**Nonlinear Functional Analysis and Applications**
**Vol. 30, No. 3 (2025), pp. 805-829**
**ISSN: 1229-1595(print), 2466-0973(online)**
https://doi.org/10.22771/nfaa.2025.30.03.10
http://nfaa.kyungnam.ac.kr/journal-nfaa

# HYBRID METHOD FOR SOLVING PSEUDO-MONOTONE EQUILIBRIUM PROBLEMS USING BREGMAN DISTANCE

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**Abstract.** In this article, we introduce two new algorithms for solving equilibrium problem involving pseudo-monotone and Bregman Lipschitz-type bifunction in reflexive Banach spaces. The algorithms are constructed around the extragradient method and the advantage of the algorithms is that it is done without the prior knowledge of Bregman Lipschitz coefficients. Theorems of strong convergence are established under mild conditions. Finally, a numerical experiments are reported to illustrate the efficiency of the proposed algorithm.

## 1. Introduction

Let $X$ be a reflexive real Banach space and $C$ be a nonempty closed and convex subset of $X$. Throughout this paper, we shall denote the dual space of $X$ by $X^*$. The norm and the duality pairing between $X$ and $X^*$ are respectively denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$. The minimization problem for a function $f : C \to \mathbb{R}$

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[0]Received December 23, 2024. Revised March 7, 2025. Accepted June 21, 2025.
[0]2020 Mathematics Subject Classification: 47H05, 47H09, 47H10.
[0]Keywords: Equilibrium problem, hybrid method, Bregman Lipschitz-type condition, pseudo-monotone bifunction, Legendre function.
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is defined as

$$\text{Find } x^* \in C \quad \text{such that } f(x^*) \leq f(y), \ \ \forall y \in C. \tag{1.1}$$

In this case, $x^*$ is called a minimizer of $f$ and

$$Argmin_{y \in C} f(y)$$

denotes the set of minimizers of $f$. Minimization problems are very useful in optimization theory as well as convex and nonlinear analysis. We introduce the equilibrium problem $(EP)$ of find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \ \ \forall y \in C, \tag{1.2}$$

where $f : C \times C \to \mathbb{R}$ is a bifunction.

The set of solutions of (1.2) is denoted by $EP(f)$. It unifies many important mathematical problems, such as optimization problems, complementarity, fixed point, Nash equilibria, optimization, saddle point, and variational inequality problems can be reformulated as equilibrium problems (cf. [10, 18, 24]).

In recent decades, many methods have been proposed and analyzed for approximating solution of equilibrium problems (cf. [23, 25, 27, 34]). However, there are only a few papers that deal with iterative procedures for solving equilibrium problems in finite as well as infinite-dimensional spaces (cf. [20, 26, 33, 42, 43]).

Recently, iterative methods for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of operators in Hilbert spaces have been developed further by several authors [3, 4, 42].

In [36], Reich and Sabach proposed two algorithms for finding common fixed points of finitely many Bregman firmly nonexpansive operators defined on a nonempty, closed and convex subset $C$ of a reflexive Banach spaces $X$. Also, they have presented several methods for solving equilibrium problems in reflexive Banach spaces, see [35, 37, 38]. Inspired by the aforementioned results, the main purpose of this paper is to extend the method in [22] to equilibrium problems (1.2) with the Bregman distance.

The paper is organized as follows: In section 2, we present some definitions and preliminaries needed in the paper. We introduce our two algorithms and prove the main result in the Section 3. Finally, sections 4 include applications of algorithms and we give a example of equilibrium problems to which our main theorem can be applied. Finally, a numerical example to support our main theorem will be exhibited in a non-Hilbertian space.

## 2. Preliminaries

In this section, we recall some definition and results that will be used in the sequel.

Let $f : X \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. The set of minimizers of $f$ is denoted by $Argmin\ f$. If $Argmin\ f$ is a singleton, its unique element is denoted by $argmin_{x\in X}\ f(x)$. Also we denote by dom $f$, the domain of $f$, that is the set $\{x \in X : f(x) < \infty\}$. Let $x \in int\ dom f$. Then subdifferential of $f$ at $x$ is the convex set defined by:

$$\partial f(x) = \{\xi \in X^* : f(x) + \langle y - x, \xi\rangle \leq f(y),\ \ \forall y \in X\},$$

and the Fenchel conjugate of $f$ is the convex function

$$f^* : X^* \to (-\infty, \infty], \qquad f^*(\xi) = \sup\{\langle x, \xi\rangle - f(x) : x \in X\}.$$

It is well known that $\xi \in \partial f(x)$ is equivalent to

$$f(x) + f^*(\xi) = \langle x, \xi\rangle. \tag{2.1}$$

It is not difficult to check that $f^*$ is proper convex and lower semicontinuous function. The function $f$ is said to be cofinite if dom $f^* = X^*$. For any convex mapping $f : X \to (-\infty, +\infty]$, we denote by $f^\circ(x, y)$ the right-hand derivative of $f$ at $x \in int\ dom f$ in the direction $y$, that is,

$$f^\circ(x, y) := \lim_{t\downarrow 0} \frac{f(x + ty) - f(x)}{t}. \tag{2.2}$$

If the limit as $t \to 0$ in (2.2) exists for each $y$, then the function $f$ is said to be Gâteaux differentiable at $x$. In this case, the gradient of $f$ at $x$ is the linear function $\nabla f(x)$, which is defined by $\langle y, \nabla f(x)\rangle : = f^\circ(x, y)$ for all $y \in X$. The function $f$ is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in int\ dom f$. When the limit as $t \to 0$ in (2.2) is attained uniformly for any $y \in X$ with $\|y\| = 1$, we say that $f$ is Fréchet differentiable at $x$. Finally, $f$ is said to be uniformly Fréchet differentiable on a subset $E$ of $X$ if the limit is attained uniformly for $x \in E$ and $\|y\| = 1$.

The function $f$ is said to be Legendre if it satisfies the following two conditions:

(L1) $int\ dom f \neq \emptyset$ and $\partial f$ is single-valued on its domain,

(L2) $int\ dom f^* \neq \emptyset$ and $\partial f^*$ is single-valued on its domain.

Because the space $X$ is assumed to be reflexive, we always have $(\partial f)^{-1} = \partial f^*$. (see[11], p. 83). This fact, when combined with conditions (L1) and (L2), implies the following equalities:

$$\nabla f = (\nabla f^*)^{-1},$$

$$ran\ \nabla f = dom\ \nabla f^* = int\ dom\ f^*,$$

$$ran\ \nabla f^* = dom\ \nabla f = int\ dom\ f.$$

Also, conditions (L1) and (L2), in conjunction with Theorem 5.4 of [8], imply that the functions $f$ and $f^*$ are strictly convex on the interior of their respective domains and $f$ is Legendre if and only if $f^*$ is Legendre. Several interesting examples of Legendre functions are presented in [6, 8]. Among them are the functions $\frac{1}{p}\|\cdot\|^p$ with $p\in(1,\infty)$, where the Banach space $X$ is smooth and strictly convex.

In 1967, Bregman [12] introduced the concept of Bregman distance, and he discovered an elegant and effective technique for the use of the Bregman distance in the process of designing and analyzing feasibility and optimization algorithms.

From now on, we assume that $f:X\to(-\infty,+\infty]$ is also Legendre. The Bregman distance with respect to $f$, or simply, Bregman distance is the bifunction $D_f:dom f\times int\ dom f\to[0,+\infty]$, defined by:

$$D_f(y,x) := f(y) - f(x) - \langle y-x, \nabla f(x)\rangle.$$

It should be noted that $D_f$ is not a distance in the usual sense of the term. Clearly, $D_f(x,x)=0$, but $D_f(y,x)=0$ may not imply $x=y$. In our case, when $f$ is Legendre this indeed holds (see [8], Theorem 7.3(vi), p. 642). In general, $D_f$ is not symmetric and does not satisfy the triangle inequality. However, $D_f$ satisfies the three point identity

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle x-y, \nabla f(z) - \nabla f(y)\rangle,$$

and four point identity

$$D_f(x,y) + D_f(w,z) - D_f(x,z) - D_f(w,y) = \langle x-w, \nabla f(z) - \nabla f(y)\rangle$$

for any $x,w\in dom f$ and $y,z\in int\ dom f$.

During the last 30 years, Bregman distances, have been studied by many researchers (see [1, 7, 8, 13, 14, 29] ).

The modulus of total convexity at $x\in int\ dom f$ is the function $\upsilon_f(x,.) : [0,\infty)\to[0,\infty]$, defined by:

$$\upsilon_f(x,t) := \inf\{D_f(y,x) : y\in dom\ f,\ \|y-x\| = t\}.$$

The function $f$ is called totally convex at $x\in int\ dom f$ if $\upsilon_f(x,t)$ is positive for any $t>0$. This notion was first introduced by Butnariu and Iusem in [14]. Let $E$ be a nonempty subset of $X$. The modulus of total convexity of $f$ on $E$ defined by:

$$\upsilon_f(E,t) = \inf\{\upsilon_f(x,t) :\ x\in E\cap int\ dom\ f\}.$$

The function $f$ is called totally convex on bounded subsets if $\upsilon_f(E,t)$ is positive for any nonempty and bounded subset $E$ and for any $t>0$.

We will use the following lemmas in the proof of our results.

**Lemma 2.1.** ([38]) *If $f:X\to\mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of $X$, then $\nabla f$ is uniformly continuous on bounded subsets of $X$ from the strong topology of $X$ to the strong topology of $X^*$.*

Recall that the function $f$ is called sequentially consistent (see [16]) if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that the first one is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \;\;\Rightarrow\;\; \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

**Lemma 2.2.** ([14]) *If dom $f$ contains at least two points, then the function $f$ is totally convex on bounded sets if and only if the function $f$ is sequentially consistent.*

**Lemma 2.3.** ([38]) *Let $f:X\to\mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0\in X$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

**Lemma 2.4.** ([39]) *Let $f:X\to\mathbb{R}$ be a Legendre function such that $\nabla f^*$ is bounded on bounded subsets of int dom$f^*$. Let $x_0\in X$, if $\{D_f(x_0, x_n)\}$ is bounded then the sequence $\{x_n\}$ is bounded too.*

The Bregman projection (see[12]) with respect to $f$ of $x\in$ *int dom$f$* onto a nonempty, closed and convex set $C\subset$ *int dom$f$* is defined as the necessarily unique vector $\overleftarrow{proj}^f_C(x)\in C$, which satisfies

$$D_f(\overleftarrow{proj}^f_C(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Similarly to the metric projection in Hilbert spaces, the Bregman projection with respect to totally convex and Gâteaux differentiable functions has a variational characterization ([16], Corollary 4.4, p. 23]).

**Lemma 2.5.** ([16]) *Suppose that $f$ is Gâteaux differentiable and totally convex on int dom $f$. Let $x\in$ int dom$f$ and $C\subset$int dom$f$ be a nonempty, closed, and convex set. If $\hat{x}\in C$, then the following conditions are equivalent:*

(i) *The vector $\hat{x}\in C$ is the Bregman projection of $x$ onto $C$ with respect to $f$.*

(ii) *The vector $\hat{x}\in C$ is the unique solution of the variational inequality*

$$\langle z - y, \nabla f(x) - \nabla f(z)\rangle \geq 0, \qquad \forall y \in C.$$

(iii) *The vector $\hat{x}$ is the unique solution of the inequality*

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x), \qquad \forall y \in C.$$

Following [2] and [17], we make use of the function $V_f{:}X\times X^*\to[0,+\infty]$ associated with $f$, which is defined by:

$$V_f(x,x^*)=f(x)-\langle x,x^*\rangle+f^*(x^*),\qquad \forall x\in X,\ x^*\in X^*.$$

Then

$$V_f(x,x^*)=D_f(x,\nabla f^*(x^*)) \tag{2.3}$$

for all $x\in X$ and $x^*\in X^*$. Moreover, by the subdifferential inequality, we have

$$V_f(x,x^*)+\langle\nabla f^*(x^*)-x,y^*\rangle\leq V_f(x,x^*+y^*) \tag{2.4}$$

for all $x\in X$ and $x^*,y^*\in X^*$ (see [28]). In addition if $f{:}X\to(-\infty,+\infty]$ is a proper lower semicontinuous function, then $f^*{:}X^*\to(-\infty,+\infty]$ is a proper weak$^*$ lower semicontinuous and convex function (see [31]). Hence $V_f$ is convex in the second variable. Thus, for all $z\in X$, we have

$$D_f\left(z,\nabla f^*\left(\sum_{i=1}^{N}t_i\nabla f(x_i)\right)\right)\leq\sum_{i=1}^{N}t_iD_f(z,x_i), \tag{2.5}$$

where $\{x_i\}_{i=1}^N\subset X$ and $\{t_i\}_{i=1}^N\subset(0,1)$ with $\sum_{i=1}^N t_i=1$.

Let $B$ and $S$ be the closed unit ball and the unit sphere of a Banach space $X$. Let $rB=\{z\in X : \|z\|\leq r\}$ for all $r>0$. Then the function $f{:}X\to\mathbb{R}$ is said to be uniformly convex on bounded subsets (see [45]) if $\rho_r(t)>0$ for all $r,t>0$, where $\rho_r{:}[0,\infty)\to[0,\infty]$ is defined by:

$$\rho_r(t)=\inf_{x,y\in rB,\|x-y\|=t,\alpha\in(0,1)}\frac{\alpha f(x)+(1-\alpha)f(y)-f(\alpha x+(1-\alpha)y)}{\alpha(1-\alpha)}$$

for all $t\geq 0$. The function $\rho_r$ is called the gauge of uniform convexity of $f$. It is known that $\rho_r$ is a nondecreasing function. If $f$ is uniformly convex, then the following lemma is known.

**Lemma 2.6.** ([30]) *Let $X$ be a Banach space, $r>0$ be a constant and $f{:}X\to\mathbb{R}$ be a uniformly convex function on bounded subsets of $X$. Then*

$$f\left(\sum_{k=0}^{n}\alpha_kx_k\right)\leq\sum_{k=0}^{n}\alpha_kf(x_k)-\alpha_i\alpha_j\rho_r(\|x_i-x_j\|)$$

*for all $i,j\in\{0,1,2,\cdots,n\}$, $x_k\in rB$, $\alpha_k\in(0,1)$ and $k=0,1,2,\cdots,n$ with $\sum_{k=0}^{n}\alpha_k=1$, where $\rho_r$ is the gauge of uniform convexity of $f$.*

The function $f$ is also said to be uniformly smooth on bounded subsets (see [45]) if $\lim_{t\downarrow 0}\dfrac{\sigma_r(t)}{t}=0$ for all $r>0$, where $\sigma_r:[0,\infty)\to[0,\infty]$ is defined by:

$$\sigma_r(t)=\sup_{x\in rB, y\in S, \alpha\in(0,1)} \frac{\alpha f(x+(1-\alpha)ty)+(1-\alpha)f(x-\alpha ty)-f(x)}{\alpha(1-\alpha)}$$

for all $t\geq 0$. A function $f$ is said to be super coercive if $\lim_{|x|\to\infty}\dfrac{f(x)}{|x|}=+\infty$.

We will use the following theorems.

**Theorem 2.7.** ([45]) *Let $f:X\to\mathbb{R}$ be a convex function which is super coercive. Then the following are equivalent:*

(i) *$f$ is bounded on bounded subsets and uniformly smooth on bounded subsets of $X$;*
(ii) *$f$ is Fréchet differentiable and $\nabla f$ is uniformly norm-to-norm continuous on bounded subsets of $X$;*
(iii) *dom $f^*=X^*$, $f^*$ is super coercive and uniformly convex on bounded subsets of $X^*$.*

**Theorem 2.8.** ([45]) *Let $f:X\to\mathbb{R}$ be a convex function which is bounded on bounded subsets of $X$. Then the following are equivalent:*

(i) *$f$ is super coercive and uniformly convex on bounded subsets of $X$;*
(ii) *dom $f^*=X^*$, $f^*$ is bounded on bounded subsets and uniformly smooth on bounded subsets of $X^*$;*
(iii) *dom $f^*=X^*$, $f^*$ is Fréchet differentiable and $\nabla f^*$ is uniformly norm-to-norm continuous on bounded subsets of $X^*$.*

**Theorem 2.9.** ([15]) *Suppose that $f:X\to(-\infty,+\infty]$ is a Legendre function. The function $f$ is totally convex on bounded subsets if and only if $f$ is uniformly convex on bounded subsets.*

**Lemma 2.10.** ([44]) *Let $C$ be a nonempty convex subset of $X$ and $f:C\to\mathbb{R}$ be a convex and subdifferentiable function on $C$. Then $f$ attains its minimum at $x\in C$ if and only if $0\in\partial f(x)+N_C(x)$, where $N_C(x)$ is the normal cone of $C$ at $x$, that is,*

$$N_C(x):=\{x^*\in X^*:\langle x-z,x^*\rangle\geq 0,\ \ \forall z\in C\}.$$

**Lemma 2.11.** ([19]) *If $f$ and $g$ are two convex functions on $X$ such that there is a point $x_0\in$dom $f\cap$dom $g$, where $f$ is continuous, then*

$$\partial(f+g)(x)=\partial f(x)+\partial g(x),\quad \forall x\in X.$$

A function $g : C\times C\to(-\infty,+\infty]$, where $C\subset X$ is a closed and convex subset, such that $g(x,x)=0$ for all $x\in C$ is called a bifunction.

**Definition 2.12.** Let $C$ be a nonempty, closed and convex subset of $X$, the bifunction $g$ is said to be

(i) monotone on $C$ if for any $x, y \in C$,

$$g(x,y) + g(y,x) \leq 0,$$

(ii) pseudo-monotone on $C$ if for any $x, y \in C$ the following implication holds:

$$g(x,y) \geq 0 \;\; \Rightarrow \;\; g(y,x) \leq 0,$$

(iii) $\gamma$-strongly pseudo-monotone if there exists a constant $\gamma\geq 0$ such that for all $x, y\in C$,

$$g(x,y) \geq 0 \;\; \Rightarrow \;\; g(y,x) \leq -\gamma\Big(D_f(y,x) + D_f(y,x)\Big).$$

(iv) Bregman-Lipschitz type continuous if there exist two positive constants $c_1, c_2$ such that

$$g(x,y) + g(y,z) \geq g(x,z) - c_1 D_f(y,x) - c_2 D_f(z,y), \quad \forall x,y,z \in C,$$

where $f{:}X\to(-\infty,+\infty]$ is a Legendre function. The constants $c_1, c_2$ are called Bregman-Lipschitz coefficients with respect to $f$.

Throughout this paper, we assume that the bifunction $g$ satisfies the following conditions:

A1: $g$ is pseudo-monotone on $C$ and $g(x,x)=0$ for all $x\in X$.
A2: $g$ is Bregman-Lipschitz-type continuous on $X$.
A3: $g(x,.)$ is convex, lower semicontinuous and subdifferentiable on $X$ for every fixed $x\in X$.
A4: $g$ is jointly weakly continuous on $X\times C$ in the sense that, if $x\in X, y\in C$ and $\{x_n\}, \{y_n\}$ converge weakly to $x, y$, respectively, then $g(x_n,y_n)\to g(x,y)$ as $n\to\infty$.
A5: $EP(g)$ is nonempty.

**Lemma 2.13.** ([21]) *Let $C$ be a nonempty, closed convex subset of a reflexive Banach space $X$, and $f{:}X\to\mathbb{R}$ be a Legendre and super coercive function. Suppose that $g : C\times C\to\mathbb{R}$ is a bifunction satisfying $A1-A4$. For the arbitrary sequences $\{x_n\}\subset C$ and $\{\lambda_n\}\subset(0,+\infty)$, let $\{w_n\}$ and $\{z_n\}$ be sequences generated by:*

$$\begin{cases} w_n = argmin\{\lambda_n g(x_n,y) + D_f(y,x_n) : y \in C\}, \\ z_n = argmin\{\lambda_n g(w_n,y) + D_f(y,x_n) : y \in C\}. \end{cases}$$

*Then for all* $x^*\in EP(g)$,

$$D_f(x^*, z_n) \leq D_f(x^*, x_n) - (1-\lambda_n c_1) D_f(w_n, x_n) - (1-\lambda_n c_2) D_f(z_n, w_n).$$

**Remark 2.14.** If $g$ satisfies $A1-A4$, then $EP(g)$ is closed and convex ([9, 12]).

Let $f{:}X\to(-\infty,+\infty]$ be a Gâteaux differentiable function, recall that the proximal mapping of a proper convex and lower semicontinuous function $g : C\to(-\infty,+\infty]$ with respect to $f$ is defined by:

$$prox_g^f(x) := argmin\{g(y) + D_f(y,x) : y \in C\}, \quad x \in X. \tag{2.6}$$

**Lemma 2.15.** ([21]) *Let* $f{:}X\to(-\infty,+\infty]$ *be a super coercive and Legendre function. Let* $x\in intdomf$, $C\subset intdomf$ *and* $g{:}C\to(-\infty,+\infty]$ *be a proper convex and lower semicontinuous function. Then the following inequality holds:*

$$g(y) - g(prox_g^f(x)) + \langle prox_g^f(x) - y, \nabla f(x) - \nabla f(prox_g^f(x))\rangle \geq 0, \quad \forall y \in C. \tag{2.7}$$

## 3. Main results

In this section, we assume that $f : X\to\mathbb{R}$ is a Legendre, super coercive and totally convex function on bounded subsets of $X$ such that $\nabla f^*$ is bounded on bounded subsets of $\mathrm{intdom} f^*$ and the bifunction $g{:}X\times X\to\mathbb{R}$ satisfies $A1-A4$. We introduce two new algorithms for solving equilibrium problems and finding the solution $x^*=\overleftarrow{proj}^f_{EP(g)}(x_0)$, where $x_0$ is a suggested starting point.

Now, we present the first algorithm and prove a convergence theorem.

**Algorithm 3.1.**

**Initialization.** Choose initial points $x_0, x_1\in X$, and two sequences $\{\lambda_n\}$ and $\{\theta_n\}$ such that

$(H1)$ $\{\lambda_n\}\subset[a,b]\subset\Big(0, \min(\frac{1}{c_1}, \frac{1}{c_2})\Big)$,

$(H2)$ $\{\theta_n\}\subset[-\theta,\theta]$ for some $\theta>0$.

Set $n=1$ and go to Step 1.

**Step 1.** Given the current iterates $x_{n-1}, x_n$ compute,

$$\begin{cases} w_n &= \nabla f^*\Big(\nabla f(x_n) + \theta_n\big(\nabla f(x_n) - \nabla f(x_{n-1})\big)\Big), \\ y_n &= prox^f_{\lambda_n g(w_n,.)}(w_n), \\ z_n &= prox^f_{\lambda_n g(y_n,.)}(w_n). \end{cases}$$

If $y_n=w_n$, then stop and $y_n$ is a solution of problem $(EP)$. Otherwise,

**Step 2.** Construct two half-spaces,

$$\begin{cases} C_n &= \{v \in X : D_f(v, z_n) \leq D_f(v, w_n)\}, \\ Q_n &= \{v \in X : \langle x_n - v, \nabla f(x_n) - \nabla f(x_0)\rangle \leqslant 0\}, \end{cases}$$

then compute

$$x_{n+1} = \overleftarrow{proj}^f_{C_n \cap Q_n}(x_0).$$

If $y_n{=}w_n$, then stop and $y_n$ is a solution. Otherwise, set $n{=}n{+}1$ and go back Step 1.

**Remark 3.2.** By Lemma 2.15 and definition of $y_n$, it is easy to show that if $y_n{=}w_n$, then $y_n$ is a solution of problem $(EP)$.

Now, we proof a strong convergence theorem for Algorithm 3.1.

**Theorem 3.3.** *Under conditions* $A1{-}A5$, *the sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *converges to a solution* $x^*{\in}EP(g)$, *where* $x^*{=}\overleftarrow{proj}^f_{EP(g)}(x_0)$.

*Proof.* We divide the proof of Theorem 3.3 into four steps.

**Step 1.** In the first step we proof the following inequality for each $n{\geq}0$ and each $y{\in}C$,

$$D_f(y, z_n) \leq D_f(y, w_n){-}(1{-}\lambda_n c_1)D_f(w_n, y_n){-}(1{-}\lambda_n c_2)D_f(y_n, z_n){+}\lambda_n g(y_n, y).$$

Note that, $y_n{=}prox^f_{\lambda_n g(w_n,\cdot)}(w_n)$. By Lemma 2.15, we obtain

$$\lambda_n\Big(g(w_n, y) - g(w_n, y_n)\Big) + \langle y_n - y, \nabla f(w_n) - \nabla f(y_n)\rangle \geq 0, \quad \forall y \in C, \tag{3.1}$$

in particular, substituting $y{=}z_n$ into the last inequality, we obtain that

$$\lambda_n\Big(g(w_n, z_n) - g(w_n, y_n)\Big) \geq \langle y_n - z_n, \nabla f(y_n) - \nabla f(w_n)\rangle \geq 0. \tag{3.2}$$

Similarly, by definition of $z_n$ and using Lemma 2.15, we get

$$\lambda_n\Big(g(y_n, y) - g(y_n, z_n)\Big) + \langle z_n - y, \nabla f(w_n) - \nabla f(z_n)\rangle \geq 0, \quad \forall y \in C,$$

thus

$$\lambda_n g(y_n, y) + \langle z_n - y, \nabla f(w_n) - \nabla f(z_n)\rangle \geq \lambda_n g(y_n, z_n), \tag{3.3}$$

the last relations and the Lipschitz-type condition of $f$, imply that

$$\begin{aligned}\lambda_n g(y_n, y){+}\langle z_n - y, \nabla f(w_n){-}\nabla f(z_n)\rangle{\geq}\lambda_n\Big(&g(w_n, z_n){-}g(w_n, y_n)\\ &-c_1 D_f(y_n, w_n){-}c_2 D_f(z_n, y_n)\Big).\end{aligned} \tag{3.4}$$

Combining (3.4) with relation (3.2), we obtain

$$\lambda_n g(y_n,y)+\langle z_n-y,\nabla f(w_n)-\nabla f(z_n)\rangle \geq\langle z_n-y_n,\nabla f(w_n)-\nabla f(y_n)\rangle \\ -\lambda_n c_1 D_f(y_n,w_n)-\lambda_n c_2 D_f(z_n,y_n). \tag{3.5}$$

By the three point identity, we obtain that

$$D_f(y,z_n)+D_f(z_n,w_n)-D_f(y,w_n)=\langle y-z_n,\nabla f(w_n)-\nabla f(z_n)\rangle, \tag{3.6}$$

and similarly, we have

$$D_f(z_n,y_n)+D_f(y_n,w_n)-D_f(z_n,w_n)=\langle z_n-y_n,\nabla f(w_n)-\nabla f(y_n)\rangle. \tag{3.7}$$

Using inequality (3.5) by taking into account equalities (3.6) and (3.7), we obtain the desired conclusion.

**Step 2.** We show that $EP(g)\subset C_n\cap Q_n$ for all $n\geq 0$.

Assume that $x^*\in EP(g)$ and substituting $y=x^*$ in Step 1, we have

$$D_f(x^*,z_n)\leq D_f(x^*,w_n)-(1-\lambda_n c_1)D_f(w_n,y_n)-(1-\lambda_n c_2)D_f(y_n,z_n)+\lambda_n g(y_n,x^*). \tag{3.8}$$

As $x^*\in EP(g)$ and $y_n\in C$, we obtain that $g(x^*,y_n)\geq 0$. By the pseudo-monotonicity of $g$, we have $g(y_n,x^*)\leq 0$. Hence, from (3.8) and hypothesis $(H1)$, we get

$$D_f(x^*,z_n)\leq D_f(x^*,w_n), \qquad \forall n\geq 0, \quad x^*\in EP(g). \tag{3.9}$$

By definition of $C_n$, we have $EP(g)\subset C_n$ for all $n\geq 0$. Next we show that $EP(g)\subset C_n\cap Q_n$ for all $n\geq 0$ by induction. It is obvious that, $EP(g)\subset Q_0=X$ and $EP(g)\subset C_0\cap Q_0$. Assume that $EP(g)\subset C_n\cap Q_n$ for some $n\geq 0$. As $EP(g)\neq\emptyset$, we have $C_n\cap Q_n\neq\emptyset$ and thus $x_{n+1}$ is well defined. By the definition of $x_{n+1}$ and Lemma 2.5 $(ii)$, we have

$$\langle x_{n+1}-v,\nabla f(x_0)-\nabla f(x_{n+1})\rangle\geq 0, \quad \forall v\in C_n\cap Q_n.$$

Since $EP(g)\subset C_n\cap Q_n$, we obtain

$$\langle x_{n+1}-v,\nabla f(x_0)-\nabla f(x_{n+1})\rangle\geq 0, \quad \forall v\in EP(g).$$

This together with definition of $x_{n+1}$ implies that $EP(g)\subset Q_{n+1}$. Therefor, $EP(g)\subset C_{n+1}\cap Q_{n+1}$ and the proof of Step 2 is complete.

**Step 3.** We show that $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} D_f(x_n,x_{n+1})=\lim_{n\to\infty} D_f(x_n,w_n)=\lim_{n\to\infty} D_f(w_n,y_n)=\lim_{n\to\infty} D_f(y_n,z_n)=0.$$

In view of $x_{n+1}=\overleftarrow{proj}^f_{C_n\cap Q_n}(x_0)$ and by the definition of Bregman projection, we get

$$D_f(x_{n+1},x_0)\leq D_f(v,x_0), \quad \forall v\in C_n\cap Q_n,$$

from Step 2 and $x^*\in EP(g)$, we have

$$D_f(x_{n+1}, x_0) \leq D_f(x^*, x_0).$$

This implies that the sequence $\{D_f(x_{n+1}, x_0)\}$ is bounded. Therefore, by Lemma 2.3 the sequence $\{x_n\}$ is bounded.

By definition of $Q_n$, we have

$$\langle v - x_n, \nabla f(x_0) - \nabla f(x_n)\rangle \geq 0, \qquad \forall v \in Q_n,$$

from Lemma 2.5, we get

$$x_n = \overleftarrow{proj}^f_{Q_n}(x_0),$$

in addition

$$D_f(v, x_n) + D_f(x_n, x_0) \leq D_f(v, x_0), \qquad \forall v \in Q_n.$$

By the definition of $x_{n+1}$ and substituting $v{=}x_{n+1}$ into the above inequality, we obtain

$$D_f(x_{n+1}, x_n) + D_f(x_n, x_0) \leq D_f(x_{n+1}, x_0), \tag{3.10}$$

since $D_f(x_{n+1}, x_n){\geq}0$, so we have

$$D_f(x_n, x_0) \leq D_f(x_{n+1}, x_0),$$

which implies that, the sequence $\{D_f(x_n, x_0)\}$ is nondecreasing. Thus, the limit of $D_f(x_n, x_0)$ exists. Passing to the limsup in inequality (3.10) as $n{\to}\infty$, we obtain

$$\limsup_{n\to\infty} D_f(x_{n+1}, x_n) = 0.$$

Therefore,

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0. \tag{3.11}$$

Since $\{x_n\}$ is bounded and $f$ is sequentially consistent, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

From definition of $w_n$, we get

$$\nabla f(w_n){-}\nabla f(x_n){=}\theta_n(\nabla f(x_n){-}\nabla f(x_{n-1}).$$

Passing to the limit in this relation and using hypothesis $(H2)$, we get

$$\lim_{n\to\infty} \|\nabla f(w_n){-}\nabla f(x_n)\|{=}0.$$

Since $\nabla f^*$ is uniformly continuous on bounded subset of $X^*$, this implies that $\lim_{n\to\infty} \|w_n - x_n\|{=}0$. Hence, $\{w_n\}$ is bounded . We have the following fact

$$\|w_n - x_{n+1}\| \leq \|w_n - x_n\| + \|x_n - x_{n+1}\|.$$

Passing to the limit in above inequality as $n{\to}\infty$, we obtain

$$\lim_{n\to\infty} \|w_n - x_{n+1}\| = 0. \tag{3.12}$$

By the three point identity, we obtain that

$D_f(x_n, w_n) + D_f(w_n, x_{n+1}) - D_f(x_n, x_{n+1}) = \langle x_n - w_n, \nabla f(x_{n+1}) - \nabla f(w_n)\rangle$,

passing to the limit in above inequality as $n \to \infty$, we obtain

$$\lim_{n\to\infty} D_f(x_n, w_n) = \lim_{n\to\infty} D_f(w_n, x_{n+1}) = 0.$$

Since $x_{n+1} \in C_n$, it follows from the definition of $C_n$ that

$$D_f(x_{n+1}, z_n) \leq D_f(x_{n+1}, w_n).$$

Thus, $\lim_{n\to\infty} D_f(x_{n+1}, z_n) = 0$ and we have

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{3.13}$$

From relations (3.12) and (3.13), we obtain that $\lim_{n\to\infty} \|z_n - w_n\| = 0$. This implies that

$$\lim_{n\to\infty} D_f(w_n, z_n) = 0. \tag{3.14}$$

This together with the three point identity, relation (3.8) and $(H1)$ imply that

$$\lim_{n\to\infty} D_f(y_n, w_n) = 0.$$

In a similar way, we obtain

$$\lim_{n\to\infty} D_f(y_n, z_n) = 0.$$

**Step 4.** Let $\omega_w(x_n)$ be the set of weak cluster points of the sequence $\{x_n\}$. First we show that $\omega_w(x_n) \subset EP(g)$.

In the following we prove that

$$D_f(x_n, x_0) \leq D_f\left(\overleftarrow{proj}^f_{EP(g)}(x_0), x_0\right)$$

for all $n > 0$ and also $x_n \to \overleftarrow{proj}^f_{EP(g)}(x_0)$ as $n \to \infty$.

We assume that $p \in \omega_w(x_n)$. Thus, there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $x_{n_m} \to p$. Using Step 3, we have

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0.$$

Therefore,

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Substituting $n = n_m$ into the last relation, we obtain that $y_{n_m} \rightharpoonup p$.

We know that the feasible set $C$ is closed and convex in $X$. Thus $C$ is weakly closed. Hence, from $\{y_{n_m}\} \subset C$, we derive that $p \in C$. From Step 1, we have

$$g(y_n, y) \geq \frac{D_f(y, z_n) - D_f(y, w_n)}{\lambda_n} + \frac{1 - \lambda_n c_1}{\lambda_n} D_f(w_n, y_n) + \frac{1 - \lambda_n c_2}{\lambda_n} D_f(y_n, z_n)$$

for each $y \in C$ and each $n \geq 0$. In particular, substituting $n = n_m$, passing to the limsup in above inequality as $m \to \infty$ and using the three point identity, we obtain

$$\limsup_{m\to\infty} g(y_{n_m}, y) \geq 0.$$

From $y_{n_m} \rightharpoonup p$ and $A4$, we get

$$g(p, y) \geq \limsup_{m\to\infty} g(y_{n_m}, y) \geq 0, \quad \forall y \in C,$$

that is implying

$$p \in EP(g).$$

On the other hand, by definition of $Q_n$, we know that $x_n \in Q_n$ and by the definition of Bregman projection, we have

$$D_f(x_n, x_0) \leq D_f(v, x_0), \quad \forall v \in Q_n.$$

Substituting $v = \overleftarrow{proj}^f_{EP(g)}(x_0)$ in the last equality, we get

$$D_f(x_n, x_0) \leq D_f(\overleftarrow{proj}^f_{EP(g)}(x_0), x_0). \tag{3.15}$$

We denote $\overline{u} := \overleftarrow{proj}^f_{EP(g)}(x_0)$. Using the there point identity, (3.15) and the definition of Bregman distance, we have

$$\begin{aligned} D_f(x_n, x_0) &= D_n(x_n, x_0) + D_f(x_0, \overline{u}) - \langle \nabla f(\overline{u}) - \nabla f(x_0), x_n - x_0 \rangle \\ &\leq D_n(\overline{u}, x_0) + D_f(x_0, \overline{u}) - \langle \nabla f(\overline{u}) - \nabla f(x_0), x_n - x_0 \rangle \\ &= \langle \nabla f(\overline{u}) - \nabla f(x_0), \overline{u} - x_0 \rangle + \langle \nabla f(\overline{u}) - \nabla f(x_0), x_n - x_0 \rangle \\ &= \langle \nabla f(\overline{u}) - \nabla f(x_0), \overline{u} - x_n \rangle. \end{aligned}$$

Substituting $n = n_m$, passing to the limsup in above inequality as $m \to \infty$, we have

$$\limsup_{m\to\infty} D_f(x_{n_m}, \overline{u}) \leq \limsup_{m\to\infty} \langle \nabla f(\overline{u}) - \nabla f(x_0), \overline{u} - x_{n_m} \rangle,$$

we note that $x_{n_m} \rightharpoonup p$ and using Lemma 2.5(ii), we obtain

$$\limsup_{m\to\infty} D_f(x_{n_m}, \overline{u}) \leq \langle \nabla f(\overline{u}) - \nabla f(x_0), \overline{u} - p \rangle \leq 0.$$

Therefore,

$$\lim_{m\to\infty} D_f(x_{n_m}, \overline{u}) = 0.$$

This implies that $\|x_{n_m} - \overline{u}\| \to 0$, so the whole sequence $\{x_n\}$ converges strongly to $\overline{u}$. Thus

$$x_n \to \overleftarrow{proj}^f_{EP(g)}(x_0).$$

This completes the proof. □

Next, we introduce the second algorithm for solving equilibrium problems. Unlike in Algorithm 3.1, we use here the shrinking projection method to design the algorithm.

**Algorithm 3.4.**

**Initialization.** Choose initial points $x_0, x_1 \in X$, and two sequences $\{\lambda_n\}$ and $\{\theta_n\}$ such that conditions $(H1)$ and $(H2)$ above hold. Assume that $C_0 = X$. Set $n=1$ and go to Step 1.

**Step 1.** Given the current iterates $x_{n-1}, x_n$ compute,

$$\begin{cases} w_n &= \nabla f^*\Big(\nabla f(x_n) + \theta_n \nabla f(x_n - x_{n-1})\Big), \\ y_n &= prox^f_{\lambda_n g(w_n,.)}(w_n), \\ z_n &= prox^f_{\lambda_n g(y_n,.)}(w_n). \end{cases}$$

If $y_n = w_n$, then stop and $y_n$ is a solution of equilibrium problem. Otherwise,

**Step 2.** Construct the half-space

$$H_n = \{v \in X : D_f(v, z_n) \leq D_f(v, w_n)\}.$$

Set $C_{n+1} = C_n \cap H_n$. Then compute

$$x_{n+1} = \overleftarrow{proj}^f_{C_{n+1}}(x_0).$$

Set $n = n+1$ and return to Step 1.

It follows from the definition of $C_n$ that, for each $n \geq 0$, $C_n$ is the intersection of finitely many half-spaces. In fact, we can write $C_n = \bigcap_{i=0}^{n} H_i$, where $C_0 = X$ and $H_i$ is defined at Step 2. Moreover,

$$X = C_0 \supset C_1 \supset C_2 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots .$$

This is the reason why Algorithm 3.4 is called the shrinking projection algorithm. In order to establish the convergence of Algorithm 3.4, we weaken hypothesis $A4$ and introduce the following weaker one.

$A4_a$. $\limsup_{n\to\infty} g(x_n, y) \leq g(x, y)$ for each sequence $\{x_n\} \subset C$ converging strongly to $x$.

The reason for considering assumption $A4_a$, which is weaker than $A4$, comes from the shrinking property of the set-sequence $\{C_n\}$. The convergence of Algorithm 3.4 is ensured by the following theorem.

**Theorem 3.5.** *Assume that hypotheses* $A1-A3, A4_a$ *and* $A5$ *hold. Then the sequence* $\{x_n\}$ *generated by Algorithm* 3.4 *converges to a solution of the equilibrium problem, that is,* $x_n \to x^* \in EP(g)$*, where* $x^* = \overleftarrow{proj}^f_{EP(g)}(x_0)$.

*Proof.* In Step 1 of the proof of Theorem 3.3, we obtain that

$$\begin{aligned} D_f(y, z_n) \leq & D_f(y, w_n) - (1-\lambda_n c_1) D_f(w_n, y_n) \\ & - (1-\lambda_n c_2) D_f(y_n, z_n) + \lambda_n g(y_n, y) \end{aligned} \tag{3.16}$$

for each $n \geq 0$ and each $y \in C$. We assume $x^* \in EP(g)$ and substitution $y = x^*$ in the above inequality, we get

$$\begin{aligned} D_f(x^*, z_n) \leq & D_f(x^*, w_n) - (1-\lambda_n c_1) D_f(w_n, y_n) \\ & - (1-\lambda_n c_2) D_f(y_n, z_n) + \lambda_n g(y_n, x^*). \end{aligned}$$

Note that $g(y_n, x^*) > 0$, so

$$D_f(x^*, z_n) \leq D_f(x^*, w_n) - (1-\lambda_n c_1) D_f(w_n, y_n) - (1-\lambda_n c_2) D_f(y_n, z_n). \tag{3.17}$$

From hypothesis $(H1)$, we obtain

$$D_f(x^*, z_n) \leq D_f(x^*, w_n), \quad \forall x^* \in EP(g).$$

By definition of $H_n$, we have $EP(g) \in H_n$ for all $n \geq 0$. Therefore, since $C_0 = X$, we obtain by induction that $EP(g) \subset C_n$ for all $n \geq 0$.

Note that, $X = C_0 \supset C_1 \supset \cdots$, $C_n = \bigcap_{i=0}^{n} H_i$ and $EP(g) \subset H_n$, therefore, $EP(g) \subset C_n$.

By $x_n = \overleftarrow{proj}^f_{C_n}(x_0)$ and Lemma 2.5 $(ii)$, we find

$$D_f(v, x_n) + D_f(x_n, x_0) \leq D_f(v, x_0), \quad \forall v \in C_n, \tag{3.18}$$

substituting $v = x^* \in EP(g)$ in above inequality, we get

$$D_f(x^*, x_n) \leq D_f(x^*, x_0), \quad \forall x^* \in EP(g), \tag{3.19}$$

this implies that the sequences $\{D_f(x_n, x^*)\}$ and $\{x_n\}$ are bounded.

In a similar way, using Step 3 of Theorem 3.3, the sequences $\{w_n\}$, $\{y_n\}$, and $\{z_n\}$ are also bounded. Since $x_{n+1} \in C_{n+1} \subset C_n$, substitution $v = x_{n+1}$ in (3.18), we get

$$D_f(x_{n+1}, x_n) + D_f(x_n, x_0) \leq D_f(x_{n+1}, x_0). \tag{3.20}$$

Then

$$D_f(x_n, x_0) \leq D_f(x_{n+1}, x_0),$$

this implies that the sequence $\{D_f(x_n, x_0)\}$ is nondecreasing. Hence,

$$\lim_{n \to \infty} D_f(x_n, x_0)$$

exists. Now, passing to the limit in relation (3.20) as $n\to\infty$, we obtain

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0, \tag{3.21}$$

then

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

this implies that, the limit of $\{x_n\}$ exists.

Similarly to the proof of Theorem 3.3, we obtain

$$\begin{aligned}\lim_{n\to\infty} D_f(x_n, w_n) &= \lim_{n\to\infty} D_f(z_n, w_n)\\ &= \lim_{n\to\infty} D_f(w_n, y_n)\\ &= \lim_{n\to\infty} D_f(y_n, z_n)\\ &= 0.\end{aligned} \tag{3.22}$$

Hence,

$$\begin{aligned}\lim_{n\to\infty} \|x_n - w_n\| &= \lim_{n\to\infty} \|z_n - w_n\|\\ &= \lim_{n\to\infty} \|w_n - y_n\|\\ &= \lim_{n\to\infty} \|y_n - z_n\|\\ &= 0.\end{aligned}$$

Let $n\geq 0$ and $k\geq 1$. Note that

$$x_{n+k}=\overleftarrow{proj}^{f}_{C_{n+k}}(x_0)\in C_{n+k}\subset C_n.$$

Thus, from the relation (3.18) and subsituting $v=x_{n+k}$, we get

$$D_f(x_{n+k}, x_n) + D_f(x_n, x_0) \leq D_f(x_{n+k}, x_0), \quad \forall k \geq 1, \forall n \geq 0. \tag{3.23}$$

Passing to the limit in (3.23) as $n, k\to\infty$ and noting that the limit of $D_f(x_n, x_0)$ exists, we obtain

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0. \tag{3.24}$$

Therefore, the sequence $\{x_n\}$ is a Cauchy sequence, and there exists $p\in X$ such that

$$\lim_{n\to\infty} x_n = p, \tag{3.25}$$

from (3.22) and (3.25), we obtain

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = p. \tag{3.26}$$

Note that $\{y_n\}\subset C$ and $C$ is a closed convex set in $X$. Therefore, we obtain $p\in C$. From relation (3.16), we get

$$g(y_n, y) \geq \frac{D_f(y, z_n) - D_f(y, w_n)}{\lambda_n} + \frac{1-\lambda_n c_1}{\lambda_n} D_f(w_n, y_n) + \frac{1-\lambda_n c_2}{\lambda_n} D_f(y_n, z_n)$$

for each $y \in C$ and each $n \geq 0$. Passing to the limit in the last inequality as $n \to \infty$ and using hypotheses $(H1)$, $A4_a$ and relation (3.22), we obtain

$$g(p, y) \geq \limsup_{n\to\infty} g(y_n, y) \geq 0, \quad \forall y \in C.$$

Thus, $p \in EP(g)$. The proof of $x_n \to \overleftarrow{proj}^f_{EP(g)}(x_0)$ is similar to the one of Theorem 3.3 and the proof of Theorem 3.4 is complete. □

## 4. Application

In this section, we consider the particular equilibrium problem corresponding to the function $g$ defined for every $x, y \in X$ by $g(x, y) = \langle y - x, Ax \rangle$ with $A: X \to X^*$ being $L$-Lipschitz continuous, that is, there exists $L > 0$ such that

$$\|Ax - Ay\| \leqslant L\|x - y\|, \quad \forall x, y \in X.$$

Doing so, we obtain the classical variational inequality:

$$\text{Find } z \in C \text{ such that } \langle y - z, Az \rangle \geq 0, \quad \forall y \in C. \tag{4.1}$$

The set of solutions of this problem is denoted by $VI(A)$. We have [21, Lemma 4.1]

$$\begin{aligned} &argmin\{\lambda_n g(x_n, y) + D_f(y, x_n) : y \in C\} \\ &\quad = argmin\{\lambda_n \langle y - y_n, Ax_n \rangle + D_f(y, x_n) : y \in C\} \\ &\quad = \overleftarrow{proj}^f_C \Big(\nabla f^*(\nabla f(x_n) - \lambda_n Ax_n)\Big). \end{aligned}$$

Therefore, we derive that

$$argmin\{\lambda_n \langle y - y_n, Ay_n \rangle + D_f(y, x_n) : y \in T_n\} = \overleftarrow{proj}^f_{T_n}(\nabla f^*(\nabla f(x_n) - \lambda_n Ay_n).$$

Let $X$ be a real Banach space. The modulus of convexity $\delta_X: [0, 2] \to [0, 1]$ is defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\right\}.$$

The space $X$ is called:

(1) uniformly convex if $\delta_X(\varepsilon) > 0$ for every $\varepsilon \in (0, 2]$,
(2) $p$-uniformly convex if $p \geq 2$ and there exists $c_p > 0$ such that $\delta_X(\varepsilon) \geq c_p \varepsilon^p$ for any $\varepsilon \in (0, 2]$,
(3) The modulus of smoothness $\rho_X(t): [0, \infty) \to [0, \infty)$ is defined by

$$\rho_X(t) = \sup\left\{\frac{\|x + ty\| + \|x - ty\|}{2} - 1 : \|x\| = \|y\| = 1\right\},$$

(4) The space $X$ is called uniformly smooth if

$$\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0.$$

For a $p$-uniformly convex space, the metric and Bregman distance have the following relation [40]:

$$\tau\|x-y\|^p \leq D_{\frac{1}{p}\|\cdot\|^p}(x,y) \leq \langle x-y, J_X^p(x) - J_X^p(x)\rangle, \tag{4.2}$$

where $\tau>0$ is a fixed number and the duality mapping $J_X^p(x):X\to 2^{X^*}$ is defined by

$$J_X^p(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^p,\ \|f\| = \|x\|^{p-1}\}$$

for every $x\in X$.

We know that $X$ is smooth if and only if $J_X^p$ is a single-valued mapping of $X$ into $X^*$. We also know that $X$ is reflexive if and only if $J_X^p$ is surjective, and $X$ is strictly convex if and only if $J_X^p$ is one-to-one. Therefore, if $X$ is a smooth, strictly convex and reflexive Banach space, then $J_X^p$ is a single-valued bijection and in this case, $J_X^p=(J_{X^*}^q)^{-1}$ where $J_{X^*}^q$ is the duality mapping of $X^*$.

For $p$=2, the duality mapping $J_X^p$, is called the normalized duality mapping and is denoted by $J$. The function $\phi:X^2\to\mathbb{R}$ is defined by:

$$\phi(y,x) = \|\ y\ \|^2 - 2\langle y, Jx\rangle + \|\ x\ \|^2$$

for all $x, y\in X$. The generalized projection $\Pi_C$ from $X$ onto $C$ is defined by:

$$\Pi_C(x) = argmin_{y\in C}\phi(y,x), \quad \forall x \in X,$$

where $C$ is a nonempty closed and convex subset of $X$.

Let $X$ be a uniformly smooth and uniformly convex Banach space, and $f = \frac{1}{2}\|\cdot\|^2$. Therefore

$$\nabla f = J,\ D_{\frac{1}{2}\|\cdot\|^2}(x,y) = \frac{1}{2}\phi(x,y) \ \text{ and } \ \overleftarrow{proj}_C^{\frac{1}{2}\|\cdot\|^2} = \Pi_C.$$

In particular, if $X$ is a Hilbert space, then

$$\nabla f = I,\ D_{\frac{1}{2}\|\cdot\|^2}(x,y) = \frac{1}{2}\|x-y\|^2 \ \text{ and } \ \overleftarrow{proj}_C^{\frac{1}{2}\|\cdot\|^2} = P_C,$$

where $P_C$ is the metric projection. In this situation, we give the following algorithms for solving variational inequalities.

**Algorithm 4.1.**

**Initialization.** Choose initial points $x_0, x_1\in X$, and two sequences $\{\lambda_n\}$ and $\{\theta_n\}$ such that $(H1)$ $\{\lambda_n\}\subset[a,b]\subset\Big(0, \min(\dfrac{1}{c_1}, \dfrac{1}{c_2})\Big)$, $(H2)$ $\{\theta_n\}\subset[-\theta,\theta]$ for some $\theta>0$. Set $n$=1 and go to Step 1.

**Step 1.** Given the current iterates $x_{n-1}, x_n$ compute,

$$\begin{cases} w_n &= J^{-1}\Big(J(x_n) + \theta_n J(x_n - x_{n-1})\Big), \\ y_n &= \Pi_C\Big(J^{-1}(J(w_n) - \lambda_n A(w_n))\Big), \\ z_n &= \Pi_C\Big(J^{-1}(J(w_n) - \lambda_n A(y_n))\Big). \end{cases}$$

If $y_n{=}w_n$, then stop and $y_n$ is a solution of variational inequality. Otherwise,

**Step 2.** Construct two half-spaces

$$\begin{cases} C_n &= \{v \in X : \|v - z_n\|_p \leq \|v - w_n\|_p\}, \\ Q_n &= \{v \in X : \langle x_n - v, J(x_n) - J(x_0)\rangle \leqslant 0\}, \end{cases}$$

then compute

$$x_{n+1} = \Pi_{C_n \cap Q_n}(x_0).$$

If $y_n{=}w_n$, then stop and $y_n$ is a solution. Otherwise, set $n{=}n{+}1$ and go back Step 1. Now, we have a strong convergence corollary for Algorithm 4.1.

**Corollary 4.2.** *Under conditions* $A1{-}A5$, *the sequence* $\{x_n\}$ *generated by Algorithm* 4.1 *converges to a solution* $x^*{\in}VI(A)$, *where* $x^*{=}\Pi_{VI(A)}(x_0)$.

Finally, we present the following algorithm and strong convergence corollary.

**Algorithm 4.3.**

**Initialization.** Choose initial points $x_0, x_1{\in}X$, and two sequences $\{\lambda_n\}$ and $\{\theta_n\}$ such that conditions $(H1)$ and $(H2)$ above hold. Assume that $C_0{=}X$. Set $n{=}1$ and go to Step 1.

**Step 1.** Given the current iterates $x_{n-1}, x_n$ compute

$$\begin{cases} w_n &= J^{-1}\Big(J(x_n) + \theta_n J(x_n - x_{n-1})\Big), \\ y_n &= \Pi_C\Big(J^{-1}(J(w_n) - \lambda_n A(w_n))\Big) \\ z_n &= \Pi_C\Big(J^{-1}(J(w_n) - \lambda_n A(y_n))\Big). \end{cases}$$

If $y_n{=}w_n$, then stop and $y_n$ is a solution of problem $(VI)$. Otherwise,

**Step 2.** Construct the half-space

$$H_n = \{v \in X : \|v - z_n\|_p \leq \|v - w_n\|_p\}.$$

Set $C_{n+1} = C_n \cap H_n$. Then compute $x_{n+1} = \Pi_{C_{n+1}}(x_0)$. Set $n{=:}n{+}1$ and return to Step 1.

**Corollary 4.4.** *Assume that hypotheses* $A1 - A3, A4_a$ *and* $A5$ *hold. Then the sequence* $\{x_n\}$ *generated by Algorithm* 4.3 *converges to a solution of the variational inequality, that is,* $x_n \to x^*{\in}VI(A)$ *where* $x^*{=}\overleftarrow{proj}^{f}_{VI(A)}(x_0)$.

## 5. Numerical experiment

In this section, numerical example is given in a non-Hilbertian space to show that our results are efficient. We will use the following theorem and lemma.

**Theorem 5.1.** ([41]) *Let* $1<p<\infty$. *The normalized duality mapping on* $L^p$ *has the following form*

$$Jf = \frac{\mid f \mid^{p-1}\, sign(f)}{\|f\|_p^{p-2}}.$$

**Lemma 5.2.** ([32]) *Let* $r\geq 0$ *and* $C=\{x\in X{:}\|x\| \leq r\}$. *Then*

$$\Pi_C(x) = \frac{r}{max\{\|x\|, r\}}x, \quad \forall x \in X.$$

**Example 5.3.** Let $p=\frac{4}{3}$, $X=L^p[0,1]$ and $C=\{x\in X{:}\|x\|\leq 1\}$. Define $A : C\to X^*$ by $Ax:=\Big(\frac{4}{3}-\|x\|\Big)Jx$. It follows from [32, Example 5.1] that $A$ is pseudo-monotone on $C$ and $\tau = \frac{p-1}{2} = \frac{1}{6}$. Furthermore, for all $x, y \in C$, we have

$$\begin{aligned}
\|Ax - Ay\| &= \Big\|(\frac{4}{3} - \|x\|)Jx - (\frac{4}{3} - \|y\|)Jy\Big\| \\
&= \Big\|(\frac{4}{3} - \|x\|)(Jx - Jy) + Jy(\|y\| - \|x\|)\Big\| \\
&\leq (\frac{4}{3} - \|x\|)\|Jx - Jy\| + \|Jy\|\Big|\|y\| - \|x\|\Big| \\
&= (\frac{4}{3} - \|x\|)\|Jx - Jy\| + \|Jy - J0\|\Big|\|y\| - \|x\|\Big| \\
&\leq \frac{4}{3\tau}\|x - y\| + \frac{1}{\tau}\|x - y\| \\
&= 14\|x - y\|.
\end{aligned}$$

This implies that $A$ is 14-Lipschitz continuous. It follows from ([21] Corollary 4.2] that $g(x, y) = \langle y - x, Ax\rangle$ is Bregman-Lipschitz-type continuous and $c_1 = c_2 = \frac{L}{2\tau} = 42$. Note that $VI(A) = \{0\}$ and $A$ is bounded on $C$ and hence all conditions of Corollaries 4.2 and 4.4 are satisfied. Using Theorem 5.1, Lemma 5.2 and Algorithms 4.1, 4.3 with the initial points $x_0 = \exp(t)$, $x_1 = 1$ and $\lambda_n = 0.02$. As seen, $\theta_n$ can be taken arbitrarily small or large value, even be negative. Therefore, by the experiment, we use these algorithms with inertial parameters of $\theta_n = 0.5$. We have the numerical results of Algorithms 4.1 and 4.3 in Figures 1 and 2.

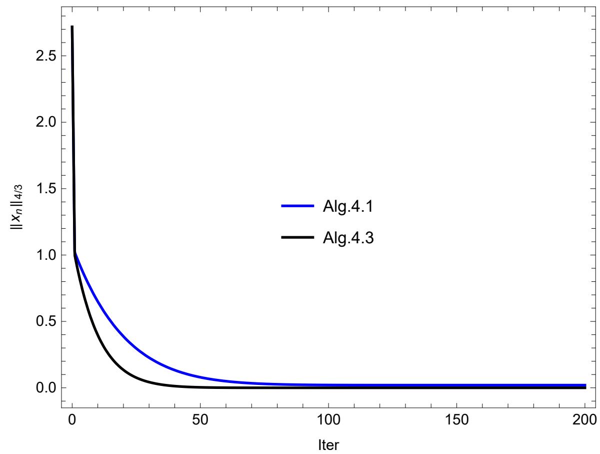


FIGURE 1. Plotting of $\|x_n\|_p$ in Example 5.3

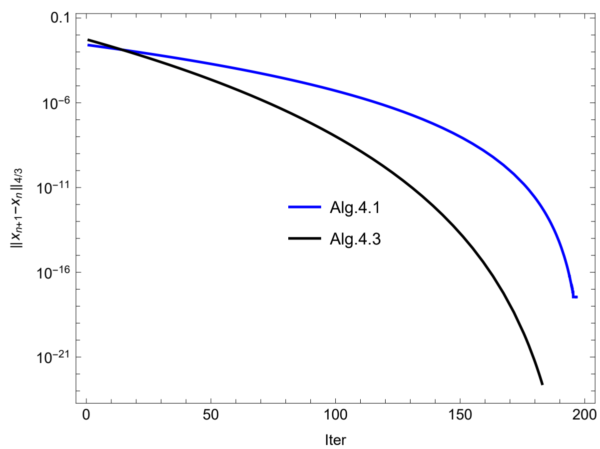


FIGURE 2. Plotting of $\|x_n - x_{n-1}\|_p$ in Example 5.3

**Acknowledgments:** The authors are grateful to the referees for their valuable comments and suggestions leading to the improvement of the paper.

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