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COMMON FIXED POINT FOR GENERALIZED WEAKLY EXPANSIVE MAPPINGS IN METRIC SPACES

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Abstract. In this paper, we prove two common fixed point theorems for a pair of generalized weakly expansive self-maps by using compatibility of type (E) and weak reciprocal continuity in metric spaces. Two examples and an application of the existence of solution for an integral equation are given to illustrate our results.

1. Introduction

The famous Banach principle has been generalized to various ways, and there are some results given under different type of contractive condition. On other hand the expansive mappings has been used and studied to obtain a fixed point or common fixed point, Wang [44] proved the existence of fixed point for expansive mappings, later Manro et al. [22] proved a common fixed point theorem for two self-mappings in a metric space by using the concept

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of weak reciprocal continuity with compatibility. Under the same conditions, Khan et al.[16] introduced the concept of ϕ-weakly expansive maps and proved some common fixed point results.

Jungck [12] introduced the notion of compatible maps, the same author Jungck and Rhoades [13] weakened the concept of compatibility to the weak compatibility. Singh et al. [42, 43] introduced new type of compatibility, called compatibility of type (E), by using this concept they obtain some results of common fixed point. Pant [24] obtained some results for non-compatible maps by using new concept called reciprocally continuous maps, which has been weakened to weakly reciprocally continuous maps by Pant et al. [25]. Other authors had presented many generalized contraction (see [4, 21, 35, 36]).

In present paper, we will define the concept of generalized weakly expansive maps and prove two common fixed point theorems, in the first one we should use compatibility of type (E), in the second we combine two concepts, compatibility of type (E) with weak reciprocal continuity, our results generalize and improve the results of Manro et al. [22], results in paper [16, 20] and several others.

2. Preliminaries

Definition 2.1. ([18]) A quasi-partial metric is a function $q : X \times X \to \mathbb{R}^+$ satisfying

(1) $q(x,x) \leq q(y,x)$ (small self-distances),
(2) $q(x,x) \leq q(x,y)$ (small self-distances),
(3) $x = y$ if and only if $q(x,x) = q(x,y)$ and $q(y,y) = q(y,x)$ (indistancy implies equality and vice versa),
(4) $q(x,z) + q(y,y) \leq q(x,y) + q(y,z)$ (triangularity),
for all $x, y, z \in X$.

Then The pair (X, q) is called a quasi-partial metric space.

Karapinar et al. [19] have taken:
(3') if $0 \leq q(x,x) = q(x,y) = q(y,y)$, then $x = y$ (equality), in place of (3).

If q satisfies all these conditions except possibly (1), then q is called a lopsided partial quasi-metric [18]. It is interesting to see here that for $q(x,y) = q(y,x)$, (X, q) becomes a partial metric space. Also for a quasi-partial metric q on X, the function $d_q : X \times X \to \mathbb{R}^+$ defined by

$$d_q(x,y) = q(x,y) + q(y,x) - q(x,x) - q(y,y)$$

is a (usual) metric on X.

Example 2.2. ([18]) The pair $(\mathbb{R}^+, q)$ with

(1) $q(x,y) = |x - y| + |x|$;

(2) $q(x,y) = \max\{y-x,0\} + x$;

are quasi-partial metric spaces.

Definition 2.3. ([19]) Let (X,q) be a quasi-partial metric space.

(1) A sequence $\{x_n\} \subset X$ in a quasi-partial metric space converges to a point $x \in X$ if $q(x,x) = \lim q(x,x_n) = \lim q(x_n,x)$.

(2) A subset E of a quasi-partial metric space (X,q) is closed if whenever $\{x_n\}$ is a sequence in E such that $\{x_n\}$ converges to some $x \in X$, then $x \in E$.

Lemma 2.4. ([19]) *Let (X,q) be a quasi-partial metric space. Then following hold:*

(1) *If $q(x,y) = 0$, then $x = y$;*

(2) *If $x \neq y$, then $q(x,y) > 0$ and $q(y,x) > 0$.*

Definition 2.5. Let (X,d) be a metric space, a self-mapping T is called to be expansive if there exists $h > 1$ such for all $x,y \in X$ we have

$$d(Tx,Ty) \geq hd(x,y).$$

Definition 2.6. ([25]) Let (X,p) be a metric space and $S,T : X \to X$ two self- maps, (S,T) is said to be weakly reciprocally continuous if

$$\lim_{n\to\infty} TSx_n = Tz$$

or

$$\lim_{n\to\infty} STx_n = Sz,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z,$$

for some $\in X$.

Definition 2.7. ([26]) Let (X,d) be a metric space and $S,T : X \to X$ two self- maps, (S,T) is said to be T weakly reciprocally continuous if

$$\lim_{n\to\infty} TSx_n = Tz$$

or

$$\lim_{n\to\infty} S^2x_n = Sz,$$

whenever $\{x_n\}$ is a sequence in X such

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$$

for some $z \in X$.

Example 2.8. Let $X = [0, 2]$ endowed with usual metric, define two maps S and T as follows:

$$Sx = \begin{cases} 1 - x, & 0 \leq x \leq 1, \\ 2, & 1 < x \leq 2, \end{cases} \quad Tx = \begin{cases} 1 + x, & 0 \leq x \leq 1, \\ 0, & 1 < x \leq 2. \end{cases}$$

Consider the sequence $\{x_n\}$ which satisfying $x_n \to 0$, as $n \to \infty$ and $0 < x_n \leq 1$ for all $n \geq 0$.

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 1,$$
$$\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} T(1 - x_n) = 2 = T1.$$

Then (S, T) is T-weakly reciprocally continuous.

3. Main results

Definition 3.1. Let (X, q) be a complete quasi-partial metric space and $T : X \to X$ be a self-mapping. Then T is said to be generalized weakly expansive, if

$$q(Tx, Ty) \geq m(x, y) + \phi(m(x, y)), \tag{3.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\phi(0) = 0$ and $\phi(t) > t$ for all $t > 0$ and

$$m(x, y) = \min\{q(Sx, Sy), q(Sx, Ty), q(Sy, Tx)\}.$$

Example 3.2. Let $X = [0, 1]$ endowed with quasi-partial metric $q(x, y) = |x - y| + |x|$, define T as $Tx = 4x$ and $\phi(x) = 2x$. Then we have:

$$q(Tx, Ty) \geq 4|x - y| \geq |x - y| + 2|x - y|.$$

Theorem 3.3. *Let (X, q) be a complete metric space and T be a self- mapping on X such that (3.1) is satisfied. If the pair (S, T) is compatible of type (E), then S and T have a unique common fixed point in X.*

Proof. Let $x_0 \in X$, since $SX \subseteq TX$ there exists $x_1 \in X$ such $Tx_1 = Sx_0$, so by continuing in this manner we can construct a sequence $\{y_n\}$ as follows:

$$y_n = Tx_{n+1} = Sx_n. \tag{3.2}$$

Suppose $y_n \neq y_{n+1}$ for all $n \geq 0$. If $q(y_{n-1}, y_n) \leq q(y_n, y_{n+1})$, by using (3.1) we get

$$q(y_{n-1}, y_n) = q(Tx_n, Tx_{n+1}) \geq m(x_n, x_{n+1}) + \phi(m(x_n, x_{n+1})),$$

it implies that

$$\begin{aligned} q(y_{n-1}, y_n) &\geq q(y_{n-1}, y_n) + \phi(q(y_{n-1}, y_n) \\ &> 2q(y_{n-1}, y_n), \end{aligned}$$

which is a contradiction. Therefore, the sequence $\{q(y_{n-1}, y_n)\}$ is decreasing and bounded at below, so it is convergent to $r = \inf\{q(y_{n-1}, y_n)\}$.

If $r > 0$, by using (3.1), we get:

$$d(y_{n-1}, y_n) = d(Tx_n, Tx_{n+1}) \geq m(x_n, x_{n+1}) + \phi(m(x_n, x_{n+1})),$$

letting $n \to \infty$ we obtain:

$$r \geq r + \phi(r) > 2r,$$

which is a contradiction. Hence $\lim_{n\to\infty} d(y_{n-1}, y_n) = 0$.

Now, we claim $\{y_n\}$ is a Cauchy sequence, if not so there exist $\varepsilon > 0$ and positive integers $n(k)), m(k)$ such for $m(k) \geq n(k) \geq k$ we have

$$d(y_n, y_m) > \varepsilon.$$

for each k. Let $m(k)$ be the least integer such $m(k) \geq n(k) \geq k$ and satisfying

$$d(y_n, y_m) > \varepsilon \ \text{ and } \ d(y_{n(k)}, y_{m(k)-1}) \leq \varepsilon. \tag{3.3}$$

By using triangular inequality we get:

$$\begin{aligned} \varepsilon &< d(y_n, y_m) \\ &\leq d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}) \\ &\leq 2d(y_{n(k)}, y_{n(k)-1}) + \varepsilon + d(y_{m(k)-1}, y_{m(k)}), \end{aligned}$$

letting $n \to \infty$, then we obtain:

$$\lim_{n\to\infty} d(y_{n(k)}, y_{m(k)}) = \varepsilon.$$

By using (3.1), we get:

$$\begin{aligned} d(y_n, y_m) &= d(Tx_{n+1}, Tx_{m+1}) \\ &\geq m(x_{n+1}, x_{m+1}) + \phi(m(x_{n+1}, x_{m+1})), \end{aligned}$$

letting $n \to \infty$ we obtain:

$$\varepsilon \geq \varepsilon + \phi(\varepsilon) > 2\varepsilon,$$

which is a contradiction. Hence $\{y_n\}$ is a Cauchy sequence, since X is a complete so the sequence $\{y_n\}$ is convergent to some $w \in X$.

Since (S, T) is compatible of type (E), we have

$$\lim_{n\to\infty} T^2x_n = \lim_{n\to\infty} TSx_n = Sw \quad \text{and} \quad \lim_{n\to\infty} S^2x_n = \lim_{n\to+\infty} STx_n = Tw.$$

Now, we show $Sw = Tw$, if not, by using (3.1) we get;

$$d(Tz, TSx_n) \geq m(w, Sx_n) + \phi(m(z, Sx_n)),$$

letting $n \to \infty$, we get: $m(z, x_{n+1}) \to d(Tw, Sw)$, so we have

$$\begin{aligned} d(Tz, Sz) &\geq d(Tw, Sw) + \phi(d(Tw, Sw)) \\ &> 2d(Tw, Sw), \end{aligned}$$

which is a contradiction. Hence $Tw = Sw$. Putting $z = Tw = Sw$, so we get,

$$Sz = STw = TSw = Tz.$$

We will prove that z is a common fixed point for S and T, if not by using (3.1) we get;

$$d(Tw, Tz) \geq m(w, z) + \phi(m(w, z)),$$

which implies that:

$$\begin{aligned} d(Tz, z) &\geq d(z, Tz) + \phi(d(Tz, z)) \\ &> 2d(Tz, z), \end{aligned}$$

which is a contradiction. Hence $z = Tz$.

For the uniqueness, suppose that t is another common fixed point for S and T, by using (3.1) we get:

$$\begin{aligned} d(z, t) = d(Tz, Tt)) &\geq m(z, t) + \phi(m(z, t)) \\ &= d(z, t) + \phi(d(z, t)) \\ &> d(Sz, Tt), \end{aligned}$$

which is a contradiction. Hence, the uniqueness is proved. □

We can get the following Corollary 3.4.

Corollary 3.4. *Let (X, d) be a complete metric space and let S and T be two self-maps on X. If there exists $q > 1$ such for all $x, y \in X$ we have*

$$d(Tx, Ty) \geq q \min\{d(Sx, Sy), d(Sx, Ty), d(Sy, Tx)\}.$$

Moreover if the pair (S, T) is compatible of type (E), then S and T have a common fixed point.

Theorem 3.5. *Let (X, d) be a complete metric space, S and T be two self-mappings on X such $SX \subseteq TX$ and satisfying (3.1). Further if the pair (S, T) is S-compatible of type (E) and T-weakly reciprocally continuous, then S and T have a unique common fixed point in X.*

Proof. As in proof of Theorem 3.3, the sequence $\{y_n\}$ which defined in (3.2) is convergent to some w in X. The pair (S, T) is S compatible of type (E) implies that

$$\lim_{n\to\infty} STx_n = \lim_{n\to\infty} TSx_n = Tw.$$

Firstly we shall show $Sw = Tw$, if not, by using (3.1) we get:

$$d(Tw, TSx_n) \geq m(w, Sx_n) + \phi(m(z, Sx_n)),$$

letting $n \to \infty$, we get $m(w, Sx_n) = m(w, x_{n+1}) \to d(Tw, Sw)$, so we have

$$\begin{aligned} d(Tw, Szw) &\geq d(Tw, Sw) + \phi(d(Tw, Sw)) \\ &> 2d(Tw, Sw), \end{aligned}$$

which is a contradiction. Then $Tw = Sw$. The pair is T-weakly reciprocally continuous, if

$$\lim_{n\to\infty} TSx_n = Tw,$$

so the pair (S, T) is compatible, therefore weakly compatible.
or

$$\lim_{n\to\infty} S^2x_n = Sw,$$

we get

$$\lim_{n\to\infty} d(TSx_n, STx_n) = 0,$$

and so (S, T) is compatible, then weakly compatible.

For $z = Sw = Tw$ implies $Sz = Tz$ and the rest of proof is similar as in proof of Theorem 3.3.

If $\lim_{n\to\infty} S^2x_n = Sw$, then also (S, T) is compatible, then it is weakly compatible. □

We can get the following two corollaries.

Corollary 3.6. *Let (X, d) be a complete metric space and let T be a surjective and a generalized weakly expansive mapping,that is, for all $x, y \in X$ we have*

$$\begin{aligned} d(Tx, Ty) \geq &\min\{d(x, y), d(x, Ty), d(y, Tx)\} \\ &+ \phi\Big(\min\{d(x, y), d(x, Ty), d(y, Tx)\}\Big), \end{aligned}$$

then T has a fixed point.

Corollary 3.7. *Let (X, d) be a complete metric space, S and T be two-self mappings on X such that $SX \subseteq TX$ and satisfying (3.1). If the pair (S, T) is compatible of type (E) and weakly reciprocally continuous, then $S$and T have a common fixed point in X.*

Example 3.8. Let $X = [0, 2]$ endowed with usual metric, define two maps S and T as follows:

$$Sx = \begin{cases} 1, & 0 \leq x \leq 1, \\ \frac{3}{4}, & 1 < x \leq 2, \end{cases} \quad Tx = \begin{cases} 2 - x, & 0 \leq x \leq 1, \\ \frac{1}{4}, & 1 < x \leq 2, \end{cases}$$

Consider the sequence $\{x_n\}$ which converges to 0 and $0 < x_n \leq 1$ for all $n \geq 0$.

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 1,$$

$$\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} ST_n) = T1 = 1.$$

$$\lim_{n\to\infty} T^2x_n = \lim_{n\to\infty} TSx_n = S1 = 1.$$

Then (S,T) is compatible of type (E).

For the inequality (3.1), we have the following cases:

(1) For $x, y \in [0,1]$, we have

$$d(Tx,Ty) = |x-y| \geq \frac{9}{4}d(Sx,Sy) = 0.$$

(2) For $x \in [0,1]$ and $1 < y \leq 2$, we have

$$d(Tx,Ty) = |\frac{7}{4} - x| \geq \frac{9}{16} = \frac{9}{4}d(Sx,Sy).$$

(3) For $x \in (1,2]$ and $y \in [0,1]$, we have

$$d(Tx,Ty) = |\frac{7}{4} - y| \geq \frac{9}{16} = \frac{9}{4}d(Sx,Sy).$$

(4) For $x, y \in (1,2]$, we have

$$d(Tx,Ty) = 0 \geq \frac{9}{4}d(Sx,Sy = 0.$$

So, it is obviously that (3.2) holds. Consequently all hypotheses of Theorem 3.3, with $\phi(t) = \frac{5}{4}t$ are satisfied, therefore 1 is the unique common fixed for S and T.

Example 3.9. Let $X = [0,\infty)$ endowed with usual metric, define two maps S and T as follows:

$$Sx = \begin{cases} \frac{1}{2}x, & 0 \leq x \leq 2, \\ 3, & x > 2, \end{cases} \qquad Tx = \begin{cases} 4x, & 0 \leq x \leq 2, \\ \frac{3}{8}, & x > 2. \end{cases}$$

Consider the sequence $\{x_n\}$ such $\lim_{n\to\infty} x_n = 0$ and $0 < x_n \leq 1$ for all $n \geq 1$.

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0,$$

$$\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} ST_n) = T0 = 0,$$

$$\lim_{n\to\infty} \lim_{n\to\infty} TSx_n = T0 = 0.$$

Then (S,T) is compatible of type (E).

For the inequality (3.1) we have the following cases:

(1) For $x, y \in [0,2]$, we have

$$d(Tx,Ty) = 4|x-y| \geq \frac{3}{2}|x-y| = 3d(Sx,Sy).$$

(2) For $x \in [0,2]$ and $y > 2$, we have

$$d(Tx,Ty) = 4|x-\frac{3}{4}| \geq \frac{3}{2}|x-\frac{3}{4}| = d(Sx,Sy).$$

(3) For $x \in (2,\infty)$ and $y \in [0,2]$, we have

$$d(Tx,Ty) = 4|y-\frac{3}{4}| \geq \frac{3}{2}|y-\frac{3}{4}| = d(Sx,Sy).$$

(4) For $x, y \in (2,\infty)$, we have

$$d(Tx,Ty) = 0 \;\; d(Sx,Sy) = 0.$$

So it is obviously (3.1) holds. Consequently all hypotheses of Theorem 3.5, with $\phi(t) = 2t$ are satisfied, therefore 0 is the unique common fixed for S and T.

4. Application

In this section, we focus on the existence of solutions of some nonlinear integral equations as an application to the results proved in the previous section. To delve further into these applications, we recommend consulting contemporary publications such as [3, 5, 9, 10, 17, 23, 32, 33, 34, 37, 38, 39]. We will utilize our results of Corollary 3.6 to prove the existence of solution for the following integral equation:

$$x(t) = f(t) + \int_0^1 K(t,s,x(s))ds, \tag{4.1}$$

where $f \in X = C([0,1],\mathbb{R})$ and $K : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous function.

For x, y define a metric as follows:

$$d(x,y) = \|x-y\|_\infty = \max_{0\leq t\leq 1}|x-y|.$$

Clearly (X,d) is a complete metric space.

Theorem 4.1. *Assume that:*

(1) *there exists a function* $\theta : [0,1]\times[0,1] \to \mathbb{R}_+$ *such that*

$$|\int_0^1 K(t,s,x(s) - K(t,s,y(s)ds| \geq \theta(t)|x(t)-y(t)|,$$

(2) $\inf_{t\in[0,1]} \theta(t) = \delta > 1.$

Then the equation (4.1) *has a solution in* X.

Proof. Define

$$Tx(t) = f(t) + \int_0^1 K(t, s, x(s))ds.$$

The equation (4.1) has a solution, if and only if the self mapping T has a fixed point in X. Since $f, x \in X$ so T is a self-mapping from X into itself. We have also,

$$\begin{aligned} |Tx(t) - Ty(t)| &= |\int_0^1 K(t, s, x(s) - K(t, s, y(s)ds| \\ &\geq \theta(t)|x(t) - y(t)| \\ &\geq \delta \min\{|x(t) - y(t)|, |Tx(t) - x(t)|, |y(t) - Ty(t)|, \\ &\qquad |x(t), Ty(t)| + |y(t) - S_1|x(t)|\}. \end{aligned}$$

Consequently, all hypotheses of Corollary 3.6 are satisfied, then the equation (4.1) has a unique solution. □

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