having v is a hypervaluation on K implies that $v(z) \geq \min\{v(x), v(y)\}$. The latter implies that

$$-v(z) \leq -\min\{v(x), v(y)\} = \max\{-v(x), -v(y)\}.$$

We get now that $q^{-v(z)} \leq \max\{q^{-v(x)}, q^{-v(y)}\}$. □

Corollary 5.9. *Let K be a finite hyperfield and v be a hypervaluation on K. Then v is the trivial hypervaluation.*

Proof. Suppose, for contradiction, that v is a non-trivial hypervaluation on K. Theorem 5.8 asserts that $/\cdot/_2 : K \longrightarrow \mathbb{R}_{\geq 0}$ as follows:

$$/x/_2 = \begin{cases} 0, & \text{if } x = \underline{0}; \\ 2^{-v(x)}, & \text{otherwise} \end{cases}$$

is a hyperabsolute value of K. Using Proposition 3.13, we get that $/\cdot/_2$ is the trivial hyperabsolute value of K. □

Corollary 5.10. *Let K be a hyperfield and v a hypervaluation on K. Then*

(1) *the set $R_v = \{x \in K : v(x) \geq 0\} = \{x \in K : /x/_q \geq 1\}$ is a Krasner hyperring;*
(2) *the set $U_v = \{x \in K : v(x) = 0\} = \{x \in K : /x/_q = 1\}$ is the group of units of R_v;*
(3) *the set $M_v = \{x \in K : v(x) > 0\} = \{x \in K : /x/_q > 1\}$ is the only maximal hyperideal of R_v.*

Proof. The proof of (1) is straightforward. To prove (2), let x be a unit in R_v with inverse x' in R_v. Then $v(xx') = v(1) = 0$. The latter implies that $v(x) + v(x') = 0$. Since $v(x) \geq 0$ and $v(x') = -v(x) \geq 0$, it follows that $v(x') = v(x) = 0$. To prove (3), let M be a hyperideal of R_v such that $M_v \subset M \subseteq R_v$. Then there exists $x \in M$ with $v(x) = 0$. Using (2), we get that x is a unit in R_v. Since M is a hyperideal of R_v and $x \in R_v$, it follows that $x'x = 1 \in M$ and hence, $M = R_v$. □

Acknowledgment: The author[1] acknowledges Abu Dhabi University for their kind support with the research grant number: 19300884. The author[2] acknowledges Yazd University for their support. The authors[3,4] thank the Manipal Institute of Technology, Manipal and Bengaluru, Manipal Academy of Higher Education, Manipal, for the kind encouragement. The author[4] acknowledges ANRF(SERB), Govt. of India for the Teachers Associateship for Research Excellence (TARE) fellowship TAR/2022/000219.

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