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A NEW ITERATIVE APPROACH FOR ESTIMATING FIXED POINTS UNDER CHATTERJEE SUZUKI CONDITION

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Abstract. Fixed point (FP) theory is an important subject of analysis that can give effective techniques for handling nonlinear issues. The existence and uniqueness of solutions to integral and differential equations are proven using FP theory. Computing the exact value of a solution to a nonlinear issue is frequently challenging. In such a case, the proposed solution's approximate value is always considered. Finding a FP using particular schematic methods requires analyzing various features of FPs, such as data dependence, convergence, and stability. Research on new iterative techniques for functional equation solving and FP analysis is active and has many useful applications. The first iterative approach for approximating a FP of a contraction mapping T on a nonempty subset S of a Banach space (BN-space) D is the Picard iterative approach. Numerous authors have created a variety of methods for estimating the FP. This paper proposes an efficient new iterative approach for approximating the FP under the Chatterjee-Suzuki-C CSC condition which is called the R* iterative approach. In the beginning, a new iterative approach is provided. Afterward, it is shown via analytical demonstration that the suggested method converges to an FP for contraction map more quickly than some well-known methods. Furthermore, some important weak and strong convergence results of the proposed iterative approach are established in the setting of BN-space. To support the primary conclusions, a brief example has been presented to demonstrate the efficiency of the recommended iterated procedure via the class of the defined mappings that fulfill CSC condition.

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1. Introduction

Many applied science fields have problems that are either extremely difficult or impossible to answer with the common analytical methods covered in this literature. The estimated value of the desired answer is always required in an instance like this. FP theory offers a number of helpful methods, including approximating the values of such solutions. For these kinds of issues, the ideal approximation solution may be expressed as the FP of a suitable operator, that is, as the resolution of an analogous FP equation $\mathrm{T}s = s$, where the self-map T is any relevant operator defined on a subset of a given space. A large number of authors have studied the FP theory in many fields and have obtained many fruitful results [2, 13, 17, 19]. In addition, several authors have recently addressed the generalization of non-expansive(Non-exp) mappings for various applications see [3, 5, 8, 14, 22].

In this context, other additional mapping classes with intriguing features were developed in the subsequent years. Suzuki [23] proposed a new type of generalized Nonexpansive mappings in 2008, which he refers to as Suzuki's generalized Nonexpansive or condition $(\mathcal{C})$. In 2018, Patir et al [15] presented an additional generalization of Nonexpansive mappings, referred to as the condition $B\gamma$, and demonstrated some convergence outcomes for these mappings in BN-spaces that are uniformly convex(UC-Space). In [24] the authors describe a new family of generalized nonexpansive operators, denoted (α, β, γ)-nonexpansive mappings.

Many authors have utilized various iterative approaches for estimating the FPs. A novel iterative approach was presented by Mann [10] in 1953.

$$\lambda_{n+1} = (1 - u_n)\ \lambda_n + u_n \mathrm{T}\lambda_n, \quad \text{for } n \geq 0, \tag{1.1}$$

where $\mathrm{T} : \mathbb{S} \ \to \mathbb{S}$ is a mapping such that $\mathbb{S}$ is a subset of BN-space $\mathcal{D}$, $u_n \in$ (0,1) and a sequence $\{\lambda_n\}$ is generated by $\lambda_\circ \in \mathbb{S}$.

Another iterative approach was developed by Noor [11].

$$\begin{cases} \omega_n = (1 - v_n)\lambda_n + v_n \mathrm{T}\lambda_n, \\ \rho_n = (1 - s_n)\lambda_n + s_n \mathrm{T}\omega_n, \\ \lambda_{n+1} = (1 - u_n)\lambda_n + u_n \mathrm{T}\rho_n, \end{cases} \tag{1.2}$$

where u_n, v_n, $s_n \in$ (0,1).

Agarwal et al. [1] presented an iterative approach as follows.

$$\begin{cases} \rho_n = (1 - s_n)\lambda_n + s_n \mathrm{T}\ \lambda_n, \\ \lambda_{n+1} = (1 - u_n)\mathrm{T}\ \lambda_n + u_n \mathrm{T}\rho_n. \end{cases} \tag{1.3}$$

Change et al. [6] presented an iterative approach called CR- iterative.

$$\begin{cases} \omega_n = (1 - v_n)\lambda_n + v_n \mathrm{T}\lambda_n, \\ \rho_n = (1 - s_n)\mathrm{T}\lambda_n + s_n \mathrm{T}\omega_n, \\ \lambda_{n+1} = (1 - u_n)\rho_n + u_n \mathrm{T}\rho_n. \end{cases} \tag{1.4}$$

Prominent mathematicians have recently proposed numerous iterative approaches that accelerate convergence to the FP [9, 16, 18, 21].

In 2011, Erdal and Kenan [7] introduced a new condition called Chatterjee–Suzuki–C ($\mathcal{CSC}$) condition which is a modification of Suzuki's C-condition.

In this study, under this condition, certain convergence findings of a novel suggested iterative approach known as $R*$-iteration approach are given. A numerical comparison is made between the new iteration approach's convergence speed and that of the other iteration approach.

2. Preliminaries

This section begins with some basic ideas and established findings that are needed in order to get at the primary findings.

For the mapping T, an FP is a point $g_\circ \in \mathbb{S}$ that fulfills the formula $g_\circ = \mathrm{T}g_\circ$. Typically, the FP set of T is represented by $\mathcal{F}_{\mathrm{T}}$.

Definition 2.1. ([7]) A mapping $\mathrm{T} : \mathbb{S} \to \mathbb{S}$ fulfills $\mathcal{CSC}$ condition if the inequality described below is valid. If $\frac{1}{2}\,||s_1 - \mathrm{T}s_2|| \leq ||s_1 - s_2||$, then

$$||\mathrm{T}s_1 - \mathrm{T}s_2|| \leq \frac{1}{2}\,(||s_1 - \mathrm{T}s_2|| + ||s_2 - \mathrm{T}s_1||).$$

Definition 2.2. ([12]) A BN-space $\mathcal{D}$ fulfills Opial's condition if and only if the sequence $\{\lambda_n\} \subseteq \mathcal{D}$ converges in the weak sense to $s_\circ \in \mathbb{S}$, and

$$\limsup_{n\to\infty} \|\lambda_n - s_\circ\| < \limsup_{n\to\infty} \|\lambda_n - a_\circ\|$$

for all $a_\circ \in \mathcal{D} - \{s_\circ\}$ is hold.

Definition 2.3. ([25]) Let $\{\lambda_n\} \subseteq \mathcal{D}$ be a bounded sequence. If $\emptyset \neq \mathbb{S} \subseteq \mathcal{D}$ (where $\mathbb{S}$ is convex and closed), then the asymptotic radius of $\{\lambda_n\}$ which corresponds to $\mathbb{S}$ is given by

$$\mathfrak{N}(\mathbb{S},\ \{\lambda_n\}) = \inf\{\limsup_{n\to\infty} \|\lambda_n - s\| \ : s \in \mathbb{S}\}.$$

Likewise, $\{\lambda_n\}$ corresponding to $\mathbb{S}$ has an asymptotic center that is specified and demonstrated via the formula

$$\mathcal{A}(\mathbb{S}, \{\lambda_n\}) = \{s \in \mathbb{S} : \limsup_{n\to\infty} \|\lambda_n - s\| = \mathfrak{N}(\mathbb{S}, \{\lambda_n\})\}. \tag{2.1}$$

Proposition 2.4. ([7]) *Suppose $\mathcal{D}$ be a BN-space and $\emptyset \neq \mathbb{S} \subseteq \mathcal{D}$ is closed. For the map $\mathrm{T}:\mathbb{S} \rightarrow \mathbb{S}$ the subsequent attributes are valid:*

(1) *If T satisfies the $\mathcal{CSC}$ condition, and $\mathcal{F}_{\mathrm{T}} \neq \emptyset$, then $\|\mathrm{T}s - g\| \leq \|s - g\|$ for each $s \in \mathbb{S}$ and $g \in \mathcal{F}_{\mathrm{T}}$.*

(2) *If T satisfies the $\mathcal{CSC}$ condition, then $\mathcal{F}_{\mathrm{T}}$ is closed. Moreover, $\mathcal{F}_{\mathbb{S}}$ is convex if $\mathbb{S}$ is convex and $\mathcal{D}$ is strictly convex.*

(3) *If T satisfies the $\mathcal{CSC}$ condition, then for arbitrary $s_1, s_2 \in \mathbb{S}$,*

$$||s_1 - \mathrm{T}s_2|| \leq 5||s_1 - \mathrm{T}s_1|| + ||s_1 - s_2||.$$

(4) *If T satisfies $\mathcal{CSC}$ condition, $\{\lambda_n\}$ is weakly convergent to g, and $\lim_{n\to\infty} ||\mathrm{T}\lambda_n - \lambda_n|| = 0$, then $g \in \mathcal{F}_{\mathrm{T}}$ given that $\mathcal{D}$ fulfills Opial's condition.*

Lemma 2.5. ([20]) *Let $0 < \gamma \leq \vartheta_n \leq \delta < 1$ and $\mathcal{D}$ is $UCB-$space. If there is $a \geq 0$ (real number) such that $\{p_n\}$ and $\{q_n\}$ in $\mathcal{D}$ fulfill*

$$\text{limsup}_{n\to\infty}\|p_n\| \leq a, \quad \text{limsup}_{n\to\infty}\|q_n\| \leq a$$

and

$$\text{limsup}_{n\to\infty}\|\vartheta_n p_n + (1-\vartheta_n)q_n\| = a,$$

then

$$\text{limsup}_{n\to\infty}\|p_n - q_n\| = 0.$$

Definition 2.6. ([4]) Let $\{p_n\}$, $\{q_n\}$ be two sequences of real numbers converging to p and q respectively. If $\lim_{n\longrightarrow\infty} \frac{|p_n - p|}{|q_n - q|} = 0$, then $\{p_n\}$ converges faster than $\{q_n\}$.

3. Main results

Inspired by the aforementioned iterative approaches, a novel iterative approach called R*- iteration is present in the following manner:

$$\begin{cases} \lambda_1 = \lambda \in \mathbb{S}, \\ \omega_n = \mathrm{T}((1-v_n)\lambda_n + v_n \mathrm{T}\lambda_n), \\ \rho_n = \mathrm{T}(\mathrm{T}\omega_n), \\ \lambda_{n+1} = \mathrm{T}((1-u_n)\mathrm{T}\omega_n + u_n \mathrm{T}\rho_n). \end{cases} \tag{3.1}$$

In this section, it is first demonstrated that the proposed method reaches the FP of the contraction map faster than some other known methods. This section of this study also presents some convergence results for the mapping that satisfies the $\mathcal{CSC}$ condition using the new iteration technique Eq.(3.1). In addition, a numerical example is provided that compares the proposed iteration speed with the other iteration methods.

The following shows that iteration Eq.(3.1) converges faster than the iteration of Mann, Noor, Agrawal and CR for contraction mappings in the sense of Berinde [4].

Initially, the following outcome is required.

Theorem 3.1. *Let $\{\lambda_n\}$ be a sequence generated using a novel approach Eq. (3.1). Then $\{\lambda_n\}$ converges to FP of* T.

Proof. Through Eq.(3.1), for every $s \in \mathcal{F}_{\mathrm{T}}$

$$\begin{aligned}\|\omega_n - s\| &= \|\mathrm{T}((1-v_n)\lambda_n + v_n\mathrm{T}\lambda_n) - s\| \\ &\le \|k((1-v_n)\lambda_n + v_n\mathrm{T}\lambda_n) - s\| \\ &\le k(1-v_n)\|\lambda_n - s\| + kv_n\|\mathrm{T}\lambda_n - s\| \\ &\le k(1-v_n)\|\lambda_n - s\| + k^2v_n\|\lambda_n - s\| \\ &\le k(1-(1-k)v_n)\|\lambda_n - s\|,\end{aligned} \tag{3.2}$$

so that

$$\begin{aligned}\|\rho_n - s\| &= \|\mathrm{T}(\mathrm{T}\omega_n) - s\| \\ &\le k\|\mathrm{T}\omega_n - s\| \\ &\le k^2\|\omega_n - s\|\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\|\lambda_{n+1} - s\| &= \|\mathrm{T}((1-u_n)\mathrm{T}\omega_n + u_n\mathrm{T}\rho_n) - s\| \\ &\le \|k((1-u_n)\mathrm{T}\omega_n + u_n\mathrm{T}\rho_n) - s\| \\ &\le k(1-u_n)\|\mathrm{T}\omega_n - s\| + ku_n\|\mathrm{T}\rho_n - s\| \\ &\le k^2(1-u_n)\|\omega_n - s\| + k^2u_n\|\rho_n - s\| \\ &\le k^3(1-u_n)(1-(1-k)v_n)\|\lambda_n - s\| \\ &\quad + k^4u_n(1-(1-k)v_n)\|\lambda_n - s\| \\ &\le k^3[1-(1-k)u_n][1-(1-k)v_n]\|\lambda_n - s\|.\end{aligned} \tag{3.4}$$

By using the fact that $0 < 1-(1-k)\,u_n \le 1$ and $0 < 1-(1-k)\,v_n \le 1$, we have

$$\|\lambda_{n+1} - s\| \quad \le k^3\,\|\lambda_n - s\|\,. \tag{3.5}$$

Inductively, we get

$$\|\lambda_{n+1} - s\| \le k^{3(n+1)}\,\|\lambda_0 - s\|\,. \tag{3.6}$$

Since $0<k<1$, $\{\lambda_n\}$ converges to s. □

Theorem 3.2. *Suppose that the sequence $\{\lambda_{1,n}\}$ is introduced by Mann Eq.*(1.1), *$\{\lambda_{2,n}\}$ by Noor Eq.*(1.2), *$\{\lambda_{3,n}\}$ by Agrawal Eq.*(1.3), *$\{\lambda_{4,n}\}$ by CR Eq.*(1.4), *and $\{\lambda_n\}$ by iterative Eq.*(3.1) *which converges to the same point s. Then* *iterative scheme Eq.*(3.1) *converges faster to a FP than all the approaches Eq.*(1.1)–*Eq.*(1.4).

Proof. Applying Eq.(3.6) of Theorem 3.1 yields,

$$\|\lambda_{n+1} - s\| \quad \leq k^{3(n+1)} \|\lambda_0 \quad - s\| = c_n. \tag{3.7}$$

From Eq.(1.2),

$$\begin{aligned} \|\omega_n - s\| &= \|(1 - v_n)\lambda_n + v_n \mathrm{T}\lambda_n - s\| \\ &\leq (1 - v_n\|\lambda_n - s\| + v_n\|\mathrm{T}\ \lambda_n - s\| \\ &\leq (1 - v_n)\|\lambda_n - s\| + kv_n\|\lambda_n - s\| \\ &\leq (1 - (1 - k)v_n)\|\lambda_n - s\|. \end{aligned} \tag{3.8}$$

It can be easily seen that $0 < 1 - (1 - k)v_n \leq \ 1$, so we obtain

$$\|\omega_n - s\| \leq \|\lambda_n \quad - s\|\,. \tag{3.9}$$

Using Eq.(1.4), we obtain that

$$\begin{aligned} \|\rho_n - s\| &= \|(1 - s_n)\lambda_n + s_n \mathrm{T}\ \omega_n - s\| \\ &\leq (1 - s_n)\|\lambda_n - s\| + s_n\|\mathrm{T}\ \omega_n - s\| \\ &\leq (1 - s_n)\|\lambda_n - s\| + ks_n|\ \omega_n - s\| \\ &\leq (1 - s_n)\,\|\lambda_n - s\| \ + ks_n(1 - (1 - k)v_n)\ \ \|\lambda_n - s\| \\ &\leq [1 - (1 - k + kv_n - k^2 v_n\ )s_n]\,\|\lambda_n - s\|\,. \end{aligned} \tag{3.10}$$

Again It can be easily seen that $0 < 1 - (1 - k + kv_n - k^2 v_n\ )s_n \leq \ 1$, then we obtain

$$\|\rho_n - s\| \leq \|\lambda_n \quad - s\|\,. \tag{3.11}$$

Using Eq.(3.11), we obtain that

$$\begin{aligned} \|\lambda_{n+1} - s\| &= \|((1 - u_n\lambda_n + u_n \mathrm{T}\ \rho_n - s\| \\ &\leq (1 - u_n)\,\|\lambda_n - s\| + u_n\,\|\mathrm{T}\ \rho_n - s\| \\ &\leq (1 - u_n\|\lambda_n \quad - s\| + ku_n\|\ \rho_n - s\| \\ &\leq (1 - u_n)\|\lambda_n - s\| \\ &\quad + ku_n(1 - (1 - k + kv_n - k^2 v_n)s_n)\ \|\lambda_n - s\| \\ &\leq [1 - (1 - k + ks_n - k^2 s_n + k^2 s_n v_n - k^3 s_n v_n)u_n]\|\lambda_n - s\|. \end{aligned} \tag{3.12}$$

Using the fact that

$$0 < 1 - \left(1 - k + ks_n - k^2 s_n + k^2 s_n v_n - k^3 s_n v_n\right) u_n \leq \ 1,$$

then we obtain

$$\|\lambda_{n+1} - s\| \leq \|\lambda_n - s\|. \tag{3.13}$$

Inductively, we get

$$\|\lambda_{n+1} - s\| \le \|\lambda_0 - s\|. \tag{3.14}$$

Let

$$\|\lambda_{2,n} - s\| \le \|\lambda_{0,n} \quad - s\| = c_{2,n}.$$

Then

$$\frac{c_n}{c_{2,n}} = \frac{k^{3(n+1)} \|\lambda_0 \quad - s\|}{\|\lambda_{2,0} \quad - s\|}.$$

Hence, $\{\lambda_n\}$ converges faster than $\{\lambda_{2,n}\}$ to s because $0 < k < 1$, then $\frac{c_n}{c_{2,n}} \longrightarrow 0$ as $n \longrightarrow \infty$.

Using an analogous approach, it can also demonstrate that iterative Eq.(3.1) possesses a faster rate of convergence to s than any other leading iterative approaches. □

The following are some convergence outcomes for the map that fulfills the $\mathcal{CSC}$ condition utilizing the novel iteration approach Eq.(3.1). First, the following main lemma will be discussed.

Lemma 3.3. *Suppose that $\mathcal{D}$ is UC-Space and $\emptyset \neq \mathbb{S} \subseteq \mathcal{D}$ (where $\mathbb{S}$ closed and convex). Consider $\mathrm{T} : \mathbb{S} \to \mathbb{S}$ fulfill $\mathcal{CSC}$ condition and $\mathcal{F}_{\mathrm{T}} \neq \emptyset$. If $\{\lambda_n\}$ produced by the iteration approach Eq.*(3.1)*, then*

$$\lim_{n\to\infty} \|\mathrm{T}\ \lambda_n - g_\circ\|$$

exists for all $g_\circ \in \mathcal{F}_{\mathrm{T}}$.

Proof. Consider $g_\circ \in \mathcal{F}_{\mathrm{T}}$. Based on Proposition 2.4(1), we get

$$\begin{aligned} \|\omega_n - g_\circ\| &= \|\mathrm{T}((1-v_n)\lambda_n + v_n \mathrm{T}\lambda_n) - g_\circ\| \\ &\le \|((1-v_n)\lambda_n + v_n \mathrm{T}\lambda_n) - g_\circ\| \\ &\le (1-v_n)\|\lambda_n - g_\circ\| + v_n\|\mathrm{T}\lambda_n - g_\circ\| \\ &\le (1-v_n)\|\lambda_n - g_\circ\| + v_n\|\lambda_n - g_\circ\| \\ &= \|\lambda_n - g_\circ\|. \end{aligned} \tag{3.15}$$

Moreover,

$$\begin{aligned} \|\rho_n - g_\circ\| &= \|\mathrm{T}(\mathrm{T}(\omega_n)) \quad - g_\circ\| \\ &\le \|\mathrm{T}(\omega_n) \quad - g_\circ\| \\ &\le \|\omega_n - g_\circ\|. \end{aligned} \tag{3.16}$$

It is evident from Eq.(3.16) that

$$\begin{aligned}\|\lambda_{n+1} - g_\circ\| &= \|\mathrm{T}((1-u_n)\mathrm{T}(\omega_n) + u_n\mathrm{T}(\rho_n)) - g_\circ\| \\ &\le \|(1-u_n)\mathrm{T}(\omega_n) + u_n\mathrm{T}(\rho_n) - g_\circ\| \\ &\le (1-u_n)\|\mathrm{T}(\omega_n) - g_\circ\| + u_n\|\mathrm{T}\rho_n - g_\circ\| \\ &\le (1-u_n)\|\omega_n - g_\circ\| + u_n\|\rho_n - g_\circ\| \\ &\le (1-u_n)\|\lambda_n - g_\circ\| + u_n\|\lambda_n - g_\circ\| \\ &= \|\lambda_n - g_\circ\|.\end{aligned}$$

Thus, $\lim_{n\to\infty}\|\lambda_n - g_\circ\|$ exists for all $g_\circ \in \mathcal{F}_\mathrm{T}$. □

Theorem 3.4. *Assume that $\mathcal{D}$ is UC-Space and $\emptyset \neq \mathbb{S} \subseteq \mathcal{D}$ (where $\mathbb{S}$ closed and convex set). Let $\mathrm{T} : \mathbb{S} \to \mathbb{S}$ satisfying $\mathcal{CSC}$ condition such that $\mathcal{F}_\mathrm{T} \neq \emptyset$. If $\{\lambda_n\}$ produced by the iteration approach Eq.(3.1), then $\mathcal{F}_\mathrm{T} \neq \emptyset$ if and only if $\lim_{n\to\infty}\|\mathrm{T}\lambda_n - \lambda_n\| = 0$ and $\{\lambda_n\}$ is bounded.*

Proof. Assume that $\mathcal{F}_\mathrm{T} \neq \emptyset$ and consider $g_\circ \in \mathcal{F}_\mathrm{T}$. According to the preceding Lemma 3.3, $\lim_{n\to\infty}\|\lambda_n - g_\circ\|$ exists and $\{\lambda_n\}$ is bounded. Let

$$\lim_{n\to\infty}\|\lambda_n - g_\circ\| = \beta. \tag{3.17}$$

To prove $\lim_{n\to\infty}\|\lambda_n - g_\circ|\ = 0$. From Eq.(3.15), we can write

$$\|\omega_n - g_\circ\| \le \|\lambda_n - g_\circ\|,$$

it implies that

$$\limsup_{n\to\infty}\|\omega_n - g_\circ\| \le \limsup_{n\to\infty}\|\lambda_n - g_\circ\| = \beta. \tag{3.18}$$

According to Proposition 2.4 (1), we obtain

$$\limsup_{n\to\infty}\|\mathrm{T}\lambda_n - g_\circ\| \le \limsup_{n\to\infty}\|\lambda_n - g_\circ\| = \beta. \tag{3.19}$$

On the other hand,

$$\begin{aligned}\|\lambda_{n+1} - g_\circ\| &= \|\mathrm{T}((1-u_n)\mathrm{T}(\omega_n) + u_n\mathrm{T}(\rho_n)) - g_\circ\| \\ &\le \|(1-u_n)\mathrm{T}(\omega_n) + u_n\mathrm{T}(\rho_n) - g_\circ\| \\ &\le (1-u_n)\|\mathrm{T}\omega_n - g_\circ\| + u_n\|\mathrm{T}\rho_n - g_\circ\| \\ &\le (1-u_n)\|\omega_n - g_\circ\| + u_n\|\omega_n - g_\circ\| \\ &\le \|\omega_n - g_\circ\|.\end{aligned}$$

Combining this with Eq.(3.17) yields that

$$\beta \le \lim_{n\to\infty} inf\|\omega_n - g_\circ\|. \tag{3.20}$$

Utilizing Eq.(3.18) and Eq.(3.20), we get

$$\lim_{n\to\infty} \|\omega_n - g_\circ\| = \beta. \tag{3.21}$$

Since

$$\|\omega_n - g_\circ\| = \|\mathrm{T}((1-v_n)(\lambda_n - g_\circ) + v_n(\mathrm{T}\lambda_n - g_\circ)\|,$$

by Eq.(3.21) we have

$$\lim_{n\to\infty} \|\mathrm{T}((1-v_n)(\lambda_n - g_\circ) + v_n(\mathrm{T}\ \lambda_n\ - g_\circ)\| = \beta. \tag{3.22}$$

Based on Eq.(3.17), Eq.(3.19), Eq.(3.22), and Lemma 3.3, we conclude that

$$\lim_{n\to\infty} \|\mathrm{T}\ \lambda_n - \lambda_n\| = 0.$$

In contrast, consider that $\lim_{n\to\infty} \|\mathrm{T}\ \lambda_n - \lambda_n\| = 0$ and $\{\lambda_n\}$ is bounded. Let $g_\circ \in \mathcal{A}(\mathbb{S},\ \{\lambda_n\}\ )$. Then according to Proposition 2.4 (3), we obtain,

$$\begin{aligned}
\mathcal{A}(\mathrm{T}g_\circ,\ \{\lambda_n\}) &= \limsup_{n\to\infty} \|\lambda_n - \mathrm{T}g_\circ\| \\
&\leq 5 \limsup_{n\to\infty} \|\mathrm{T}\lambda_n - \lambda_n\| + \|\lambda_n - g_\circ\| \\
&\leq \limsup_{n\to\infty} \|\lambda_n - g_\circ\| \\
&= \ \mathcal{A}(g_\circ, \{\lambda_n\}).
\end{aligned}$$

This illustrates that $\mathrm{T}g_\circ \in \mathcal{A}(\mathbb{S},\ \{\lambda_n\})$. Thus $\mathrm{T}g_\circ = g_\circ$ and $\mathcal{F}_\mathrm{T} \neq \emptyset$. □

Theorem 3.5. *Suppose $\mathcal{D}$ is UC-Space and $\emptyset \neq \mathbb{S} \subseteq \mathcal{D}$ (where $\mathbb{S}$ weakly compact set and convex). If $\mathrm{T} : \mathbb{S} \to \mathbb{S}$ satisfying $\mathcal{CSC}$ condition with $\mathcal{F}_\mathrm{T} \neq \emptyset$ and $\{\lambda_n\}$ is a sequence of R∗ iteration Eq.(3.1), then $\{\lambda_n\}$ converges weakly in $\mathcal{F}_\mathrm{T}$ given that $\mathcal{D}$ fulfills Opial's condition.*

Proof. Because $\mathbb{S}$ is weakly compact, then there is $\{\lambda_{n_j}\}$ a subsequence of $\{\lambda_n\}$ and $\lambda_\circ \in \mathbb{S}$ with $\{\lambda_{n_j}\}$ converges weakly to $\lambda_\circ$. From the perspective of Theorem 3.4, it is evident that,

$$\lim_{j\to\infty} ||\lambda_{n_j} - \mathrm{T}\lambda_{n_j}|| = 0.$$

As a result, $\lambda_\circ \in \mathcal{F}_\mathrm{T}$ since each of the prerequisites of Proposition 2.4(2) are fulfilled. Now to show that $\lambda_\circ$ is only a weak limit of $\{\lambda_n\}$. Suppose that $\lambda_\circ$ is not the weak limit for $\{\lambda_n\}$, this means, there is another subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$, with a weak limit, $\lambda_\circ{}^* \neq \lambda_\circ$. Again using Theorem 3.4, observe that

$$\lim_{k\to\infty} ||\lambda_{n_k} - \mathrm{T}\lambda_{n_k}|| = 0.$$

All Proposition 2.4(2) criteria have been made accessible, therefore $\lambda_\circ{}^* \in \mathcal{F}_{\mathrm{T}}$. Utilizing Opial's condition of $\mathcal{D}$ together with Lemma 3.3, we get

$$\begin{aligned}\lim_{n\to\infty}\|\lambda_n-\lambda_\circ\| &< \lim_{j\to\infty}\left\|\lambda_{n_j}-\lambda_\circ\right\| \\ &< \lim_{j\to\infty}\left\|\lambda_{n_j}-\lambda_\circ{}^*\right\| \\ &= \lim_{n\to\infty}\|\lambda_n-\lambda_\circ{}^*\| \\ &= \lim_{k\to\infty}\|\lambda_{n_k}-\lambda_\circ{}^*\| \\ &< \lim_{k\to\infty}\|\lambda_{n_k}-\lambda_\circ\| \\ &= \lim_{n\to\infty}\|\lambda_n-\lambda_\circ\|.\end{aligned} \tag{3.23}$$

Thus, $\lim_{n\to\infty}\|\lambda_n-\lambda_\circ\| < \lim_{n\to\infty}\|\lambda_n-\lambda_\circ\|$, which is a contradiction. This terminates the proof. □

The following results demonstrate the strong convergence.

Theorem 3.6. *Suppose that $\mathcal{D}$ is UC-Space and $\emptyset\neq\mathbb{S}\subseteq\mathcal{D}$ (where $\mathbb{S}$ is compact and convex). If $\mathrm{T}:\mathbb{S}\to\mathbb{S}$ satisfying $\mathcal{CSC}$ condition with $\mathcal{F}_{\mathrm{T}}\neq\emptyset$ and $\{\lambda_n\}$ is a sequence of R^* iteration Eq.*3.1*, then $\{\lambda_n\}$ converges s strongly in $\mathcal{F}_{\mathrm{T}}$.*

Proof. Because $\mathbb{S}$ is compact and $\{\lambda_n\}\subset\mathbb{S}$, then $\{\lambda_n\}$ has a strongly convergent subsequence $\{\lambda_{n_j}\}$ such that

$$\lim_{j\to\infty}\left\|\lambda_{n_j}-s_\circ\right\|=0,\quad s_\circ\in\mathbb{S}.$$

Therefore, $\lim_{j\to\infty}||\lambda_{n_j}-\mathrm{T}\lambda_{n_j}||=0$ can be deduced from Theorem 3.4. Now employing Proposition 2.4(3), we get

$$||\lambda_{n_j}-\mathrm{T}s_\circ||\leq 5||\lambda_{n_j}-\mathrm{T}\lambda_{n_j}||+||\lambda_{n_j}-s_\circ||,$$

which implies that

$$\lambda_{n_j}\to\mathrm{T}s_\circ \quad as \quad j\to\infty.$$

Also, we get $\mathrm{T}s_\circ=s_\circ$, that is, $s_\circ\in\mathcal{F}_{\mathrm{T}}$. In addition $\lim_{j\to\infty}\|\lambda_j-s_\circ\|$ exists according to Lemma 3.3. Therefore, $s_\circ$ is a strong limit point for $\{\lambda_n\}$. □

4. Results and discussion

In support of the main findings, an example of a map that meets condition $\mathcal{CSC}$ is provided. In addition, the convergence of the $R*$ iterative approach was assessed in contrast to other iterative approaches.

Example 4.1. Let $\mathbb{S} = [6, 12]$ with $\|.\| = |.|$ and $\mathrm{T} : \mathbb{S} \to \mathbb{S}$ be a function defined by

$$\mathrm{T}(s) = \begin{cases} 6 & \text{if } s = 12, \\ \frac{s+6}{2} & \text{otherwise.} \end{cases} \tag{4.1}$$

First to show that T is meet condition $\mathcal{CSC}$. Consider the following cases:

Case 1: If $s_1 = 12 = s_2$, then $|\mathrm{T}s_1 - \mathrm{T}s_2| = 0$. Hence,

$$\frac{1}{2}\left(|s_1 - \mathrm{T}s_2| + |s_2 - \mathrm{T}s_1|\right) \geq 0 = |\mathrm{T}s_1 - \mathrm{T}s_2|.$$

Case 2: If $6 \leq s_1, s_2 < 12$, then $|\mathrm{T}s_1 - \mathrm{T}s_2| = |\frac{s_1 - s_2}{2}|$. Hence,

$$\begin{aligned}
\frac{1}{2}\left(|s_1 - \mathrm{T}s_2| + |s_2 - \mathrm{T}s_1|\right) &= |\frac{s_1 - (\frac{s_2+6}{2})}{2}| + |\frac{s_2 - (\frac{s_1+6}{2})}{2}| \\
&\geq |\frac{(s_1 - (\frac{s_2+6}{2})) - (s_2 - (\frac{s_1+6}{2})}{2}| \\
&= |\frac{3s_1 - 3s_2}{4}| \\
&\geq |\frac{s_1 - s_2}{2}| \\
&= |\mathrm{T}s_1 - \mathrm{T}s_2|.
\end{aligned}$$

Case 3: If $s_1 = 12$ and $6 \leq s_2 < 12$, then, $|\mathrm{T}s_1 - \mathrm{T}s_2| = |\frac{s_1-6}{2}|$. Hence

$$\begin{aligned}
\frac{1}{2}\left(|s_1 - \mathrm{T}s_2| + |s_2 - \mathrm{T}s_1|\right) &= |\frac{s_1 - 6}{2}| + |\frac{s_2 - (\frac{s_1+6}{2})}{2}| \\
&\geq |\frac{s_1 - 6}{2}| \\
&= |\mathrm{T}s_1 - \mathrm{T}s_2|.
\end{aligned}$$

Case 4: If $s_2 = 12$ and $6 \leq s_1 < 12$, then, $|\mathrm{T}s_1 - \mathrm{T}s_2| = |\frac{s_2-6}{2}|$. Hence

$$\begin{aligned}
\frac{1}{2}\left(|s_1 - \mathrm{T}s_2| + |s_2 - \mathrm{T}s_1|\right) &= |\frac{s_1 - \left(\frac{s_2+6}{2}\right)}{2}| + |\frac{s_2 - 6}{2}| \\
&\geq |\frac{s_2 - 6}{2}| \\
&= |\mathrm{T}s_1 - \mathrm{T}s_2|.
\end{aligned}$$

In all the above cases, we get

$$\frac{1}{2}\left(|s_1 - \mathrm{T}s_2| + |s_2 - \mathrm{T}s_1|\right) \geq |\mathrm{T}s_1 - \mathrm{T}s_2|.$$

Thus, T satisfies the condition $\mathcal{CSC}$.

Now to demonstrate the effectiveness of the iteration Eq.(3.1) choosing in this example the initial value $\lambda_1 = 8.5$ with parameters $v_n = 0.86$, $u_n = 0.8$, and s_n =0.8.

Table 1 and Figure 1 show how the new iterative approach's rate of convergence compares to other iterative approaches.

Table 1. Comparison of convergence rates for various iteration approaches

step	Mann	Noor	Agarwal	CR	R* iteration
1	8.5	8.5	8.5	8.5	8.5
2	7.425000	6.928000	6.820000	6.513000	6.071250
3	6.812250	6.344473	6.268960	6.105267	6.002030
4	6.462982	6.127868	6.088218	6.021600	6.000057
5	6.263900	6.047464	6.028935	6.004432	6.000001
6	6.150423	6.017618	6.009490	6.000909	6.000000
7	6.085741	6.006540	6.003113	6.000186	
8	6.048872	6.002427	6.001021	6.000038	
9	6.027857	6.000901	6.000334	6.000007	
10	6.015878	6.000334	6.000109	6.000001	
11	6.009050	6.000124	6.000036	6.000000	
12	6.005158	6.000046	6.000011		
13	6.002940	6.000017	6.000003		
14	6.001676	6.000006	6.000001		
15	6.000955	6.000002	6.000000		
16	6.000544	6.000000			
17	6.000310				
18	6.000176				
19	6.000100				
20	6.000057				
21	6.000032				
22	6.000018				
23	6.000010				
24	6.000006				
25	6.000003				
26	6.000001				
27	6.000001				
28	6.000000				

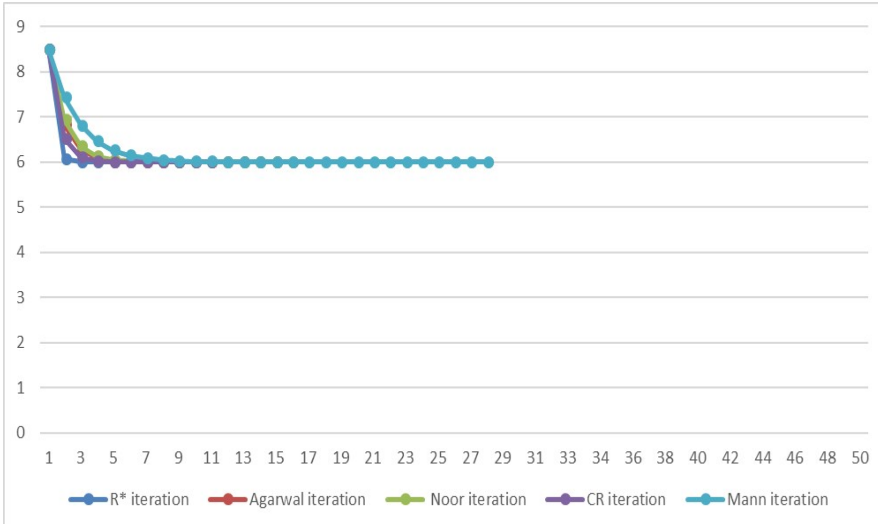

Figure 1. Graphic illustration of the convergence of iterative approaches

Table 2 indicates the number of iterations required for certain iterative approaches to reach the FP. The new iterative approach R* converges more quickly than the other methods, as shown in the data.

Table 2. Number of iterations

Iterative approaches	Number of iterations
R* iteration	6
CR iteration	11
Agarwal iteration	15
Noor iteration	16
Mann iteration	28

Furthermore, iterative approaches are compared another time under various choices of beginning and set of parameters. In any case, the iteration process Eq.3.1 appears to converge faster than the other iteration processes, as indicated in Table 3 below:

Table 3. Comparison of iteration with various initial points and parameters

Iterative approaches	**Number of iterations**
Initial Points **:** $\lambda_1 = 10$, $v_n = 0.6$, $u_n = 0.7$, $s_n = 0.7$	
R* iteration	7
CR iteration	14
Agarwal iteration	18
Noor iteration	23
Mann iteration	32
Initial Points **:** $\lambda_1 = 11.5$, $v_n = 0.87$, $u_n = 0.66$, $s_n = 0.77$	
R* iteration	6
CR iteration	12
Agarwal iteration	17
Noor iteration	20
Mann iteration	29

5. Conclusion

This paper presents a new iterative approach to estimating the FPs of maps that meets the $\mathcal{CSC}$ requirement, called the R*−iterative Eq.3.1. Through analytical proof, it is shown that the suggested method gets to an FP for the contraction map faster than some other well-known methods. In addition, certain convergence findings of a novel iterative method are also given under condition $\mathcal{CSC}$. Numerically, it is shown that the innovative iterative approach converges faster than other iterative approaches in the literature.

Declarations

This work was written strictly in accordance with the guidelines established by the Universities of Technology. This paper has not been sent to any other university or non-academic organization with the goal of earning a different degree or certification.

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