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COMMON FIXED POINT RESULTS IN COMPLETE NMS BY UTILIZING GERAGHTY FUNCTIONS

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Abstract. Fixed point theory is an essential tool in both applied and theoretical mathematics, owing to its extensive applications. This research presents common fixed point theorems pertaining to Geraghty neutrosophic contractions within the context of neutrosophic metric spaces. Additionally, we derive several results concerning fixed points that are grounded in our principal theorem.

1. Introduction

The notion of Fuzzy Sets (FSs), first introduced by Zadeh [43], has significantly influenced a wide range of scientific fields since its emergence. While

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this framework is highly pertinent to practical applications, it has not consistently offered satisfactory solutions to various problems over the years. As a result, there has been a renewed focus on research aimed at resolving these difficulties. In this context, Atanassov [10] proposed Intuitionistic Fuzzy Sets (IFSs) to tackle such challenges. Furthermore, the Neutrosophic Set (NS), created by Smarandache [41], serves as a sophisticated extension of traditional set theory. Fuzzy sets and its generalizations were extensively studied by many authors see for example [5, 6, 18, 19, 44].

Neutrosophic sets demonstrate a wide array of applications across various fields. For example, Ashika *et al.* [9] developed an enhanced neutrosophic set approach combined with machine learning for improved breast cancer prediction. For a more in-depth investigation into the applications of neutrosophic sets and their extensive uses, it is advisable to refer to the literature cited in [4, 8, 22] and the related references therein.

The Banach fixed-point theorem [12], commonly known as the Banach contraction principle, represents a pivotal theorem in mathematics, especially within the realm of metric spaces. This theorem guarantees both the existence and uniqueness of fixed points for certain self-maps in metric spaces, thereby offering a structured approach to locating these fixed points. Essentially, the Banach fixed-point theorem provides a comprehensive foundation for Picard's method of successive approximations.

Recent work has extended fixed point theory to generalized metric spaces. Malkawi et al. [27, 28, 29] developed fundamental results in MR-metric spaces, while [21, 30, 31, 32] expanded to M*-metric and Mb-metric spaces. Related contributions include fractional calculus [7] and symmetry analysis [3].

A multitude of mathematicians has explored various extensions and generalizations across diverse mathematical fields of Banach's result, as indicated by the references in [1, 2, 11, 13, 14, 15, 16, 17, 23, 24, 25, 34, 35, 36, 37, 38, 39, 42].

A notable instance of this is the concept of neutrosophic metric space (NMS), first introduced by Kirisci and Simsek [26]. This framework has been employed to examine a variety of fixed point theorems. Particularly, fixed-point theory plays a significant role in statistics, particularly through the application of iterative algorithms and methodologies that focus on identifying fixed points to address statistical challenges.

2. Preliminary

Triangular norms (abbreviated as TN), first proposed by Menger [33], represent a crucial concept in the realm of mathematical analysis. Menger's pioneering methodology utilized probability distributions to evaluate the distance between two elements in a defined space, thereby transcending the conventional dependence on numerical values. This approach enables the generalization of the triangle inequality within metric spaces via the implementation of triangular norms.

Conversely, triangular conorms (abbreviated as CN) act as the dual counterparts to t-norms. Both TN and CN play vital roles in fuzzy operations, particularly in relation to intersections and unions. Through this manuscript, $\mathbb{R}^+ = (0, +\infty),\ I = [0, 1]$.

Definition 2.1. ([33]) Consider an operation $\otimes : I \times I \to I$. This operation is classified as continuous t-norm (CTN) if it meets the following criteria: for any elements $\sigma, \sigma', t, t' \in I$,

(1) $\sigma \otimes 1 = \sigma$,
(2) if $\sigma \leq \sigma'$ and $t \leq t'$, then $\sigma \otimes t \leq \sigma' \otimes t'$,
(3) $\otimes$ is continuous,
(4) $\otimes$ is commutative and associate.

Definition 2.2. ([33]) Consider an operation $\oplus : I \times I \to I$. This operation is classified as continuous t-conorm (CTC) if it meets the following criteria: for all elements $\sigma, \sigma', t, t' \in I$,

(1) $\sigma \oplus 0 = \sigma$,
(2) if $\sigma \leq \sigma'$ and $t \leq t'$, then $\sigma \oplus t \leq \sigma' \oplus t'$,
(3) $\oplus$ is continuous,
(4) $\oplus$ is commutative and associate.

Definition 2.3. ([26]) Let F represent an arbitrary set, and define

$$N = \{< \xi, \Pi_U(\xi), \mho_U(\xi), \Theta_U(\xi) >: \xi \in F\}$$

as a neutrosophic structure such that $N : F \times F \times \mathbb{R}^+ \to I$. The symbols $\otimes$ and $\oplus$ denote the operations of (CTN) and (CTC), respectively. The quadruple $V = (F, N, \otimes, \oplus)$ is referred to as a neutrosophic metric space (NMS) when the following conditions hold for all $\xi, \omega, c, \in F$.

(1) $0 \leq \Pi(\xi, \omega, \lambda) \leq 1,\ 0 \leq \mho(\xi, \omega, \lambda) \leq 1,\ 0 \leq \Theta(\xi, \omega, \lambda) \leq 1,\ \forall \lambda \in \mathbb{R}^+$,
(2) $0 \leq \Pi(\xi, \omega, \lambda) + \mho(\xi, \omega, \lambda) + \Theta(\xi, \omega, \lambda) \leq 3,\ \forall \lambda \in \mathbb{R}^+$,
(3) $\Pi(\xi, \omega, \lambda) = 1$ for $\lambda > 0$ iff $\xi = \omega$,
(4) $\Pi(\xi, \omega, \lambda) = H(\omega, \xi, \lambda)$ for $\lambda > 0$,
(5) $\Pi(\xi, \omega, \lambda) \otimes \Pi(\omega, c, \rho) \leq \Pi(\xi, c, \lambda + \rho)$ for $\rho, \lambda > 0$
(6) $\Pi(\xi, \omega, \cdot) : \mathbb{R}^+ \to I$ is continuous,
(7) $\lim_{\lambda \to +\infty} \Pi(\xi, \omega, \lambda) = 1$,
(8) $\mho(\xi, \omega, \lambda) = 0$ for $\lambda > 0$ iff $\xi = \omega$,

(9) $\mho(\xi,\omega,\lambda) = \mho(\omega,\xi,\lambda)$ for $\lambda > 0$,
(10) $\mho(\xi,\omega,\lambda) \oplus \mho(\omega,c,\rho) \geq \mho(\xi,c,\lambda+\rho)$ for $\rho,\lambda > 0$,
(11) $\mho(\xi,\omega,\cdot) : \mathbb{R}^+ \to I$ is continuous,
(12) $\lim_{\lambda\to+\infty} \mho(\xi,\omega,\lambda) = 0$,
(13) $\Theta(\xi,\omega,\lambda) = 0$ for $\lambda > 0$ iff $\xi = \omega$,
(14) $\Theta(\xi,\omega,\lambda) = \Theta(\omega,\xi,\lambda)$ for $\lambda > 0$,
(15) $\Theta(\xi,\omega,\lambda) \oplus \Theta(\omega,c,\rho) \geq \Theta(\xi,c,\lambda+\rho)$ for $\rho,\lambda > 0$,
(16) $\Theta(\xi,\omega,\cdot) : \mathbb{R}^+ \to I$ is continuous,
(17) $\lim_{\lambda\to+\infty} \Theta(\xi,\omega,\lambda) = 0$,
(18) If $\lambda \leq 0$, then $\Pi(\xi,\omega,\lambda) = 0$, $\mho(\xi,\omega,\lambda) = \Theta(\xi,\omega,\lambda) = 1$.

Then, the triplet $N = (\Pi,\mho,\Theta)$ is referred to as a neutrosophic metric (NM) on the set F. The functions $\Pi(\xi,\omega,\lambda)$, $\mho(\xi,\omega,\lambda)$, and $\Theta(\xi,\omega,\lambda)$ represent the degrees of nearness, neutralness, and non-nearness between the elements ξ and ω in relation to the parameter λ, respectively.

Definition 2.4. ([26]) Let V denote a NMS, $0 < \epsilon < 1$, $\lambda > 0$, $\xi \in F$. The set

$$\mathcal{O}(\xi,\epsilon,\lambda) = \{\omega \in F : \ \Pi(\xi,\omega,\lambda) > 1-\epsilon, \ \mho(\xi,\omega,\lambda) < \epsilon, \ \Theta(\xi,\omega,\lambda) < \epsilon\}$$

is referred to as the open ball (OB) centered at the point ξ with a radius of ϵ in relation to the parameter λ.

Definition 2.5. ([26]) Let $\{\xi_n\}$ be a sequence in $V = (F,N,\otimes,\oplus)$. Then

(1) $\{\xi_n\}$ converges to $\xi \in F$, if for a given $\epsilon \in (0,1)$, $\lambda > 0$, there is $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $\Pi(\xi_n,\xi,\lambda) > 1-\epsilon$, $\mho(\xi_n,\xi,\lambda) < \epsilon$, $\Theta(\xi_n,\xi,\lambda) < \epsilon$, that is,
$$\lim_{n\to+\infty} \Pi(\xi_n,\xi,\lambda) = 1, \ \lim_{n\to+\infty} \mho(\xi_n,\xi,\lambda) = 0$$
and
$$\lim_{n\to+\infty} \Theta(\xi_n,\xi,\lambda) = 0.$$

(2) $\{\xi_n\}$ is called Cauchy, if for a given $\epsilon \in (0,1)$, $\lambda > 0$, there is $n_0 \in \mathbb{N}$ such that for each $n,m \geq n_0$,
$$\Pi(\xi_n,\xi_m,\lambda) > 1-\epsilon, \ \mho(\xi_n,\xi_m,\lambda) < \epsilon, \ \Theta(\xi_n,\xi_m,\lambda) < \epsilon$$
that is,
$$\lim_{n.m\to+\infty} \Pi(\xi_n,\xi_m,\lambda) = 1, \ \lim_{n,m\to+\infty} \mho(\xi_n,\xi_m,\lambda) = 0$$
and
$$\lim_{n,m\to+\infty} \Theta(\xi_n,\xi_m,\lambda) = 0.$$

(3) V is called complete if each Cauchy sequence is convergent to an element in F.

In [40], Simsek and Kirisci defined NC-contractions on neutrosophic metric spaces and showed that every NC-contraction has a unique fixed point under special considerations.

Definition 2.6. ([40]) Let $V = (F, N, \otimes, \oplus)$ be a NMS. A mapping $f : F \to F$ is called neutrosophic contraction if there is $k \in (0,1)$ such that for each $\xi, \omega \in F$ and $\lambda > 0$, we have

$$\frac{1}{\Pi(f\xi, f\omega, \lambda)} - 1 \leq k\left(\frac{1}{\Pi(\xi, \omega, \lambda)} - 1\right),$$

$$\frac{1}{\mho(f\xi, f\omega, \lambda)} - 1 \geq k\left(\frac{1}{\mho(\xi, \omega, \lambda)} - 1\right)$$

and

$$\frac{1}{\Theta(f\xi, f\omega, \lambda)} - 1 \geq k\left(\frac{1}{\Theta(\xi, \omega, \lambda)} - 1\right).$$

In this scholarly endeavor, we commence by articulating a pivotal lemma that will serve as a foundation for the ensuing discussions. Subsequently, we employ Geraghty functions to forge an innovative category of contractions within the realm of neutrosophic metric spaces. We introduce the (UC) property pertinent to fuzzy sets. Assuming that all fuzzy sets embody the (UC) property, we establish that each Geraghty neutrosophic contraction is endowed with a singular fixed point within a complete neutrosophic metric space.

In conclusion, we unveil a series of fixed point theorems that emerge from our principal discoveries.

3. Main result

Definition 3.1. In this context, we define a real-valued function of three variables on $\mathcal{F}^2 \times (0, +\infty)$, where $\mathcal{F}$ is any nonempty set, denoted as $\mathcal{G}$, to possess the property (UC) if, for any sequences $\{\xi_n\}$ and $\{\omega_n\}$ in $\mathcal{F}$, the following equality holds:

$$\lim_{\lambda \to \lambda_0} \lim_{n \to +\infty} \mathcal{G}(\xi_n, \omega_n, \lambda) = \lim_{n \to +\infty} \lim_{\lambda \to \lambda_0} \mathcal{G}(\xi_n, \omega_n, \lambda),$$

whenever the two limits are exist.

Throughout the remainder of this study, we will assume that each of the fuzzy sets $\Pi, \mho, \Theta$ exhibits the (UC) property.

We begin by the following useful lemmas:

Lemma 3.2. *Let $V = (F, N, \otimes, \oplus)$ be a NMS. Then*

(1) $\Pi(\xi, \omega, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ *is non-decreasing,*
(2) $\mho(\xi, \omega, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ *is non-increasing,*
(3) $\Theta(\xi, \omega, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ *is non-increasing.*

Proof. (1) Let $\lambda_1, \lambda_2 > 0$ with $\lambda_1 > \lambda_2$. Then, there is $\delta > 0$ such that $\lambda_1 = \lambda_2 + \delta$. From Definition 2.3-(5), we get

$$\begin{aligned} \Pi(\xi, \omega, \lambda_1) &= \Pi(\xi, \omega, \lambda_2 + \delta) \\ &\geq \Pi(\xi, \omega, \lambda_2) \otimes \Pi(\omega, \omega, \delta) \\ &= \Pi(\xi, \omega, \lambda_2). \end{aligned}$$

The proofs for (2) and (3) are identical to that of (1). □

Lemma 3.3. *Let $V = (F, N, \otimes, \oplus)$ be a NMS and let $\{\xi_n\}$ be a sequence such that for $\lambda > 0$*

$$\begin{aligned} &\Pi(\xi_p, \xi_q, \lambda) \geq \Pi(\xi_{p-1}, \xi_{q-1}, \lambda), \\ &\mho(\xi_p, \xi_q, \lambda) \leq \mho(\xi_{p-1}, \xi_{q-1}, \lambda), \\ &\Theta(\xi_p, \xi_q, \lambda) \leq \Theta(\xi_{p-1}, \xi_{q-1}, \lambda) \end{aligned} \tag{3.1}$$

and

$$\lim_{n\to+\infty} \Pi(\xi_n, \xi_{n+1}, \lambda) = 1, \ \lim_{n\to+\infty} \mho(\xi_n, \xi_{n+1}, \lambda) = 0,$$
$$\text{and } \lim_{n\to+\infty} \Theta(\xi_n, \xi_{n+1}, \lambda) = 0. \tag{3.2}$$

If $\{\xi_n\}$ is not Cauchy, then there exist an $\epsilon > 0$ and $\lambda > 0$ along with two subsequences $\{\xi_{n_k}\}$ and $\{\xi_{m_k}\}$ derived from $\{\xi_n\}$ such that

$$\lim_{k\to+\infty} \Pi(\xi_n, \xi_m, \lambda) = 1 - \epsilon,$$
$$\lim_{k\to+\infty} \mho(\xi_n, \xi_m, \lambda) = \epsilon,$$
$$\lim_{k\to+\infty} \Theta(\xi_n, \xi_m, \lambda) = \epsilon.$$

Proof. If (ξ_n) is not Cauchy, then

$$\lim_{n,m\to+\infty} \Pi(\xi_n, \xi_m, \lambda) \neq 1, \text{ or } \lim_{n,m\to+\infty} \mho(\xi_n, \xi_m, \lambda) \neq 0,$$
$$\text{or } \lim_{n,m\to+\infty} \Theta(\xi_n, \xi_m, \lambda) \neq 0.$$

Case 1: If $\lim_{n,m\to+\infty} \Pi(\xi_n, \xi_m, \lambda) \neq 1$, then there are $\lambda > 0$ and $\epsilon > 0$ along with two subsequences $\{\xi_{n_k}\}$ and $\{\xi_{m_k}\}$ derived from $\{\xi_n\}$, where (m_k) is selected as the smallest index satisfying the condition.

$$\Pi(\xi_{n_k}, \xi_{m_k}, \lambda) \leq 1 - \epsilon, \quad m_k > n_k > k. \tag{3.3}$$

This implies that

$$\Pi(\xi_{n_k}, \xi_{m_k-1}, \lambda) > 1 - \epsilon. \tag{3.4}$$

Chose $\delta > 0$. Then

$$\begin{aligned}\Pi(\xi_{n_k}, \xi_{m_k}, \lambda + \delta) &\geq \Pi(\xi_{n_k}, \xi_{m_k-1}, \lambda) \otimes \Pi(\xi_{m_k-1}, \xi_{m_k}, \delta) \\ &> (1 - \epsilon) \otimes \Pi(\xi_{m_k-1}, \xi_{m_k}, \delta).\end{aligned}$$

Using Eq. (3.2), we get

$$\liminf_{k\to+\infty} \Pi(\xi_{n_k}, \xi_{m_k}, \lambda + \delta) \geq (1 - \epsilon).$$

Also,

$$\begin{aligned}(1 - \epsilon) &\leq \lim_{\delta\to0^+} \liminf_{k\to+\infty} \Pi(\xi_{n_k}, \xi_{m_k}, \lambda + \delta) \\ &= \liminf_{k\to+\infty} \Pi(\xi_{n_k}, \xi_{m_k}, \lambda).\end{aligned}$$

Also, from (3.3), it follows

$$\limsup_{k\to+\infty} \Pi(\xi_{n_k}, \xi_{m_k}, \lambda) \leq (1 - \epsilon).$$

So, we get

$$\lim_{k\to+\infty} \Pi(\xi_{n_k}, \xi_{m_k}, \lambda) = (1 - \epsilon).$$

Again, we have

$$\begin{aligned}\Pi(\xi_{n_k-1}, \xi_{m_k-1}, \lambda + \delta) &\geq \Pi(\xi_{n_k-1}, \xi_{n_k}, \delta) \otimes \Pi(\xi_{n_k}, \xi_{m_k-1}, \lambda) \\ &> \Pi(\xi_{n_k-1}, \xi_{n_k}, \delta) \otimes (1 - \epsilon).\end{aligned}$$

Using Eq. (3.2), we get

$$\liminf_{k\to+\infty} \Pi(\xi_{n_k-1}, \xi_{m_k-1}, \lambda + \delta) \geq (1 - \epsilon).$$

Also,

$$\begin{aligned}(1 - \epsilon) &\leq \lim_{\delta\to0^+} \liminf_{k\to+\infty} \Pi(\xi_{n_k-1}, \xi_{m_k-1}, \lambda + \delta) \\ &= \liminf_{k\to+\infty} \Pi(\xi_{n_k-1}, \xi_{m_k-1}, \lambda).\end{aligned}$$

From Eq.(3.3), we get

$$\Pi(\xi_{n_k-1}, \xi_{m_k-1}, \lambda) \leq \Pi(\xi_{n_k}, \xi_{m_k}, \lambda) \leq (1 - \epsilon).$$

So,

$$\limsup_{k\to+\infty} \Pi(\xi_{n_k-1}, \xi_{m_k-1}, \lambda) \leq (1 - \epsilon).$$

Hence,

$$\lim_{k\to+\infty} \Pi(\xi_{n_k-1}, \xi_{m_k-1}, \lambda) = (1 - \epsilon).$$

The demonstration for the remaining cases is Similar to that of Case (1). □

To facilitate our main result we need the following class of function which defined by Geraghty in [20].

Definition 3.4. ([20]) Let S be the class of all functions $\alpha : \mathbb{R}^+ \to [0,1)$ that satisfy the following implication:

$$\alpha(t_n) \to 1 \implies t_n \to 0.$$

We will now provide the definition of a Geraghty-neutrosophic contraction.

Definition 3.5. Let $V = (F, N, \otimes, \oplus)$ be a NMS, $\alpha \in S$. A pair of mappings (f, g) on F are called Geraghty-neutrosophic contraction if for each $\xi, \omega \in F$ and each $\lambda > 0$, we have

$$\frac{1}{\Pi(f\xi, g\omega, \lambda)} - 1 \leq \alpha\left(\frac{1}{\Pi(\xi,\omega,\lambda)} - 1\right)\left(\frac{1}{\Pi(\xi,\omega,\lambda)} - 1\right),$$

$$\mho(f\xi, g\omega, \lambda) \leq \alpha(\mho(\xi,\omega,\lambda))(\mho(\xi,\omega,\lambda))$$

and

$$\Theta(f\xi, g\omega, \lambda) \leq \alpha(\Theta(\xi,\omega,\lambda))(\Theta(\xi,\omega,\lambda)).$$

Theorem 3.6. *Let $V = (F, N, \otimes, \oplus)$ be a complete NMS. Suppose that there is $\alpha \in S$ such that the pair (f, g) are Geraghty-Neutrosophic contraction. Then, the functions f and g possesses a unique common fixed point.*

Proof. Let $\xi_0 \in F$ represent an arbitrary point. We examine the sequence $\{\xi_n\}$ characterized by the relation $\xi_{2n+1} = f\xi_{2n},\ \xi_{2n+2} = g\xi_{2n+1},\ n \geq 0$.

Let $n \in \mathbb{N}$. If n is even, then $n = 2s,\ s \in \mathbb{N}$. By Definition 3.5, and part (4) of Definition 2.3, we have

$$\begin{aligned}
\frac{1}{\Pi(\xi_n, \xi_{n+1}, \lambda)} - 1 &= \frac{1}{\Pi(\xi_{2s}, \xi_{2s+1}, \lambda)} - 1 \\
&= \frac{1}{\Pi(g\xi_{2s-1}, f\xi_{2s}, \lambda)} - 1 \\
&\leq \alpha\left(\frac{1}{\Pi(\xi_{2s-1}, \xi_{2s}, \lambda)} - 1\right)\left(\frac{1}{\Pi(\xi_{2s-1}\xi_{2s}, \lambda)} - 1\right) \\
&= \alpha\left(\frac{1}{\Pi(\xi_{n-1}, \xi_n, \lambda)} - 1\right)\left(\frac{1}{\Pi(\xi_{n-1}, \xi_n, \lambda)} - 1\right).
\end{aligned}$$

By the same way, we get

$$\mho(\xi_n, \xi_{n+1}, \lambda) \leq \alpha(\mho(\xi_{n-1}, \xi_n, \lambda))(\mho(\xi_{n-1}, \xi_n, \lambda))$$

and

$$\Theta(\xi_n, \xi_{n+1}, \lambda) \leq \alpha(\Theta(\xi_{n-1}, \xi_n, \lambda))(\Theta(\xi_{n-1}, \xi_n, \lambda)).$$

If n is odd, then $n = 2l+1,\ l \in \mathbb{N} \cup \{0\}$. By Definition 3.5, and Definition 2.3 (4), we have

$$\begin{aligned}
\frac{1}{\Pi(\xi_n, \xi_{n+1}, \lambda)} - 1 &= \frac{1}{\Pi(\xi_{2l+1}, \xi_{2l+2}, \lambda)} - 1 \\
&= \frac{1}{\Pi(g\xi_{2l}, f\xi_{2l+1}, \lambda)} - 1 \\
&\leq \alpha\left(\frac{1}{\Pi(\xi_{2l}, \xi_{2l+1}, \lambda)} - 1\right)\left(\frac{1}{\Pi(\xi_{2l}, \xi_{2l+1}, \lambda)} - 1\right) \\
&= \alpha\left(\frac{1}{\Pi(\xi_{n-1}, \xi_n, \lambda)} - 1\right)\left(\frac{1}{\Pi(\xi_{n-1}, \xi_n, \lambda)} - 1\right).
\end{aligned}$$

By the same way, we get

$$\mho(\xi_n, \xi_{n+1}, \lambda) \leq \alpha(\mho(\xi_{n-1}, \xi_n, \lambda))(\mho(\xi_{n-1}, \xi_n, \lambda))$$

and

$$\Theta(\xi_n, \xi_{n+1}, \lambda) \leq \alpha(\Theta(\xi_{n-1}, \xi_n, \lambda))(\Theta(\xi_{n-1}, \xi_n, \lambda)).$$

Thus, for all $n \in \mathbb{N}$, we conclude

$$\frac{\frac{1}{\Pi(\xi_n, \xi_{n+1}, \lambda)} - 1}{\left(\frac{1}{\Pi(\xi_{n-1}, \xi_n, \lambda)} - 1\right)} \leq \alpha\left(\frac{1}{\Pi(\xi_{n-1}, \xi_n, \lambda)} - 1\right), \tag{3.5}$$

$$\frac{\mho(\xi_n, \xi_{n+1}, \lambda)}{(\mho(\xi_{n-1}, \xi_n, \lambda))} \leq \alpha(\mho(\xi_{n-1}, \xi_n, \lambda)), \tag{3.6}$$

$$\frac{\Theta(\xi_n, \xi_{n+1}, \lambda)}{(\Theta(\xi_{n-1}, \xi_n, \lambda))} \leq \alpha(\Theta(\xi_{n-1}, \xi_n, \lambda)). \tag{3.7}$$

And also, we get

$$\frac{1}{\Pi(\xi_n, \xi_{n+1}, \lambda)} - 1 < \frac{1}{\Pi(\xi_{n-1}, \xi_n, \lambda)} - 1,$$

$$\mho(\xi_n, \xi_{n+1}, \lambda) < (\mho(\xi_{n-1}, \xi_n, \lambda))$$

and

$$\Theta(\xi_n, \xi_{n+1}, \lambda) < (\Theta(\xi_{n-1}, \xi_n, \lambda)).$$

So, we have

(1) the sequence $\{\Pi(\xi_n, \xi_{n+1}, \lambda) : n \in \mathbb{N}\}$ is nondecreasing in [0,1], and so, there is $r_\Pi \leq 1$ such that r_Π is the limit of this sequence.

(2) the sequence $\{\mho(\xi_n, \xi_{n+1}, \lambda) : n \in \mathbb{N}\}$ is nonincreasing in [0,1], and so, there is $r_{\mho} \geq 0$ such that $r_{\mho}$ is the limit of this sequence.

(3) the sequence $\{\Theta(\xi_n, \xi_{n+1}, \lambda) : n \in \mathbb{N}\}$ is nonincreasing in [0,1], and so, there is $r_{\Theta} \geq 0$ such that r_{Θ} is the limit of this sequence.

Case 1: If $r_{\Pi} > 0$, by taking the limit in Eq.(3.5), we get

$$\lim_{n\to+\infty} \alpha\left(\frac{1}{\Pi(\xi_{n-1}, \xi_n, \lambda)} - 1\right) = 1,$$

which implies that

$$\lim_{n\to+\infty} \frac{1}{\Pi(\xi_n, \xi_{n+1}, \lambda)} - 1 = 0,$$

which is a contradiction. So $r_{\Pi} = 0$. By the same way we conclude that $r_{\mho} = 0$ and $r_{\Theta} = 0$.

Now, we claim that $\{\xi_n\}$ is Cauchy by proving that $\{\xi_{2n}\}$ is so. If not then by Lemma 3.3, there exist an $\epsilon > 0$ and $\lambda > 0$ along with two subsequences $\{\xi_{2n_k}\}$ and $\{\xi_{2m_k}\}$ derived from $\{\xi_{2n}\}$ such that one of the following holds

$$\lim_{k\to+\infty} \Pi(\xi_{2n_k}, \xi_{2m_k}, \lambda) = 1 - \epsilon,$$
$$\lim_{k\to+\infty} \mho(\xi_{2n_k}, \xi_{2m_k}, \lambda) = \epsilon,$$
$$\lim_{k\to+\infty} \Theta(\xi_{2n_k}, \xi_{2m_k}, \lambda) = \epsilon.$$

Using Definition 3.5, we deduce that one of the following holds

$$\frac{1}{\Pi(\xi_{2n_k}, \xi_{2m_k}, \lambda)} - \leq \alpha\left(\frac{1}{\Pi(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda)} - 1\right)\left(\frac{1}{\Pi(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda)} - 1\right),$$

$$\mho(\xi_{2n_k}, \xi_{2m_k}, \lambda) \leq \alpha(\mho(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda))(\mho(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda)),$$

or

$$\Theta(\xi_{2n_k}, \xi_{2m_k}, \lambda) \leq \alpha(\Theta(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda))(\Theta(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda)).$$

So,

$$\frac{\frac{1}{\Pi(\xi_{2n_k}, \xi_{2m_k}, \lambda)} - 1}{\left(\frac{1}{\Pi(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda)} - 1\right)} \leq \alpha\left(\frac{1}{\Pi(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda)} - 1\right),$$

$$\frac{\mho(\xi_{2n_k}, \xi_{2m_k}, \lambda)}{(\mho(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda))} \leq \alpha(\mho(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda)),$$

or

$$\frac{\Theta(\xi_{2n_k}, \xi_{2m_k}, \lambda)}{(\Theta(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda))} \leq \alpha(\Theta(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda)).$$

Hence, by taking the limit on $k \to +\infty$, we get

$$\lim_{k\to+\infty} \alpha\left(\frac{1}{\Pi(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda)} - 1\right) = 1,$$

$$\lim_{k\to+\infty} \alpha\left(\mho(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda)\right) = 1,$$

or

$$\lim_{k\to+\infty} \alpha\left(\Theta(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda)\right) = 1,$$

which implies that

$$\lim_{k\to+\infty} \left(\frac{1}{\Pi(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda)} - 1\right) = 0,$$

$$\lim_{k\to+\infty} \left(\mho(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda)\right) = 0,$$

or

$$\lim_{k\to+\infty} \left(\Theta(\xi_{2n_k-1}, \xi_{2m_k-1}, \lambda)\right) = 0,$$

which leads to a contradiction in each single case. Hence $\{\xi_{2n}\}$ is a Cauchy sequence, and hence $\{\xi_n\}$ is so, thus, there is $u \in F$ such that $\xi_n \to u$.

Definition 3.5 gives that

$$\frac{1}{\Pi(gu, f\xi_n, \lambda)} - 1 \leq \alpha\left(\frac{1}{\Pi(u, \xi_n, \lambda)} - 1\right)\left(\frac{1}{\Pi(u, \xi_n, \lambda)} - 1\right) \to 0 \text{ as } n \to +\infty,$$

$$\mho(fu, \xi_{n+1}, \lambda) \leq \alpha(\mho(u, \xi_n, \lambda))(\mho(u, \xi_n, \lambda)) \to 0 \text{ as } n \to +\infty$$

and

$$\Theta(fu, \xi_{n+1}, \lambda) \leq \alpha(\Theta(u, \xi_n, \lambda))(\Theta(u, \xi_n, \lambda)) \to 0 \text{ as } n \to +\infty,$$

which implies that $\{\xi_{n+1}\}$ converges to gu, hence $u = gu$. By the same steps above, we get $u = fu$. Hence, u is a common fixed point for f and g.

Let $v \in F$ with $v = fv$ and $v = gv$. If $u \neq v$, then from Definition 3.5, it follows that

$$\begin{aligned}\frac{1}{\Pi(u,v,\lambda)} - 1 &= \frac{1}{\Pi(fu,gv,\lambda)} - 1 \\ &\leq \alpha\left(\frac{1}{\Pi(u,v,\lambda)} - 1\right)\left(\frac{1}{\Pi(u,v,\lambda)} - 1\right) \\ &< \frac{1}{\Pi(u,v,\lambda)} - 1,\end{aligned}$$

$$\mho(u,v,\lambda) = \mho(fu,gv,\lambda) \leq \alpha(\mho(u,v,\lambda))(\mho(u,v,\lambda)) < \mho(u,v,\lambda)$$

and

$$\Theta(u,v,\lambda) = \Theta(fu,gv,\lambda) \leq \alpha(\Theta(u,v,\lambda))(\Theta(u,v,\lambda)) < \Theta(u,v,\lambda),$$

which are contradiction. So $u = v$. □

Using Theorem 3.6, if we establish the function $\alpha(s) = q$, where q is constrained within the interval $[0,1)$, we arrive at the subsequent conclusion:

Corollary 3.7. *Let $V = (F, N, \otimes, \oplus)$ be a complete NMS. Suppose that $f, g : F \to F$ satisfy the following, for each $\xi, \omega \in F$ and each $\lambda > 0$, we have*

$$\frac{1}{\Pi(f\xi,g\omega,\lambda)} - 1 \leq q\left(\frac{1}{\Pi(\xi,\omega,\lambda)} - 1\right),$$

$$\mho(f\xi,g\omega,\lambda) \leq q\mho(\xi,\omega,\lambda)$$

and

$$\Theta(f\xi,g\omega,\lambda) \leq q\Theta(\xi,\omega,\lambda).$$

Then, the function f and g possesses a unique common fixed point.

If we take $g = f$ in Theorem 3.6, we get that

Corollary 3.8. *Let $V = (F, N, \otimes, \oplus)$ be a complete NMS. Suppose that $f : F \to F$ satisfies the following, for each $\xi, \omega \in F$ and each $\lambda > 0$, we have*

$$\frac{1}{\Pi(f\xi,f\omega,\lambda)} - 1 \leq \alpha\left(\frac{1}{\Pi(\xi,\omega,\lambda)} - 1\right)\left(\frac{1}{\Pi(\xi,\omega,\lambda)} - 1\right),$$

$$\mho(f\xi,f\omega,\lambda) \leq \alpha(\mho(\xi,\omega,\lambda))(\mho(\xi,\omega,\lambda))$$

and

$$\Theta(f\xi,f\omega,\lambda) \leq \alpha(\Theta(\xi,\omega,\lambda))(\Theta(\xi,\omega,\lambda)).$$

Then, the function f possesses a unique fixed point.

In Corollary 3.8, if we take the function $\alpha(s) = q$, where q is constrained within the interval $[0, 1)$, we arrive at the subsequent conclusion:

Corollary 3.9. *Let $V = (F, N, \otimes, \oplus)$ be a complete NMS. Suppose that $f : F \to F$ satisfies the following, for each $\xi, \omega \in F$ and each $\lambda > 0$, we have*

$$\frac{1}{\Pi(f\xi, f\omega, \lambda)} - 1 \leq q \left(\frac{1}{\Pi(\xi, \omega, \lambda)} - 1 \right),$$

$$\mho(f\xi, f\omega, \lambda) \leq q\mho(\xi, \omega, \lambda)$$

and

$$\Theta(f\xi, f\omega, \lambda) \leq q\Theta(\xi, \omega, \lambda).$$

Then, the function f possesses a unique fixed point.

4. Conclusion

Fixed point theorems represent essential principles in mathematics, especially within the domains of analysis, topology, and applied mathematics. These theorems define the criteria under which a function possesses a fixed point, which is a point that remains unchanged when the function is applied. Conversely, neutrosophic metric spaces are a sophisticated extension of metric spaces. This framework is particularly beneficial for addressing uncertainty and imprecision across various mathematical and practical scenarios. In this research, we present fixed point theorems associated with i contractions of Geraghty type, situated within the advanced framework of neutrosophic metric spaces. Furthermore, we have derived several fixed point results pertinent to this specific context.

Future directions include generalization to bipolar neutrosophic spaces, applications to nonlinear dynamical systems with uncertain parameters, and the development of computational algorithms for real-world decision-making under uncertainty, potentially enhancing predictive modeling in fields like medical diagnostics and security analysis.

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