Nonlinear Functional Analysis and Applications
Vol. 30, No. 3 (2025), pp. 769-784
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2025.30.03.08
http://nfaa.kyungnam.ac.kr/journal-nfaa

APPLICATION OF FOURTH POWER TYPE CONTRACTION IN $\mathscr{G}$-METRIC SPACES

S. Saravanan[1] and K. Kumara Swamy[2]

[1]Department of Mathematics,
Nehru Institute of Engineering and Technology, Coimbatore-641105,
Tamil Nadu, India
e-mail: sarodhana87@gmail.com

[2]Department of Mathematics, Basic Sciences and Humanities,
GMR Institute of Technology, Rajam-532127, A.P, India
e-mail: kumaraswamy.k@gmrit.edu.in

Abstract. Fixed point theory is an important discipline in mathematics because of its results which are utilized to investigate the existence of solutions for the problems in applied sciences and engineering. Many fixed point results have been widely generalized throughout the years in various directions by introducing new metric spaces and setting of new contraction mappings. The results in fixed point theory can be noticed in geometry, computational algorithms, economics, fluid dynamics, micro-structures, nonlinear sciences, medical fields and optimization theory. Recently, Feng Gu and Hongqing Ye introduced third power type contraction mappings and also established related fixed point theorems under G-metric. In this article, we introduce new fourth power type contraction conditions in $\mathscr{G}$-metric space. Further, the obtained fixed point of a fourth power contraction is proved as a $\mathscr{G}$-contractive fixed point. In addition, eigen value problem will be solved as an application of fourth power contraction.

1. Introduction

Definition 1.1. ([15]) Let $\mathscr{M}$ be a set which is nonempty and $\mathscr{G} : \mathscr{M} \times \mathscr{M} \times \mathscr{M} \to [0, \infty)$ such that

[0]Received December 26, 2024. Revised April 9, 2025. Accepted June 12, 2025.
[0]2020 Mathematics Subject Classification: 30E25, 34A08, 47H10, 54H25, 93C42, 93C70.
[0]Keywords: Fixed Point, $\mathscr{G}$-metric Space, $\mathscr{G}$-contractive fixed point, characteristic value, characteristic vector.
[0]Corresponding author: K. K. Swamy(kumaraswamy.k@gmrit.edu.in).

($\mathbb{G}1$) $\mathscr{G}(\phi, \xi, \psi) = 0$ for $\phi, \xi, \delta \in \mathscr{M}$ with $\phi = \xi = \psi$,
($\mathbb{G}2$) $\mathscr{G}(\phi, \phi, \xi) > 0$, when $\phi, \xi \in \mathscr{M}$ with $\phi \neq \xi$,
($\mathbb{G}3$) $\mathscr{G}(\phi, \phi, \xi) \leq \mathscr{G}(\phi, \xi, \psi)$, whenever $\phi, \xi, \psi \in \mathscr{M}$ with $\xi \neq \psi$,
($\mathbb{G}4$) $\mathscr{G}(\phi, \xi, \psi) = \mathscr{G}(\pi(\phi, \xi, \psi))$, whenever $\phi, \xi, \psi \in \mathscr{M}$ and $\pi(\phi, \xi, \psi)$denotes permutation of elements in the set $\{\phi, \gamma, \psi\}$,
($\mathbb{G}5$) $\mathscr{G}(\phi, \xi, \psi) \leq \mathscr{G}(\phi, \kappa, \kappa) + \mathscr{G}(\kappa, \xi, \psi)$ for $\phi, \xi, \psi, \kappa \in \mathscr{M}$.

Then $\mathscr{G}$ is a called $\mathscr{G}$-metric and $(\mathscr{M}, \mathscr{G})$ is called a $\mathscr{G}$-metric space. Here ($\mathbb{G}4$) reveals that $\mathscr{G}$ is symmetric under the permutation of the arguments ϕ, ξ, ψ and ($\mathbb{G}5$) denotes the rectangle inequality of $\mathscr{G}$.

The following propositions are used to establish our proofs:

Proposition 1.2. ([13]) *If $(\mathscr{M}, \mathscr{G})$ is a $\mathscr{G}$-metric space, then we have*

$$\mathscr{G}(\beta, \kappa, \kappa) \leq 2\mathscr{G}(\beta, \beta, \kappa) \text{ for } \beta, \kappa \in \mathscr{M}. \tag{1.1}$$

Proposition 1.3. ([13]) *Suppose that $(\mathscr{M}, \mathscr{G})$ is a $\mathscr{G}$-metric space and $\mathscr{G}(\beta, \kappa, \kappa) = 0$ for $\beta, \kappa \in \mathscr{M}$. Then $\beta = \kappa$.*

Definition 1.4. ([15]) If,

$$\mathscr{G}(\beta, \kappa, \kappa) = \mathscr{G}(\beta, \beta, \kappa) \text{ for } \beta, \kappa \in \mathscr{M}, \tag{1.2}$$

then $(\mathscr{M}, \mathscr{G})$ is called a symmetric $\mathscr{G}$-metric space.

The following definitions and lemmas in [15] are used to prove our main results:

Definition 1.5. ([15]) If $(\mathscr{M}, \mathscr{G})$ is a $\mathscr{G}$-metric space, then a $\mathscr{G}$-ball in $\mathscr{M}$ is in the form of

$$\mathscr{B}_{\mathscr{G}}(\xi, k) = \{\phi \in \mathscr{M} : \mathscr{G}(\xi, \phi, \phi) < k\}.$$

It is very clear that the family of these balls forms a topology which is known as $\mathscr{G}$-metric topology on $\mathscr{M}$ and is denoted by $\tau(\mathscr{G})$.

Definition 1.6. ([15]) In $(\mathscr{M}, \mathscr{G})$, $\langle \alpha_\zeta \rangle_{\zeta=1}^{\infty}$ is said to be $\mathscr{G}$-*convergent* to an element $\omega \in \mathscr{M}$, if it converges to ω in $\tau(\mathscr{G})$ where $\tau(\mathscr{G})$ is a $\mathscr{G}$-metric topology.

Lemma 1.7. ([15]) *The following are equivalent in $(\mathscr{M}, \mathscr{G})$:*

(G_a) $\langle \alpha_\zeta \rangle_{n=1}^{\infty} \subset \mathscr{M}$ *$\mathscr{G}$-converges to an element $\omega \in \mathscr{M}$,*
(G_b) $\lim_{\zeta \to \infty} \mathscr{G}(\alpha_\zeta, \alpha_\eta, \omega) = 0$,
(G_c) $\lim_{\zeta \to \infty} \mathscr{G}(\alpha_\zeta, \omega, \omega) = 0$.

Definition 1.8. ([15]) In $(\mathscr{M},\mathscr{G})$, $\langle\alpha_\zeta\rangle_{\zeta=1}^{\infty}$ is said to be $\mathscr{G}$-Cauchy, if for every $\epsilon > 0$ there is a positive integer $\mathscr{N}$ such that $\mathscr{G}(\alpha_\zeta,\alpha_\eta,\alpha_\psi) < \epsilon$ for all $\zeta,\eta,\psi \geq \mathscr{N}$.

Definition 1.9. ([15]) If every Cauchy sequence in $\mathscr{M}$ is $\mathscr{G}$-convergent in it, then $(\mathscr{M},\mathscr{G})$ is called $\mathscr{G}$-complete.

Lemma 1.10. ([15]) *Suppose that $(\mathscr{M},\mathscr{G})$ is a $\mathscr{G}$-metric space. Then $\mathscr{G}$-metric is jointly continuous in all of its variables.*

In [7], Karapinar et al. proved the fixed point theorem under Banach type contraction in a $\mathscr{G}$-metric space. For different fixed poitn results on $\mathscr{G}$-metric space in recent years, refer to [1], [2], [10], [11], [14] and [19].

It is very clear that in $(\mathscr{M},\mathscr{G})$, every complete $\mathscr{G}$-metric space is f-orbitally complete at every element in $\mathscr{M}$. Converse of the above is not true due to the following:

Example 1.11. Let $\mathscr{M} = \mathbb{Q}$ and

$$\mathscr{G}(\beta,\gamma,\delta) = \max\{|\beta-\gamma|, |\gamma-\delta|, |\delta-\beta|\} \text{ for all } \beta,\gamma,\delta \in \mathscr{M}. \tag{1.3}$$

Then we know that $(\mathbb{Q}, |\cdot|)$ is not complete. Therefore, $(\mathscr{M},\mathscr{G})$ is not $\mathscr{G}$-complete. Define $\mathbb{F}: \mathscr{M} \to \mathscr{M}$ by $\mathbb{F}\alpha = \alpha/2$ for all $\alpha \in \mathscr{M}$. Then

$$O_{\mathbb{F}}(\alpha_0) = \langle\alpha_0, \alpha_0/2, ..., \alpha_0/2^n, ...\rangle$$

convergent to $0 \in \mathscr{M}$. Hence $\mathscr{M}$ is $\mathbb{F}$-orbitally complete but not $\mathscr{G}$-complete.

2. Main results

Note that $(\mathscr{M},\mathscr{G})$ represents complete $\mathscr{G}$-metric space in all of our results which are given below.

Theorem 2.1. *Suppose that $\mathbb{F}$ is a mapping on $(\mathscr{M},\mathscr{G})$ with*

$$\mathscr{G}^4(\mathbb{F}\alpha,\mathbb{F}\beta,\mathbb{F}\gamma) \leq \lambda\mathscr{G}(\alpha,\beta,\mathbb{F}\alpha)\mathscr{G}(\beta,\mathbb{F}\alpha,\mathbb{F}\beta)\mathscr{G}(\gamma,\mathbb{F}\alpha,\mathbb{F}\beta)\mathscr{G}(\beta,\mathbb{F}\gamma,\mathbb{F}\gamma) \tag{2.1}$$

for $\alpha,\beta,\gamma \in \mathscr{M}$ and $0 < \lambda < 1/4$. Then $\mathbb{F}$ has unique fixed point.

Proof. Let $\alpha_0 \in \mathscr{M}$. Define $\langle\alpha_\eta\rangle_{\eta=1}^{\infty} \subset \mathscr{M}$ by

$$\alpha_\eta = f\alpha_{\eta-1} \quad \text{for } \eta \geq 1. \tag{2.2}$$

Now substituting $\alpha = \alpha_{\eta-1}$ and $\beta = \gamma = \alpha_\eta$ in (2.1), and employing (1.1), we observe that

$$\mathscr{G}^4(\alpha_\eta,\alpha_{\eta+1},\alpha_{\eta+1}) = \mathscr{G}^4(\mathbb{F}\alpha_{\eta-1},\mathbb{F}\alpha_\eta,\mathbb{F}\alpha_\eta)$$
$$\leq \lambda\mathscr{G}(\alpha_{\eta-1},\alpha_\eta,\mathbb{F}\alpha_{\eta-1})\mathscr{G}(\alpha_\eta,\mathbb{F}\alpha_{\eta-1},\mathbb{F}\alpha_\eta)\mathscr{G}(\alpha_\eta,\mathbb{F}\alpha_{\eta-1},\mathbb{F}\alpha_\eta)\mathscr{G}(\alpha_\eta,\mathbb{F}\alpha_\eta,\mathbb{F}\alpha_\eta)$$

$$= \lambda\mathscr{G}(\alpha_{\eta-1},\alpha_\eta,\alpha_\eta)\mathscr{G}(\alpha_\eta,\alpha_\eta,\alpha_{\eta+1})\mathscr{G}(\alpha_\eta,\alpha_\eta,\alpha_{\eta+1})\mathscr{G}(\alpha_\eta,\alpha_{\eta+1},\alpha_{\eta+1})$$
$$\leq 4\lambda\mathscr{G}(\alpha_{\eta-1},\alpha_\eta,\alpha_\eta)\mathscr{G}^3(\alpha_\eta,\alpha_{\eta+1},\alpha_{\eta+1}),$$

it implies that

$$\mathscr{G}(\alpha_\eta,\alpha_{\eta+1},\alpha_{\eta+1}) \leq \mathbb{K}\mathscr{G}(\alpha_{\eta-1},\alpha_\eta,\alpha_\eta) \quad \text{for } \eta \geq 1 \quad \text{and} \quad \mathbb{K} = 4\lambda.$$

By induction, we have

$$\mathscr{G}(\alpha_\eta,\alpha_{\eta+1},\alpha_{\eta+1}) \leq \mathbb{K}^\eta\mathscr{G}(\alpha_0,\alpha_1,\alpha_1) \quad \text{for} \quad \eta \geq 1. \tag{2.3}$$

For $\zeta > \eta$, using ($\mathscr{G}5$) and (2.3), we get

$$\begin{aligned}\mathscr{G}(\alpha_\eta,\alpha_\zeta,\alpha_\zeta) &\leq \mathscr{G}(\alpha_\eta,\alpha_{\eta+1},\alpha_{\eta+1}) + \mathscr{G}(\alpha_{\eta+1},\alpha_{\eta+2},\alpha_{\eta+2}) \\ &\quad + \cdots + \mathscr{G}(\alpha_{\zeta-1},\alpha_\zeta,\alpha_\zeta) \\ &\leq (\underbrace{\mathbb{K}^\eta + \mathbb{K}^{\eta+1} + \mathbb{K}^{\eta+2} + \cdots + \mathbb{K}^{\eta+(\zeta-\eta-1)}}_{\zeta-\eta \text{ terms}})\mathscr{G}(\alpha_0,\alpha_1,\alpha_1) \\ &\leq \mathbb{K}^\eta \cdot \tfrac{1-\mathbb{K}^{\zeta-\eta}}{1-\mathbb{K}} \cdot \mathscr{G}(\alpha_0,\alpha_1,\alpha_1) \\ &\leq \tfrac{\mathbb{K}^\eta}{1-\mathbb{K}} \cdot \mathscr{G}(\alpha_0,\alpha_1,\alpha_1). \end{aligned} \tag{2.4}$$

Now employing $\eta \to \infty$, we get $\mathscr{G}(\alpha_\eta,\alpha_\zeta,\alpha_\zeta) \to 0$ which follows that $\langle\alpha_\eta\rangle_{\eta=1}^\infty$ is $\mathscr{G}$-Cauchy in $\mathscr{M}$. Due to completeness of$\mathscr{M}$, there exists an element $\omega \in \mathscr{M}$ such that

$$\lim_{\eta\to\infty} \alpha_{\eta-1} = \lim_{\eta\to\infty} \alpha_\eta = \omega. \tag{2.5}$$

Now, writing $\alpha = \alpha_{\eta-1}$ and $\beta = \gamma = \omega$ in (2.1), we get

$$\begin{aligned}&\mathscr{G}^4(\mathbb{F}\alpha_{\eta-1},\mathbb{F}\omega,\mathbb{F}\omega) \\ &\quad = \mathscr{G}^4(\alpha_\eta,\mathbb{F}\omega,\mathbb{F}\omega) \\ &\quad \leq \lambda\mathscr{G}(\alpha_{\eta-1},\omega,\mathbb{F}\alpha_{\eta-1})\mathscr{G}(\omega,\mathbb{F}\alpha_{\eta-1},\mathbb{F}\omega)\mathscr{G}(\omega,\mathbb{F}\alpha_{\eta-1},\mathbb{F}\omega)\mathscr{G}(\omega,\mathbb{F}\omega,\mathbb{F}\omega) \\ &\quad \leq \lambda\mathscr{G}(\alpha_{\eta-1},p,\alpha_\eta)\mathscr{G}(\omega,\alpha_\eta,\mathbb{F}\omega)\mathscr{G}(\omega,\alpha_\eta,\mathbb{F}\omega)G(\omega,\mathbb{F}\omega,\mathbb{F}\omega) \\ &\quad = \lambda\mathscr{G}(\alpha_{\eta-1},\omega,\alpha_\eta)\mathscr{G}^2(\omega,\alpha_\eta,\mathbb{F}\omega)\mathscr{G}(\omega,\mathbb{F}\omega,\mathbb{F}\omega).\end{aligned}$$

Taking $\eta \to \infty$ together with Lemma 1.10 and (2.5), we have

$$\mathscr{G}^4(\omega,\mathbb{F}\omega,\mathbb{F}\omega) \leq \lambda\mathscr{G}(\omega,\omega,\omega)\mathscr{G}^2(\omega,\omega,\mathbb{F}\omega)\mathscr{G}(\omega,\mathbb{F}\omega,\mathbb{F}\omega) \tag{2.6}$$

so that $\mathscr{G}(\omega,\mathbb{F}\omega,\mathbb{F}\omega) = 0$, which implies that $\omega = \mathbb{F}\omega$ in view of Proposition 1.3. Hence, ω is a fixed point of $\mathbb{F}$.

Suppose that δ is a point in $\mathscr{M}$ such that $\mathbb{F}\delta = \delta$. Now taking $\alpha = \omega$, $\beta = \gamma = \delta$ in (2.1) and using (1.1), we get

$$\begin{aligned}\mathscr{G}^4(\omega,\delta,\delta) &= \mathscr{G}^4(\mathbb{F}\omega,\mathbb{F}\delta,\mathbb{F}\delta) \\ &\leq \lambda\mathscr{G}(\omega,\delta,\mathbb{F}p)\mathscr{G}(\delta,\mathbb{F}\omega,\mathbb{F}\delta)\mathscr{G}(\delta,\mathbb{F}\omega,\mathbb{F}\delta)\mathscr{G}(\delta,\mathbb{F}\delta,\mathbb{F}\delta)\end{aligned}$$

$$= \lambda\mathscr{G}(\omega,\delta,\omega)\mathscr{G}(\delta,\omega,\delta)\mathscr{G}(\delta,\omega,\delta)\mathscr{G}(\delta,\delta,\delta),$$

it implies that $\mathscr{G}^4(\omega,\delta,\delta) \leq 0$ and hence, $\omega = \delta$. Thus ω is a unique fixed point of $\mathbb{F}$. □

The following is the example to Illustrate Theorem 2.1.

Example 2.2. Let $\mathscr{M} = \{0, \frac{1}{2}, 1\}$, with a mapping $\mathscr{G} : \mathscr{M} \times \mathscr{M} \times \mathscr{M} \to [0,\infty)$ defined by $\mathscr{G}(0,1,1) = 6$, $\mathscr{G}(1,0,0) = \frac{3}{2}$, $\mathscr{G}(0,\frac{1}{2},\frac{1}{2}) = 4, \mathscr{G}(\frac{1}{2},0,0) = 4, \mathscr{G}(\frac{1}{2},1,1) = 5, \mathscr{G}(1,\frac{1}{2},\frac{1}{2}) = 5, \mathscr{G}(0,\frac{1}{2},1) = \frac{15}{2}$ and $\mathscr{G}(\phi,\phi,\phi) = 0$ for all $\phi \in \mathscr{M}$. Then $(\mathscr{M},\mathscr{G})$is a complete $\mathscr{G}$-metric space.

Define $\mathbb{F} : \mathscr{M} \to \mathscr{M}$ by $\mathbb{F}0 = 0$, $\mathbb{F}\frac{1}{2} = 1$, $\mathbb{F}1 = 0$. We show that $\mathbb{F}$ satisfies the inequality (2.1) with $\lambda = \frac{1}{5}$.

$$\mathscr{G}^4(\mathbb{F}\alpha,\mathbb{F}\beta,\mathbb{F}\gamma) = \mathscr{G}^4(\mathbb{F}0,\mathbb{F}\frac{1}{2},\mathbb{F}1) = \mathscr{G}^4(0,1,0) = \frac{81}{16},$$

$$\mathscr{G}(\alpha,\beta,\mathbb{F}\alpha) = \mathscr{G}(0,\frac{1}{2},\mathbb{F}0) = \mathscr{G}(0,\frac{1}{2},0) = 4,$$

$$\mathscr{G}(\beta,\mathbb{F}\alpha,\mathbb{F}\beta) = \mathscr{G}(\frac{1}{2},\mathbb{F}0,\mathbb{F}\frac{1}{2}) = \mathscr{G}(\frac{1}{2},0,1) = \frac{15}{2},$$

$$\mathscr{G}(\gamma,\mathbb{F}\alpha,\mathbb{F}\beta) = \mathscr{G}(1,\mathbb{F}0,\mathbb{F}\frac{1}{2}) = \mathscr{G}(1,0,1) = 6,$$

$$\mathscr{G}(\beta,\mathbb{F}\gamma,\mathbb{F}\gamma) = \mathscr{G}(\frac{1}{2},\mathbb{F}1,\mathbb{F}1) = \mathscr{G}(\frac{1}{2},0,0) = 4.$$

Hence, we have

$$\lambda\mathscr{G}(\alpha,\beta,\mathbb{F}\alpha)\mathscr{G}(\beta,\mathbb{F}\alpha,\mathbb{F}\beta)\mathscr{G}(\gamma,\mathbb{F}\alpha,\mathbb{F}\beta)\mathscr{G}(\beta,\mathbb{F}\gamma,\mathbb{F}\gamma) = \frac{1}{5}\times 4\times\frac{15}{2}\times 6\times 4 = 144$$

and

$$\mathscr{G}^4(\mathbb{F}\alpha,\mathbb{F}\beta,\mathbb{F}\gamma) = \frac{81}{16}.$$

Therefore, we have

$$\mathscr{G}^4(\mathbb{F}\alpha,\mathbb{F}\beta,\mathbb{F}\gamma) \leq \frac{1}{5}\mathscr{G}(\alpha,\beta,\mathbb{F}\alpha)\mathscr{G}(\beta,\mathbb{F}\alpha,\mathbb{F}\beta)\mathscr{G}(\gamma,\mathbb{F}\alpha,\mathbb{F}\beta)\mathscr{G}(\beta,\mathbb{F}\gamma,\mathbb{F}\gamma)$$

for $\alpha,\beta,\gamma \in \mathscr{M}$, which implies that $\mathbb{F}$ satisfies all the conditions of Theorem 2.1. Hence, we have that 0 is unique fixed point of $\mathbb{F}$.

Theorem 2.3. *If $\mathbb{F}$ is a mapping on $\mathscr{M}$ with the condition*

$$\mathscr{G}^4(\mathbb{F}\alpha,\mathbb{F}\beta,\mathbb{F}\gamma) \leq \lambda\mathscr{G}(\alpha,\beta,\gamma)\mathscr{G}(\beta,\mathbb{F}\beta,\mathbb{F}\gamma)\mathscr{G}(\beta,\mathbb{F}\alpha,\mathbb{F}\gamma)\mathscr{G}(\gamma,\mathbb{F}\alpha,\mathbb{F}\beta) \quad (2.7)$$

for all $\alpha,\beta,\gamma \in \mathscr{M}$ with $0 < \lambda < 1/2$, then $\mathbb{F}$ has a fixed point which is unique.

Proof. Let $\alpha_0 \in \mathscr{M}$. Define $\langle\alpha_\eta\rangle_{n=1}^{\infty} \subset \mathscr{M}$ by (2.2). Writing with $\alpha = \alpha_{\eta-1}$ and $\beta = \gamma = \alpha_\eta$ in (2.7), and using (1.1), we find that

$$\begin{aligned}
&\mathscr{G}^4(\alpha_\eta, \alpha_{\eta+1}, \alpha_{\eta+1})\\
&\quad= \mathscr{G}^4(\mathbb{F}\alpha_{\eta-1}, \mathbb{F}\alpha_\eta, \mathbb{F}\alpha_\eta)\\
&\quad\le \lambda\mathscr{G}(\alpha_{\eta-1}, \alpha_\eta, \alpha_\eta)\mathscr{G}(\alpha_\eta, \mathbb{F}\alpha_\eta, \mathbb{F}\alpha_\eta)\mathscr{G}(\alpha_\eta, \mathbb{F}\alpha_{\eta-1}, \mathbb{F}\alpha_\eta)\mathscr{G}(\alpha_\eta, \mathbb{F}\alpha_{\eta-1}, \mathbb{F}\alpha_\eta)\\
&\quad= \lambda\mathscr{G}(\alpha_{\eta-1}, \alpha_\eta, \alpha_\eta)\mathscr{G}(\alpha_\eta, \alpha_{\eta+1}, \alpha_{\eta+1})\mathscr{G}(\alpha_\eta, \alpha_\eta, \alpha_{\eta+1})\mathscr{G}(\alpha_\eta, \alpha_\eta, \alpha_{\eta+1})\\
&\quad\le 2\lambda\mathscr{G}(\alpha_{\eta-1}, \alpha_\eta, \alpha_\eta)\mathscr{G}^3(\alpha_\eta, \alpha_{\eta+1}, \alpha_{\eta+1}),
\end{aligned}$$

simplifying this and we get,

$$\mathscr{G}(\alpha_\eta, \alpha_{\eta+1}, \alpha_{\eta+1}) \le \mathbb{K}\mathscr{G}(\alpha_{\eta-1}, \alpha_\eta, \alpha_\eta) \quad \text{where} \quad \eta \ge 1 \text{ and } \mathbb{K} = 2\lambda.$$

By induction, we have

$$\mathscr{G}(\alpha_\eta, \alpha_{\eta+1}, \alpha_{\eta+1}) \le \mathbb{K}^\eta\mathscr{G}(\alpha_0, \alpha_1, \alpha_1) \quad \text{for} \quad \eta \ge 1 \tag{2.8}$$

Now using ($\mathscr{G}5$) and (2.8) with $\zeta > \eta$, we see that

$$\begin{aligned}
\mathscr{G}(\alpha_\eta, \alpha_\zeta, \alpha_\zeta) &\le \mathscr{G}(\alpha_\eta, \alpha_{\eta+1}, \alpha_{\eta+1}) + \mathscr{G}(\alpha_{\eta+1}, \alpha_{\eta+2}, \alpha_{\eta+2})\\
&\qquad + \cdots + \mathscr{G}(\alpha_{\zeta-1}, \alpha_\zeta, \alpha_\zeta) \ (\zeta - \eta \text{ terms})\\
&\le (\underbrace{\mathbb{K}^\eta + \mathbb{K}^{\eta+1} + \mathbb{K}^{\eta+2} + \cdots + \mathbb{K}^{\eta+(\zeta-\eta-1)}}_{\zeta-\eta \text{ terms}})\mathscr{G}(\alpha_0, \alpha_1, \alpha_1)\\
&\le \mathbb{K}^\eta \cdot \tfrac{1-\mathbb{K}^{\zeta-\eta}}{1-\mathbb{K}} \cdot \mathscr{G}(\alpha_0, \alpha_1, \alpha_1)\\
&\le \tfrac{\mathbb{K}^\eta}{1-\mathbb{K}} \cdot \mathscr{G}(\alpha_0, \alpha_1, \alpha_1).
\end{aligned} \tag{2.9}$$

Applying $\eta \to \infty$, in (2.9), we get $\mathscr{G}(\alpha_\eta, \alpha_\zeta, \alpha_\zeta) \to 0$ which implies that $\langle\alpha_\eta\rangle_{\eta=1}^{\infty}$ is $\mathscr{G}$-Cauchy in $\mathscr{M}$. Due to completeness of $\mathscr{M}$, there exists an element $\xi \in \mathscr{M}$ such that

$$\lim_{\eta\to\infty} \alpha_\eta = \xi. \tag{2.10}$$

By taking $\alpha = \alpha_{\eta-1}$, $\beta = \gamma = \xi$ in (2.7) and applying ($\mathscr{G}5$) and (2.10), we have

$$\begin{aligned}
&\mathscr{G}^4(\mathbb{F}\alpha_{\eta-1}, \mathbb{F}\xi, \mathbb{F}\xi)\\
&\quad= \mathscr{G}^4(\alpha_\eta, \mathbb{F}\xi, \mathbb{F}\xi)\\
&\quad\le \lambda\mathscr{G}(\alpha_{\eta-1}, \xi, \xi)\mathscr{G}(\xi, \mathbb{F}\xi, \mathbb{F}\xi)\mathscr{G}(\xi, \mathbb{F}\alpha_{\eta-1}, \mathbb{F}\xi)\mathscr{G}(\xi, \mathbb{F}\alpha_{\eta-1}, \mathbb{F}\xi)\\
&\quad= \lambda\mathscr{G}(\alpha_{\eta-1}, \xi, \xi)\mathscr{G}(\xi, \mathbb{F}\xi, \mathbb{F}\xi)\mathscr{G}(\xi, \alpha_\eta, \mathbb{F}\xi)\mathscr{G}(\xi, \alpha_\eta, \mathbb{F}\xi)\\
&\quad= \lambda\mathscr{G}(\alpha_{\eta-1}, \xi, \xi)\mathscr{G}(\xi, \mathbb{F}\xi, \mathbb{F}\xi)\mathscr{G}^2(\xi, \alpha_\eta, \mathbb{F}\xi).
\end{aligned}$$

Employing the limit as $\eta \to \infty$, applying (2.10) and continuity of $\mathscr{G}$ on all three variables, we get $\mathscr{G}(\xi, \mathbb{F}\xi, \mathbb{F}\xi) = 0$. This implies that $\xi = \mathbb{F}\xi$ in view of Proposition 1.3. Therefore, $\mathbb{F}$ has a fixed point ξ. To test the uniqueness, let us assume that ξ ,δ are any two points in $\mathscr{M}$ such that $\xi = \mathbb{F}\xi$ and $\delta = \mathbb{F}\delta$. Now, writing $\alpha = \xi$ and $\beta = \gamma = \delta$ in (2.7), we get

$$\begin{aligned}\mathscr{G}^4(\xi,\delta,\delta) &= \mathscr{G}^4(\mathbb{F}\xi, f\delta, f\delta)\\ &\le \lambda\mathscr{G}(\xi,\delta,\delta)\mathscr{G}(\delta,\mathbb{F}\delta,\mathbb{F}\delta)\mathscr{G}(\delta,\mathbb{F}\xi,\mathbb{F}\delta)\mathscr{G}(\delta,\mathbb{F}\xi,\mathbb{F}\delta)\\ &= \lambda\mathscr{G}^3(\xi,\delta,\delta)\mathscr{G}(\delta,\delta,\delta).\end{aligned}$$

Simplifying this and we obtain that $\mathscr{G}^4(\xi,\delta,\delta) \le 0$, hence, we have $\xi = \delta$. Thus, $\mathbb{F}$ has a unique fixed point ξ. □

The following is the example to Illustrate Theorem 2.3.

Example 2.4. Let $\mathscr{M} = [0,1]$ with $\mathscr{G}$-metric

$$\mathscr{G}(\alpha,\beta,\gamma) = \max\{|\alpha-\beta|, |\beta-\gamma|, |\gamma-\alpha|\}.$$

Then $(\mathscr{M},\mathscr{G})$ is a complete $\mathscr{G}$-metric space. Define $\mathbb{F} : \mathscr{M} \to \mathscr{M}$ by

$$\mathbb{F}\alpha = \begin{cases} 0, & 0 \le \alpha < \frac{1}{2},\\ \frac{1}{10}, & \frac{1}{2} \le \alpha \le 1.\end{cases}$$

We show that $\mathbb{F}$ satisfies the inequality (2.7) with $\lambda = \frac{1}{3}$. We discuss four cases:

Case (i): $\alpha,\beta \in [0,\frac{1}{2})$; $\gamma \in [0,\frac{1}{2})$,

$$\begin{aligned}\mathscr{G}^4(\mathbb{F}\alpha,\mathbb{F}\beta,\mathbb{F}\gamma) &= \mathscr{G}^4(0,0,0)\\ &= 0\\ &\le \lambda\mathscr{G}(\alpha,\beta,\gamma)\mathscr{G}(\beta,\mathbb{F}\beta,\mathbb{F}\gamma)\mathscr{G}(\beta,\mathbb{F}\alpha,\mathbb{F}\gamma)\mathscr{G}(\gamma,\mathbb{F}\alpha,\mathbb{F}\beta).\end{aligned}$$

Case (ii): $\alpha,\beta \in [\frac{1}{2},1]$; $\gamma \in [\frac{1}{2},1]$,

$$\begin{aligned}\mathscr{G}^4(\mathbb{F}\alpha,\mathbb{F}\beta,\mathbb{F}\gamma) &= \mathscr{G}^4(\frac{1}{10},\frac{1}{10},\frac{1}{10})\\ &= 0\\ &\le \lambda\mathscr{G}(\alpha,\beta,\gamma)\mathscr{G}(\beta,\mathbb{F}\beta,\mathbb{F}\gamma)\mathscr{G}(\beta,\mathbb{F}\alpha,\mathbb{F}\gamma)\mathscr{G}(\gamma,\mathbb{F}\alpha,\mathbb{F}\beta).\end{aligned}$$

Case (iii): $\alpha,\beta \in [0,\frac{1}{2})$; $\gamma \in [\frac{1}{2},1]$,

$$\mathscr{G}^4(\mathbb{F}\alpha,\mathbb{F}\beta,\mathbb{F}\gamma) = \mathscr{G}^4(0,0,\frac{1}{10}) = \frac{1}{10000},$$

$$\mathscr{G}(\alpha,\beta,\gamma) = \max\{|\alpha-\beta|, |\beta-\gamma|, |\gamma-\alpha|\} \ge \max\left\{0,\frac{1}{2},\frac{1}{2}\right\} = \frac{1}{2},$$

$$\mathscr{G}(\beta,\mathbb{F}\beta,\mathbb{F}\gamma) = \mathscr{G}(\beta,0,\frac{1}{10}) = \max\left\{|\beta-0|, \left|0-\frac{1}{10}\right|, \left|\frac{1}{10}-\beta\right|\right\}$$

$$\geq \max\left\{0, \frac{1}{10}, \frac{2}{5}\right\} = \frac{2}{5},$$

$$\begin{aligned}\mathscr{G}(\beta, \mathbb{F}\alpha, \mathbb{F}\gamma) &= \mathscr{G}(\beta, 0, \frac{1}{10})\\ &= \max\left\{|\beta - 0|, \left|0 - \frac{1}{10}\right|, \left|\frac{1}{10} - \beta\right|\right\}\\ &\geq \max\left\{0, \frac{1}{10}, \frac{2}{5}\right\} = \frac{2}{5},\end{aligned}$$

$$\begin{aligned}\mathscr{G}(\gamma, \mathbb{F}\alpha, \mathbb{F}\beta) &= \mathscr{G}(\gamma, 0, 0)\\ &= \max\{|\gamma - 0|, |0 - 0|, |0 - \gamma|\}\\ &\geq \max\left\{\frac{1}{2}, 0, \frac{1}{2}\right\} = \frac{1}{2}.\end{aligned}$$

Hence, we have

$$\lambda\mathscr{G}(\alpha, \beta, \gamma)\mathscr{G}(\beta, \mathbb{F}\beta, \mathbb{F}\gamma)\mathscr{G}(\beta, \mathbb{F}\alpha, \mathbb{F}\gamma)\mathscr{G}(\gamma, \mathbb{F}\alpha, \mathbb{F}\beta) \geq \frac{1}{3} \times \frac{1}{2} \times \frac{2}{5} \times \frac{2}{5} \times \frac{1}{2} = \frac{1}{75}.$$

Therefore,

$$\begin{aligned}\mathscr{G}^4(\mathbb{F}\alpha, \mathbb{F}\beta, \mathbb{F}\gamma) &= \mathscr{G}^4(0, 0, \frac{1}{10})\\ &= \frac{1}{10000}\\ &< \frac{1}{75}\\ &\leq \lambda\mathscr{G}(\alpha, \beta, \gamma)\mathscr{G}(\beta, \mathbb{F}\beta, \mathbb{F}\gamma)\mathscr{G}(\beta, \mathbb{F}\alpha, \mathbb{F}\gamma)\mathscr{G}(\gamma, \mathbb{F}\alpha, \mathbb{F}\beta).\end{aligned}$$

Case (iv): $\alpha, \beta \in [\frac{1}{2}, 1]$; $\gamma \in [0, \frac{1}{2})$, $\mathscr{G}^4(\mathbb{F}\alpha, \mathbb{F}\beta, \mathbb{F}\gamma) = \mathscr{G}^4(\frac{1}{10}, \frac{1}{10}, 0) = \frac{1}{10000}$, $\mathscr{G}(\alpha, \beta, \gamma) = \max\{|\alpha - \beta|, |\beta - \gamma|, |\gamma - \alpha|\} \geq \max\{0, \frac{1}{2}, \frac{1}{2}\} = \frac{1}{2}$,

$$\begin{aligned}\mathscr{G}(\beta, \mathbb{F}\beta, \mathbb{F}\gamma) = \mathscr{G}(\beta, \frac{1}{10}, 0) &= \max\left\{\left|\beta - \frac{1}{10}\right|, \left|\frac{1}{10} - 0\right|, |0 - \beta|\right\}\\ &\geq \max\left\{\frac{2}{5}, \frac{1}{10}, \frac{1}{2}\right\} = \frac{1}{2},\end{aligned}$$

$$\begin{aligned}\mathscr{G}(\beta, \mathbb{F}\alpha, \mathbb{F}\gamma) = \mathscr{G}(\beta, \frac{1}{10}, 0) &= \max\left\{\left|\beta - \frac{1}{10}\right|, \left|\frac{1}{10} - 0\right|, |0 - \beta|\right\}\\ &\geq \max\left\{\frac{2}{5}, \frac{1}{10}, \frac{1}{2}\right\} = \frac{1}{2},\end{aligned}$$

$$\mathscr{G}(\gamma,\mathbb{F}\alpha,\mathbb{F}\beta)=\mathscr{G}(\gamma,\frac{1}{10},\frac{1}{10})=\max\left\{\left|\gamma-\frac{1}{10}\right|,\left|\frac{1}{10}-\frac{1}{10}\right|,\left|\frac{1}{10}-\gamma\right|\right\}$$
$$\geq\max\left\{\frac{1}{10},0,\frac{1}{10}\right\}=\frac{1}{10},$$

we have

$$\lambda\mathscr{G}(\alpha,\beta,\gamma)\mathscr{G}(\beta,\mathbb{F}\beta,\mathbb{F}\gamma)\mathscr{G}(\beta,\mathbb{F}\alpha,\mathbb{F}\gamma)\mathscr{G}(\gamma,\mathbb{F}\alpha,\mathbb{F}\beta)\geq\frac{1}{3}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{10}=\frac{1}{240}.$$

Therefore, we have

$$\begin{aligned}\mathscr{G}^4(\mathbb{F}\alpha,\mathbb{F}\beta,\mathbb{F}\gamma)&=\mathscr{G}^4(\frac{1}{10},\frac{1}{10},0)\\&=\frac{1}{10000}\\&<\frac{1}{240}\\&\leq\lambda\mathscr{G}(\alpha,\beta,\gamma)\mathscr{G}(\beta,\mathbb{F}\beta,\mathbb{F}\gamma)\mathscr{G}(\beta,\mathbb{F}\alpha,\mathbb{F}\gamma)\mathscr{G}(\gamma,\mathbb{F}\alpha,\mathbb{F}\beta).\end{aligned}$$

From all the above cases, we observe that

$$\mathscr{G}^4(\mathbb{F}\alpha,\mathbb{F}\beta,\mathbb{F}\gamma)\leq\frac{1}{3}\mathscr{G}(\alpha,\beta,\gamma)\mathscr{G}(\beta,\mathbb{F}\beta,\mathbb{F}\gamma)\mathscr{G}(\beta,\mathbb{F}\alpha,\mathbb{F}\gamma)\mathscr{G}(\gamma,\mathbb{F}\alpha,\mathbb{F}\beta)$$

for all $\alpha,\beta,\gamma\in\mathscr{M}$, which implies that $\mathbb{F}$ satisfies all the conditions of Theorem 2.3. It is clear that 0 is unique fixed point of $\mathbb{F}$.

Corollary 2.5. *If $\mathbb{F}$ is a mapping on $\mathscr{M}$ satisfying the condition*

$$\begin{aligned}\mathscr{G}^4(\mathbb{F}^\eta\alpha,\mathbb{F}^\eta\beta,\mathbb{F}^\eta\gamma)\leq{}&\lambda\mathscr{G}(\alpha,\mathbb{F}^\eta\alpha,\mathbb{F}^\eta\alpha)\mathscr{G}(\mathbb{F}^\eta\alpha,\mathbb{F}^\eta\alpha,\mathbb{F}^\eta\beta)\\&\times\mathscr{G}(\mathbb{F}^\eta\beta,\mathbb{F}^\eta\alpha,\mathbb{F}^\eta\beta)\mathscr{G}(\mathbb{F}^\eta\alpha,\mathbb{F}^\eta\gamma,\mathbb{F}^\eta\gamma)\end{aligned}\tag{2.11}$$

for all $\alpha,\beta,\gamma\in\mathscr{M}$ and $\eta\in\mathbb{N}$, with $0<\lambda<0.25$, then $\mathbb{F}$ has fixed point (say σ) which is unique and $\mathbb{F}^\eta$ is $\mathscr{G}$-continuous at σ .

Proof. It can be proved that $\mathbb{F}^\eta\sigma=\sigma$, and $\mathbb{F}^\eta$ is $\mathscr{G}$-continuous at σ by using Theorem 2.1. Then $\mathbb{F}\sigma=\mathbb{F}\mathbb{F}^\eta\sigma=\mathbb{F}^{\eta+1}\sigma=\sigma$, which implies that $\mathbb{F}\sigma$ is also a fixed point for $\mathbb{F}^\eta$. Hence, $\mathbb{F}\sigma=\sigma$ in view of uniqueness of fixed point for $\mathbb{F}^\eta$. □

Theorem 2.6. *Suppose that $\mathbb{F}$ is a mapping on $\mathscr{M}$ with the condition*

$$\begin{aligned}\mathscr{G}^4(\mathbb{F}\beta,\mathbb{F}^2\beta,\mathbb{F}^3\beta)\leq{}&\lambda\mathscr{G}(\beta,\mathbb{F}\beta,\mathbb{F}\beta)\mathscr{G}(\mathbb{F}\beta,\mathbb{F}\beta,\mathbb{F}^2\beta)\\&\times\mathscr{G}(\mathbb{F}^2\beta,\mathbb{F}\beta,\mathbb{F}^2\beta)\mathscr{G}(\mathbb{F}\beta,\mathbb{F}^3\beta,\mathbb{F}^3\beta)\end{aligned}\tag{2.12}$$

for all $\beta\in\mathscr{M}$ and $0<\lambda<1/4$. Then $\mathbb{F}$ has a fixed point σ.

Proof. Let $\Theta_0 \in \mathscr{M}$. Now define $\{\Theta_\eta\}$ by $\Theta_\eta = \mathbb{F}^\eta\Theta_0 = \mathbb{F}\Theta_{\eta-1}$, $\eta \in \mathbb{N}$. Writing $\beta = \Theta_{\eta-1}$ in (2.12), we get

$$\begin{aligned}
&\mathscr{G}^4(\mathbb{F}\Theta_{\eta-1}, \mathbb{F}^2\Theta_{\eta-1}, \mathbb{F}^3\Theta_{\eta-1})\\
&\quad= \mathscr{G}^4(\Theta_\eta, \Theta_{\eta+1}, \Theta_{\eta+2})\\
&\quad\leq \lambda\mathscr{G}(\Theta_{\eta-1}, \mathbb{F}\Theta_{\eta-1}, \mathbb{F}\Theta_{\eta-1})\mathscr{G}(\mathbb{F}\Theta_{\eta-1}, \mathbb{F}\Theta_{\eta-1}, \mathbb{F}^2\Theta_{\eta-1})\\
&\qquad\times \mathscr{G}(\mathbb{F}^2\Theta_{\eta-1}, \mathbb{F}\Theta_{\eta-1}, \mathbb{F}^2\Theta_{\eta-1})\mathscr{G}(f\Theta_{\eta-1}, \mathbb{F}^3\Theta_{\eta-1}, \mathbb{F}^3\Theta_{\eta-1})\\
&\quad= \lambda\mathscr{G}(\Theta_{\eta-1}, \Theta_\eta, \Theta_\eta)\mathscr{G}(\Theta_\eta, \Theta_\eta, \Theta_{\eta+1})\mathscr{G}(\Theta_{\eta+1}, \Theta_\eta, \Theta_{\eta+1})\mathscr{G}(\Theta_\eta, \Theta_{\eta+2}, \Theta_{\eta+2}).
\end{aligned}$$

On the other hand, using ($\mathscr{G}3$), we have

$$\begin{aligned}
\mathscr{G}(\Theta_{\eta-1}, \Theta_\eta, \Theta_\eta) &\leq \mathscr{G}(\Theta_{\eta-1}, \Theta_\eta, \Theta_{\eta+1}),\\
\mathscr{G}(\Theta_\eta, \Theta_{\eta+1}, \Theta_{\eta+1}) &\leq \mathscr{G}(\Theta_\eta, \Theta_{\eta+1}, \Theta_{\eta+2}),\\
\mathscr{G}(\Theta_\eta, \Theta_{\eta+2}, \Theta_{\eta+2}) &\leq \mathscr{G}(\Theta_\eta, \Theta_{\eta+1}, \Theta_{\eta+2}).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\mathscr{G}^4(\Theta_\eta, \Theta_{\eta+1}, \Theta_{\eta+2})\\
&\qquad\leq \lambda\mathscr{G}(\Theta_{\eta-1}, \Theta_\eta, \Theta_{\eta+1})\mathscr{G}^2(\Theta_\eta, \Theta_{\eta+1}, \Theta_{\eta+2})\mathscr{G}(\Theta_\eta, \Theta_{\eta+1}, \Theta_{\eta+2})\\
&\qquad= \lambda\mathscr{G}(\Theta_{\eta-1}, \Theta_\eta, \Theta_{\eta+1})\mathscr{G}^3(\Theta_\eta, \Theta_{\eta+1}, \Theta_{\eta+2}),
\end{aligned}$$

simplifying this and we get,

$$\mathscr{G}(\Theta_\eta, \Theta_{\eta+1}, \Theta_{\eta+2}) \leq \lambda\mathscr{G}(\Theta_{\eta-1}, \Theta_\eta, \Theta_{\eta+1}) \quad \text{for} \quad \eta \geq 1.$$

By induction, we get

$$\mathscr{G}(\Theta_\eta, \Theta_{\eta+1}, \Theta_{\eta+2}) \leq \lambda\mathscr{G}(\Theta_{\eta-1}, \Theta_\eta, \Theta_{\eta+1}) \leq \cdots \leq \lambda^\eta\mathscr{G}(\Theta_0, \Theta_1, \Theta_2). \tag{2.13}$$

For $\zeta > \eta$, by using ($\mathbb{G}5$) and (2.13), we get

$$\begin{aligned}
\mathscr{G}(\Theta_\eta, \Theta_\zeta, \Theta_\zeta) &\leq \mathscr{G}(\Theta_\eta, \Theta_{\eta+1}, \Theta_{\eta+1}) + \mathscr{G}(\Theta_{\eta+1}, \Theta_{\eta+2}, \Theta_{\eta+2})\\
&\qquad + \cdots + \mathscr{G}(\Theta_{\zeta-1}, \Theta_\zeta, \Theta_\zeta)\\
&\leq (\underbrace{\mathbb{K}^\eta + \mathbb{K}^{\eta+1} + \mathbb{K}^{\eta+2} + \cdots + \mathbb{K}^{\eta+(\zeta-\eta-1)}}_{\zeta-\eta \text{ terms}})\mathscr{G}(\Theta_0, \Theta_1, \Theta_1)\\
&\leq \mathbb{K}^\eta \cdot \tfrac{1-\mathbb{K}^{\zeta-\eta}}{1-\mathbb{K}} \cdot \mathscr{G}(\Theta_0, \Theta_1, \Theta_1)\\
&\leq \tfrac{\mathbb{K}^\eta}{1-\mathbb{K}} \cdot \mathscr{G}(\Theta_0, \Theta_1, \Theta_1).
\end{aligned} \tag{2.14}$$

Applying the $\eta \to \infty$ in (2.14), we get $\mathscr{G}(\Theta_\eta, \Theta_\zeta, \Theta_\zeta) \to 0$, which implies that the sequence $\langle\Theta_\eta\rangle_{\eta=1}^\infty$ is $\mathscr{G}$-Cauchy in $\mathscr{M}$. Due to completeness of$\mathscr{M}$, there is an element $\sigma \in \mathscr{M}$ such that

$$\lim_{\eta\to\infty} \Theta_\eta = \lim_{\eta\to\infty} \Theta_{\eta+1} = \lim_{\eta\to\infty} \Theta_{\eta+2} = \sigma. \tag{2.15}$$

It is clear that $\mathbb{F}$ is $\mathscr{G}$-continuous from $\Theta_{\eta+1} = \mathbb{F}\Theta_\eta$. Now taking $\eta \to \infty$ on $\Theta_{\eta+1} = \mathbb{F}\Theta_\eta$ and using $\mathscr{G}$-continuity of $\mathbb{F}$ together with (2.15), we get $\sigma = \mathbb{F}\sigma$. Therefore, σ is a fixed point of $\mathbb{F}$. □

Theorem 2.7. *Let $\mathbb{F}$ be a mapping on $\mathscr{M}$ such that*

$$\begin{aligned}\mathscr{G}^4(\mathbb{F}\beta, \mathbb{F}^2\gamma, \mathbb{F}^3\delta) \leq & \lambda\mathscr{G}(\beta, \mathbb{F}\beta, \mathbb{F}\beta)\mathscr{G}(\mathbb{F}\beta, \mathbb{F}\beta, \mathbb{F}^2\gamma) \\ & \times \mathscr{G}(\mathbb{F}\gamma, \mathbb{F}\beta, \mathbb{F}^2\gamma)\mathscr{G}(\mathbb{F}\beta, \mathbb{F}^3\delta, \mathbb{F}^3\delta)\end{aligned} \tag{2.16}$$

with $0 < \lambda < 0.25$ for all $\beta, \gamma, \delta \in \mathscr{M}$. Then $\mathbb{F}$ has a fixed point κ which is unique.

Proof. Let $\Theta_0 \in \mathscr{M}$. Now define $\langle\Theta_\eta\rangle_{\eta=1}^\infty \subset \mathscr{M}$ by (2.2). Writing with $\beta = \Theta = \Theta_{\eta-1}$ and $\delta = \Theta_{\eta-2}$ in (2.16), and using ($\mathscr{G}$3), we observe that

$$\begin{aligned}&\mathscr{G}^4(\mathbb{F}\Theta_{\eta-1}, \mathbb{F}^2\Theta_{\eta-1}, \mathbb{F}^3\Theta_{\eta-2}) \\ &\quad = \mathscr{G}^4(\Theta_\eta, \Theta_{\eta+1}, \alpha_{\eta+2}) \\ &\quad \leq \lambda\mathscr{G}(\Theta_{\eta-1}, \mathbb{F}\Theta_{\eta-1}, \mathbb{F}\Theta_{\eta-1})\mathscr{G}(\mathbb{F}\Theta_{\eta-1}, \mathbb{F}\Theta_{\eta-1}, \mathbb{F}^2\Theta_{\eta-1}) \\ &\qquad \times \mathscr{G}(\mathbb{F}\Theta_{\eta-1}, \mathbb{F}\Theta_{\eta-1}, \mathbb{F}^2\Theta_{\eta-1})\mathscr{G}(\mathbb{F}\Theta_{\eta-1}, \mathbb{F}^3\Theta_{\eta-2}, \mathbb{F}^3\Theta_{\eta-2}) \\ &\quad = \lambda\mathscr{G}(\Theta_{\eta-1}, \Theta_\eta, \Theta_\eta)\mathscr{G}(\Theta_\eta, \Theta_\eta, \Theta_{\eta+1})\mathscr{G}(\Theta_\eta, \Theta_\eta, \Theta_{\eta+1})\mathscr{G}(\Theta_\eta, \Theta_{\eta+2}, \Theta_{\eta+2}) \\ &\quad \leq \lambda\mathscr{G}(\Theta_{\eta-1}, \Theta_\eta, \Theta_{\eta+1})\mathscr{G}^3(\Theta_\eta, \Theta_{\eta+1}, \Theta_{\eta+2}),\end{aligned}$$

simplifying this and we get

$$\mathscr{G}(\Theta_\eta, \Theta_{\eta+1}, \Theta_{\eta+2}) \leq \lambda\mathscr{G}(\Theta_{\eta-1}, \Theta_\eta, \Theta_{\eta+1}) \quad \text{for} \quad \eta \geq 1.$$

By induction, we get

$$\mathscr{G}(\Theta_\eta, \Theta_{\eta+1}, \Theta_{\eta+2}) \leq \lambda\mathscr{G}(\Theta_{\eta-1}, \Theta_\eta, \Theta_{\eta+1}) \leq \cdots \leq \lambda^\eta\mathscr{G}(\Theta_0, \Theta_1, \Theta_2). \tag{2.17}$$

Now using for ($\mathscr{G}$5) and (2.17) for $\zeta > \eta$, we have

$$\begin{aligned}\mathscr{G}(\Theta_\eta, \Theta_\zeta, \Theta_\zeta) &\leq \mathscr{G}(\Theta_\eta, \Theta_{\eta+1}, \Theta_{\eta+1}) + \mathscr{G}(\Theta_{\eta+1}, \Theta_{\eta+2}, \Theta_{\eta+2}) \\ &\qquad + \cdots + \mathscr{G}(\Theta_{\zeta-1}, \Theta_\zeta, \Theta_\zeta) \\ &\leq (\underbrace{\mathbb{K}^\eta + \mathbb{K}^{\eta+1} + \mathbb{K}^{\eta+2} + \cdots + \mathbb{K}^{\eta+(\zeta-\eta-1)}}_{\zeta-\eta \text{ terms}})\mathscr{G}(\Theta_0, \Theta_1, \Theta_1) \\ &\leq \mathbb{K}^\eta \cdot \tfrac{1-\mathbb{K}^{\zeta-\eta}}{1-\mathbb{K}} \cdot \mathscr{G}(\Theta_0, \Theta_1, \Theta_1) \\ &\leq \tfrac{\mathbb{K}^\eta}{1-\mathbb{K}} \cdot \mathscr{G}(\Theta_0, \Theta_1, \Theta_1).\end{aligned} \tag{2.18}$$

Applying $\eta \to \infty$, in (2.18), we find that $\mathscr{G}(\Theta_\eta, \Theta_\zeta, \Theta_\zeta) \to 0$, which means that the sequence $\langle\Theta_\eta\rangle_{\eta=1}^\infty$ is $\mathscr{G}$-Cauchy in $\mathscr{M}$. Due to completeness of $\mathscr{M}$, there exists an element $\kappa \in \mathscr{M}$ such that

$$\lim_{\eta\to\infty} \Theta_\eta = \lim_{\eta\to\infty} \Theta_{\eta+1} = \lim_{\eta\to\infty} \Theta_{\eta+2} = \kappa. \tag{2.19}$$

By writing $\beta = \kappa$, $\Theta = \Theta_{\eta-1}$ and $\delta = \Theta_{\eta-2}$ in (2.16), we get

$$\begin{aligned}
&\mathscr{G}^4(\mathbb{F}\kappa, \mathbb{F}^2\Theta_{\eta-1}, \mathbb{F}^3\Theta_{\eta-2}) \\
&\quad = \mathscr{G}^4(\mathbb{F}\kappa, \Theta_{\eta+1}, \Theta_{\eta+1}) \\
&\quad \leq \lambda\mathscr{G}(\kappa, \mathbb{F}\Theta_{\eta-1}, \mathbb{F}\Theta_{\eta-1})\mathscr{G}(\mathbb{F}\kappa, \mathbb{F}\kappa, \mathbb{F}^2\Theta_{\eta-1}) \\
&\qquad \times \mathscr{G}(\mathbb{F}\Theta_{\eta-1}, \mathbb{F}\kappa, \mathbb{F}^2\Theta_{\eta-1})\mathscr{G}(\mathbb{F}\kappa, \mathbb{F}^3\Theta_{\eta-2}, \mathbb{F}^3\Theta_{\eta-2}) \\
&\quad = \lambda\mathscr{G}(\kappa, \kappa_\eta, \Theta_\eta)\mathscr{G}(\mathbb{F}\kappa, \mathbb{F}\kappa, \Theta_{\eta+1})\mathscr{G}(\Theta_\eta, \mathbb{F}\kappa, \Theta_{\eta+1})\mathscr{G}(\mathbb{F}\kappa, \Theta_{\eta+1}, \Theta_{\eta+1}).
\end{aligned}$$

Now, taking $\eta \to \infty$ and using continuity of $\mathscr{G}$ on all its variables together with (2.19), we have $G^4(\mathbb{F}\kappa, \kappa, \kappa) \leq 0$ which gives that $\kappa = \mathbb{F}\kappa$ in view of Proposition 1.3. Thus κ is a fixed point of $\mathbb{F}$.

Let us suppose that there are two points κ and ϕ in $\mathscr{M}$ such that $\kappa = \mathbb{F}\kappa$ and $\phi = \mathbb{F}\phi$. Then writing $\beta = \gamma = \kappa$ and $\delta = \phi$ in (2.16), we have

$$\begin{aligned}
&\mathscr{G}^4(\mathbb{F}\kappa, \mathbb{F}^2\kappa, \mathbb{F}^3\phi) \\
&\quad = \mathscr{G}^4(\kappa, \kappa, \phi) \\
&\quad \leq \lambda\mathscr{G}(\kappa, \mathbb{F}\kappa, \mathbb{F}\kappa)\mathscr{G}(\mathbb{F}\kappa, \mathbb{F}\kappa, \mathbb{F}^2\kappa)\mathscr{G}(\mathbb{F}\kappa, \mathbb{F}\kappa, \mathbb{F}^2\kappa)\mathscr{G}(\mathbb{F}\kappa, \mathbb{F}^3\phi, \mathbb{F}^3\phi) \\
&\quad = \lambda\mathscr{G}(\kappa, \kappa, \kappa)\mathscr{G}(\kappa, \kappa, \kappa)\mathscr{G}(\kappa, \kappa, \kappa)\mathscr{G}(\kappa, \phi, \phi).
\end{aligned}$$

Simplifying this and we obtain that, $\mathscr{G}^4(\kappa, \kappa, \phi) \leq 0$ and then $\kappa = \phi$. Hence, κ is a fixed point of $\mathbb{F}$ which is unique. □

3. $\mathscr{G}$-contractive fixed point

Let $(\mathscr{M}, \mathscr{G})$ be a $\mathscr{G}$-metric space, σ be a fixed point of $\mathbb{F}$. Then σ is called $\mathscr{G}$-contractive fixed point of $\mathbb{F}$, if each orbit $O_{\mathbb{F}}(\alpha_0) \subset \mathscr{M}$ is $\mathscr{G}$-convergent to σ.

Example 3.1. Let $(\mathscr{M}, \rho)$ be a metric space with $\mathscr{M} = \mathbb{R}$ and $\rho(\beta, \gamma) = |\beta - \gamma|$. Consider $\mathscr{G}(\beta, \gamma, \delta) = \max\{|\beta - \gamma|, |\gamma - \delta|, |\delta - \gamma|\}$ for $\beta, \gamma, \delta \in \mathscr{M}$. Then $(\mathscr{M}, \mathscr{G})$ is a $\mathscr{G}$-metric space. Define $\mathbb{F} : \mathscr{M} \to \mathscr{M}$ such that $\mathbb{F}\beta = \frac{\beta}{2}$ for any $\beta \in \mathscr{M}$, we observe that fixed point of $\mathbb{F}$ is 0 only. Also for each $\beta_0 \in \mathscr{M}$, we have $O_{\mathbb{F}}(\beta) = \left\{\beta_0, \beta_0/2, \beta_0/2^2, ...\right\}$. Since for any $\beta_0 \in \mathscr{M}$, $\beta_0/2^\eta \to 0$ as $\eta \to \infty$, we see that $O_{\mathbb{F}}(\beta_0) \to 0$ for all β_0. In other words, every $\mathbb{F}$-orbit converges to the fixed point 0. Therefore, 0 is the $\mathscr{G}$-contractive fixed point.

The concept of contractive fixed point under $\mathscr{G}$-metric was indroduced by Phaneendra et al. [16]. Recently, Saravanan and Phaneendra obtained $\mathscr{G}$-contractive fixed point for various types of contractions ([17], [18]). Now we find $\mathscr{G}$-contractive fixed points of $\mathbb{F}$ with conditions (2.1), (2.7).

Theorem 3.2. *Suppose that ξ is a fixed point of $\mathbb{F}$ with the condition* (2.1),*where* $0 < \lambda < 1/4$. *Then ξ is a $\mathscr{G}$-contractive fixed point of $\mathbb{F}$.*

Proof. Taking $\alpha = \mathbb{F}^{\eta-1}\alpha$ and $\beta = \gamma = \xi$ in (2.1), we get

$$\begin{aligned}\mathscr{G}^4(\mathbb{F}^{\eta}\alpha, \xi, \xi) &= \mathscr{G}^4(\mathbb{F}^{\eta}\alpha, \mathbb{F}\xi, \mathbb{F}\xi)\\ &\leq \lambda\mathscr{G}(\mathbb{F}^{\eta-1}\alpha, \xi, \mathbb{F}^{\eta}\alpha)\mathscr{G}(\xi, \mathbb{F}^{\eta}\alpha, \mathbb{F}\xi)\mathscr{G}(\xi, \mathbb{F}^{\eta}\alpha, \mathbb{F}\xi)\mathscr{G}(\xi, \mathbb{F}\xi, \mathbb{F}\xi)\\ &= \lambda\mathscr{G}(\mathbb{F}^{\eta-1}\alpha, \xi, \mathbb{F}^{\eta}\alpha)\mathscr{G}^2(\xi, \mathbb{F}^{\eta}\alpha, \mathbb{F}\xi)\mathscr{G}(\xi, \mathbb{F}\xi, \mathbb{F}\xi)\\ &\leq \lambda\mathscr{G}(\mathbb{F}^{\eta-1}\alpha, \xi, \xi)\mathscr{G}^2(\xi, \xi, \mathbb{F}\xi)\mathscr{G}(\xi, \mathbb{F}\xi, \mathbb{F}\xi)\\ &= \lambda\mathscr{G}(\mathbb{F}^{\eta-1}\alpha, \xi, \xi)\mathscr{G}^2(\xi, \xi, \xi).\end{aligned}$$

Applying $\eta \to \infty$, we get $\mathscr{G}(\mathbb{F}^{\eta}\alpha, \xi, \xi) \to 0$ for each $\alpha \in \mathscr{M}$. Thus, ξ is a $\mathscr{G}$-contractive fixed point of $\mathbb{F}$. □

Theorem 3.3. *Let ξ be a fixed point of $\mathbb{F}$ with the condition* (2.7)*, where* $0 < \lambda < 0.5$*. Then the $\mathscr{G}$-contractive fixed point of $\mathbb{F}$ is ξ.*

Proof. By taking $\alpha = \mathbb{F}^{\eta-1}\alpha$ and $\beta = \gamma = \xi$ in (2.7), we have

$$\begin{aligned}\mathscr{G}^4(\mathbb{F}^{\eta}\alpha, \xi, \xi) &= \mathscr{G}^4(\mathbb{F}^{\eta}\alpha, \mathbb{F}\xi, \mathbb{F}\xi)\\ &\leq \lambda\mathscr{G}(\mathbb{F}^{\eta-1}\alpha, \xi, \xi)\mathscr{G}(\xi, \mathbb{F}\xi, \mathbb{F}\xi)\mathscr{G}(\xi, \mathbb{F}^{\eta}\alpha, \mathbb{F}\xi)\mathscr{G}(\xi, \mathbb{F}^{\eta}\alpha, \mathbb{F}\xi)\\ &= \lambda\mathscr{G}(\mathbb{F}^{\eta-1}\alpha, \xi, \xi)\mathscr{G}(\xi, \mathbb{F}\xi, \mathbb{F}\xi)\mathscr{G}(\xi, \xi, \mathbb{F}\xi)\mathscr{G}(\xi, \xi, \mathbb{F}\xi)\\ &= \lambda\mathscr{G}(\mathbb{F}^{\eta-1}\alpha, \xi, \xi)\mathscr{G}^3(\xi, \xi, \xi).\end{aligned}$$

Applying $\eta \to \infty$, we get $\mathscr{G}(\mathbb{F}^{\eta}\alpha, \xi, \xi) \to 0$ for each $\alpha \in \mathscr{M}$. Thus, ξ is a $\mathscr{G}$-contractive fixed point of $\mathbb{F}$. □

4. Application

Recently Younis et al. applied their fixed point results to solve differential equations ([21], [23], [24]).

In this paper, we solve characteristic value problem as an application of our fixed point results.

Definition 4.1. ([20]) A number η is said to be characteristic value of an $n \times n$ matrix T if $TY = \eta Y$ for a non-zero vector Y. Here Y is called characteristic vector of a matrix T.

Now writing $\delta = \gamma$ in (2.16), we get

$$\begin{aligned}\mathscr{G}^4(\mathbb{F}\beta, \mathbb{F}^2\gamma, \mathbb{F}^3\gamma) \leq{}& \lambda\mathscr{G}(\beta, \mathbb{F}\beta, \mathbb{F}\beta)\mathscr{G}(\mathbb{F}\beta, \mathbb{F}\beta, \mathbb{F}^2\gamma)\\ &\times \mathscr{G}(\mathbb{F}\gamma, \mathbb{F}\beta, \mathbb{F}^2\gamma)\mathscr{G}(\mathbb{F}\beta, \mathbb{F}^3\gamma, \mathbb{F}^3\gamma)\end{aligned}$$

or

$$\begin{aligned}\mathscr{G}^4(\mathbb{F}\beta, \mathbb{F}^2\gamma, \mathbb{F}^3\gamma) \leq{}& 2\lambda\mathscr{G}(\beta, \mathbb{F}\beta, \mathbb{F}\beta)(\mathscr{G}\mathbb{F}\beta, \mathbb{F}^2\gamma, \mathbb{F}^2\gamma)\\ &\times \mathscr{G}(\mathbb{F}\beta, \mathbb{F}\gamma, \mathbb{F}^2\gamma)\mathscr{G}(\mathbb{F}\beta, \mathbb{F}^3\gamma, \mathbb{F}^3\gamma) \qquad (4.1)\end{aligned}$$

for all $\beta, \gamma \in \mathscr{M}$ with $\beta \neq \gamma$ and $0 < \lambda < \frac{1}{4}$.

We solve a characteristic value problem which related to the following result:

Theorem 4.2. *Let $\mathbb{F}$ be a mapping on $(\mathscr{M}, \mathscr{G})$ which satisfies the condition (4.1). If $\langle \beta_\eta \rangle_{\eta=1}^{\infty}$ is a sequence in $\mathscr{M}$ such that*

$$\lim_{n\to\infty} \mathbb{F}\beta_\eta = \lim_{\eta\to\infty} \beta_\eta = \beta \quad \text{for some } \beta \in \mathscr{M}, \tag{4.2}$$

then $\mathbb{F}$ has a fixed point which is unique.

Proof. The proof of this theorem is omitted, as it closely follows the argument used in Theorem 2.7. □

Proposition 4.3. *Suppose that$(\mathscr{M}, \|\cdot\|)$ is a normed linear space which is complete, $\mathbb{F} : \mathscr{M} \to \mathscr{M}$ is a mapping such that $\mathbb{F}\boldsymbol{0} \neq \boldsymbol{0}$. If there exists $\langle \beta_k \rangle_{k=1}^{\infty} \subset \mathscr{M}$ such that*

$$\lim_{k\to\infty} \mathbb{F}_\eta \beta_k = \lim_{k\to\infty} \beta_k = y \quad \text{for some } y \in \mathscr{M}, \tag{4.3}$$

where

$$\mathbb{F}_\eta = \frac{1}{E_\eta}\mathbb{F} \text{ where } E_\eta = \frac{1}{\left(1 - \frac{1}{\eta}\right)} \quad \text{for} \quad \eta \geq 2 \tag{4.4}$$

and

$$\|\mathbb{F}\beta - \mathbb{F}\gamma\|^4 \leq 2\lambda \|\beta - \mathbb{F}_\eta\beta\| \left\|\mathbb{F}_\eta\beta - \mathbb{F}_\eta^2\gamma\right\| \|\mathbb{F}_\eta\gamma - \mathbb{F}_\eta\beta\| \left\|\mathbb{F}_\eta\beta - \mathbb{F}_\eta^3\boldsymbol{y}\right\| \tag{4.5}$$

holds for all $\beta, \gamma \in \mathscr{M}$ with $\beta \neq \gamma$, then E_η is a characteristic value and β_η is a characteristic vector of $\mathbb{F}$ for each $\eta > 1$.

Proof. From (4.4), we have

$$\|\mathbb{F}_\eta\beta - \mathbb{F}_\eta\gamma\| = \left(1 - \frac{1}{\eta}\right) \|\mathbb{F}\beta - \mathbb{F}\gamma\| < \|\mathbb{F}\beta - \mathbb{F}\gamma\| \quad \text{for } \eta > 1. \tag{4.6}$$

Then, from (4.6) and (4.5), we get

$$\|\mathbb{F}_\eta\beta - \mathbb{F}_\eta\beta\|^4 < 2\lambda \|\beta - \mathbb{F}_\eta\alpha\| \left\|\mathbb{F}_\eta\beta - \mathbb{F}_\eta^2\gamma\right\| \|\mathbb{F}_\eta\gamma - \mathbb{F}_\eta\beta\| \left\|\mathbb{F}_\eta\beta - \mathbb{F}_\eta^3\gamma\right\| \tag{4.7}$$

Now, define the $\mathscr{G}$-metric $\mathscr{G}$ on $\mathscr{M} = \mathbb{R}$ as

$$\mathscr{G}(\beta, \gamma, \delta) = \tfrac{1}{3}[\|\beta - \gamma\| + \|\gamma - \delta\| + \|\delta - \beta\|] \quad \text{for all } \beta, \gamma, \delta \in \mathscr{M}.$$

With $\delta = \gamma$, this gives

$$G(\beta, \gamma, \gamma) = \tfrac{2}{3} \|\beta - \gamma\| \quad \text{for all } \beta, \gamma \in \mathscr{M}. \tag{4.8}$$

Therefore, (4.7) is a particular case of (4.1) in view of (4.8). Thus by Theorem 4.2, we can find a unique $\beta_\eta \in \mathscr{M}$ such that $\mathbb{F}_\eta\beta_\eta = \beta_\eta$ for each $\eta \geq 2$. Therefore,

$$\left(1 - \tfrac{1}{\eta}\right) \mathbb{F}\beta_\eta = \beta_\eta, \text{ that is, } \mathbb{F}\beta_\eta = \beta_\eta / \left(1 - \tfrac{1}{\eta}\right).$$

In other words $\mathbb{F}\beta_\eta = E_\eta\beta_\eta$ for every $\eta > 1$. Suppose that $\beta_\eta = \mathbf{0}$ for $\eta > 1$. Then $\mathbb{F}\mathbf{0} = \mathbf{0}$, which contradicts the fact that $\mathbb{F}\mathbf{0} \neq \mathbf{0}$. Thus, β_η can not be zero for any $\eta > 1$. That is, β_η is an characteristic vector and E_η is an characteristic value of $\mathbb{F}$ for each $\eta > 1$. □

5. Conclusion

We established some fixed point theorems under newly introduced fourth power contractions in $\mathscr{G}$-metric spaces. Moreover, we proved that the obtained fixed point is also $\mathscr{G}$-contractive fixed point in $\mathscr{G}$-metric spaces. In addition, some examples are given to support our results. Finally, we solved characteristic value problem as an application of our fixed point results in $\mathscr{G}$-metric spaces.

References

[1] M. Abbas and B.E. Rhoades, *Common fixed point results for noncommuting mappings without continuity in generalized metric spaces,* Appl. Math. Comput., **215**(1) (2009), 262-269.

[2] H. Aydi, E. Karapinar and W. Sathanawi, *Tripled Fixed Point Results in Generalized Metric spaces,* J. Appl. Math., **2012** (2012), 1-9, https://doi.org/10.1155/2012/314279.

[3] Lj.B. Ciric, *A generalization of Banach's contraction principle,*, Proc. Amer. Math. Soc., **45**(2) (1974), 267-273.

[4] K. Dasu Naidu, K. Kumara Swamy and P. Sumati Kumari, *Utilizing fixed point Methods on Mathematical Modelling,* Nonlinear Funct. Anal. Appl., **28**(2) (2023), 473-495.

[5] F. Gu and H. Ye, *Fixed point theorems for a third power type contraction mappings in G-metric spaces,* Hacettepe J. Math. Stati., **42**(5) (2013), 495-500.

[6] M. Jleli and B. Samet, *Remarks on G-metric spaces and fixed point theorems,* Fixed Point Theory Appl., **2012**(210) (2012), 1-7.

[7] E. Karapinar and A. Ravi Paul, *Further Remarks on G-metric spaces*, Fixed point theory Appl., **2013**(154) (2013), 1-19.

[8] A.R. Khan and A.A. Domlo, *Coincidence and fixed points of nonself contractive maps with applications,* Indian J. Math., **49**(1) (2007), 17-13.

[9] K. Kumara Swamy, Swatmaram, B. Hazarika and P.S. Kumari, *Unique Common Tripled Fixed Point for Three Mappings in Gb-metric Space,* Math. and Stati., **9**(5) (2021), 835-852.

[10] S.K. Mohanta and S.A. Mohanta, *Common Fixed Point Theorem G-metric spaces,* CUBO A Math. J., **14**(3) (2012), 85-101.

[11] Z. Mustafa, *Some New Common Fixed Point Theorems under Strict Contractive Conditions in G-MetricSpaces,* J. Appl. Math., **2012**:248397, (2012).

[12] Z. Mustafa and H. Obiedat, *A fixed point theorem of Reich in G-Metric Spaces,* CUBO A Math. J., **12**(1) (2010), 83-93.

[13] Z. Mustafa, H. Obiedat and F. Awawdeh, *Some fixed point theorem for mapping on complete G-Metric Spaces,* Fixed Point Theory Appl., **2008**:189870, (2008).

[14] Z. Mustafa, W. Shatanawi and M. Bataineh, *Existence of Fixed Point Results in G-Metric Spaces*, Int. J. Math. Math. Sci., **2009**:283028, (2009).

[15] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces,* J. Nonlinear Convex Anal., **7**(2) (2006), 289-297.

[16] T. Phaneendra and K. Kumara Swamy, *Unique fixed point in G-metric space through greatest lower bound properties,* Novi Sad J. Math., **43**(2) (2013), 107-115.

[17] T. Phaneendra and S. Saravanan, *Fixed Point Under General Contraction Conditions in G-metric space,* The J. Anal., **25**(2) (2017), 215-234.

[18] S. Saravanan and T. Phaneendra, *Fixed point as a G-Contractive fixed point,* Int. J. Appl. Engg. Res., **11**(1) (2016), 316-319.

[19] W. Shatanawi, *Fixed Point Theory for Contractive Mappings Satisfying ϕ-Maps in G-Metric Spaces,* Fixed Point Theory Appl., **2010**:181650, (2010), 1-9.

[20] G. Strang, *Introduction to Linear Algebra*, Wellesley-Cambridge Press, 5th edition.

[21] M. Younis and A.A.N. Abdou, *Novel fuzzy contractions and applications to Engineering science.,* Fractal and Fractional, **8**(1) (2023), 28.

[22] M. Younis, D. Singh, L. Chen and M. Metwali, *A study on the solutions of notable engineering models,* Math. Model. Anal., **27**(3) (2022), 492-509.

[23] M. Younis, D. Singh, D. Gopal, A. Goyal and M.S. Rathore, *On applications of generalized F-contraction to differential equations,* Nonlinear Funct. Anal. Appl., **24**(1) (2019), 155-174.

[24] M. Younis, A. Stretenović and S. Radenović, *Some critical remarks on Some new fixed point results in rectangular metric spaces with an application to fractional order functional differential equations*, Nonlinear Anal. Model. Control, **27**(1) (2022), 163-178.