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# WOVEN $b$-FRAMES IN HILBERT SPACES

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**Abstract.** This paper aims to study woven $b$-frames, which are a generalization of woven frames in Hilbert spaces, where the frames in Hilbert spaces generated by the bilinear mapping are considered. The $b$-frame operator is defined, and the conditions for the existence of a $b$-frame in Hilbert spaces are obtained. We will define woven, weaving $b$-frames and woven, weaving $K$-$b$-frames and present initial results. We will also study and explore the stability and preservation of both woven and weaving $K$-$b$-frames.

## 1. Introduction

The theory of frames in Hilbert spaces is a powerful tool for analyzing and representing signals. They were introduced by Duffin and Schaeffer in 1952 [9]. Frames have applications not only in signal processing but also in various other fields, such as physics, image processing, and engineering; (see [1, 2, 4, 5, 6, 7, 11, 13, 14, 17, 18, 19, 20]).

In 2016, Bemrose, Casazza, Gröchenig, Lammers and Lynch introduced woven frames in separable Hilbert spaces in [3, 4, 5]. Their motivation was a question in distributed signal processing, where frames play an important role.

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Another recent development is the concept of $b$-frames. The idea is to take a sequence from a Banach space and see how it can be a frame for a Hilbert space, first introduced by Ismailov et al. in [12]. In 2024, Mezzat and Kabbaj generalized this concept, provided examples, and studied the stability and preservation of $b$-frames in [15]. $b$-frames allow for the construction of frames in Hilbert spaces using a bilinear mapping, offering greater flexibility compared to classical frames.

In this paper, based on the weaving frames and K-frames, we propose the notion of weaving $b$-frames. First, we recall several definitions about $b$-frames, and we will explore woven $b$-frames. The motivation behind this exploration is to answer the same question answered in [16]. Except that in our study the frames considered to measure a signal $z$ don't belong to the same space. This paper is structured as follows. Section 2 provides preliminaries and notations used in our research. Section 3 formally defines woven and weaving $b$-frames and woven and weaving $K$-$b$-frames and gives alternatives to $b$-synthesis, $b$-analysis, and $b$-frame operators and establishes their connection to existing frame theory. The last section we discuss transitivity of weaving $b$-frames and studies the stability and preservation of woven $b$-frames, $K$-$b$-frames, weaving $b$-frames, and $K$-$b$-frames.

## 2. Preliminaries and notations

We begin by giving the following notations. Let $\mathcal{Z}$ and $\mathcal{H}$ be two Hilbert spaces. The set of bounded linear operators from $\mathcal{H}$ to $\mathcal{Z}$ is denoted by $\mathcal{L}(\mathcal{H},\mathcal{Z})$. If $K$ belongs to $\mathcal{L}(\mathcal{H},\mathcal{Z})$, then the adjoint operator of $K$, denoted $K^*$, belongs to $\mathcal{L}(\mathcal{Z},\mathcal{H})$ and satisfies the relation $\langle Kh, z\rangle_{\mathcal{Z}} = \langle h, K^*z\rangle_{\mathcal{H}}$ for all $h$ in $\mathcal{H}$ and $z$ in $\mathcal{Z}$. The identity operator on $\mathcal{H}$ is denoted by $Id_{\mathcal{H}}$, the range of an operator $K$ is denoted as $\mathcal{R}(K)$, and the kernel of an operator $K$ is denoted by $\ker(K)$.

**Lemma 2.1.** *Let $\mathcal{H}$ be a Hilbert space and let $K \in \mathcal{L}(\mathcal{H})$, such that $\mathcal{R}(K)$ is closed. $K^\dagger$ will denote the pseudo-inverse (or the moore-penrose inverse) of $K$ verifying :*

(1) $(KK^\dagger)^* = KK^\dagger; \quad (K^\dagger K)^* = K^\dagger K; \quad Ker(K^\dagger) = (\mathcal{R}(K))^\perp,$
(2) $(KerK)^\perp = \mathcal{R}(K^\dagger)$.

**Theorem 2.2.** ([8]) *Let $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$ be Hilbert spaces, and let $P \in \mathcal{L}(\mathcal{F}_1, \mathcal{F})$, and $Q \in \mathcal{L}(\mathcal{F}_2, \mathcal{F})$. Then the following statements are equivalent:*

(1) $\mathcal{R}(P) \subset \mathcal{R}(Q)$.
(2) $PP^* \leq \lambda^2 QQ^*$ *for some* $\lambda > 0$.
(3) *there exists* $X \in \mathcal{L}(\mathcal{F}_1, \mathcal{F}_2)$ *such that* $P = QX$.

**Theorem 2.3.** ([12]) *Let $K \in \mathcal{L}(\mathcal{H}, \mathcal{Z})$, where $\mathcal{H}$ and $\mathcal{Z}$ are two Hilbert spaces. Then:*

(1) $K^* \in \mathcal{L}(\mathcal{Z}, \mathcal{H})$, *and* $\|K^*\| = \|K\|$.
(2) $\mathcal{R}(K)$ *is closed if and only if* $\mathcal{R}(K^*)$ *is closed.*
(3) $K$ *is surjective if and only if there exists a* $\delta > 0$ *such that*

$$\|K^* z\|_{\mathcal{Z}} \geq \delta \|z\|_{\mathcal{Z}}, \ \forall\, z \in \mathcal{Z}.$$

Let $\mathcal{Z}, \mathcal{H}$ be two Hilbert spaces and let $\langle .,. \rangle_{\mathcal{Z}}$, $\langle .,. \rangle_{\mathcal{H}}$ be their corresponding scalar products respectively. We denote by $\|.\|_{\mathcal{Z}}$ (resp. $\|.\|_{\mathcal{H}}$) the norm of $\mathcal{Z}$ (resp. $\mathcal{H}$).

Let $\mathcal{B}$ be a Banach space with norm $\|.\|$ and the bilinear mapping :
$b : \mathcal{H} \times \mathcal{B} \longrightarrow \mathcal{Z}$, satisfying the condition :

$$\exists\, A > 0 : \ \|b(h,x)\|_{\mathcal{Z}} \leqslant A \|h\|_{\mathcal{H}} \|x\|, \ \ \forall\, h \in \mathcal{H}, \ x \in \mathcal{B}. \tag{2.1}$$

Note that the inequality (2.1) means that $b$ is bounded and continuous.
Fix $x \in \mathcal{B}$ and $z \in \mathcal{Z}$, and consider the linear functional $\zeta_x^z : \mathcal{H} \longrightarrow \mathbb{C}$, defined by

$$\zeta_x^z(h) = \langle b(h,x), z \rangle_{\mathcal{Z}}\,.$$

Then note that $\zeta_x^z$ is linear, also by :(2.1) we have, for all $h \in \mathcal{H}$

$$\begin{aligned} |\zeta_x^z(h)| &= |\, \langle b(h,x), z \rangle_{\mathcal{Z}} \,| \leqslant \|b(h,x)\|_{\mathcal{Z}} \|z\|_{\mathcal{Z}} \\ &\leqslant A \|h\|_{\mathcal{H}} \|x\| \|z\|_{\mathcal{Z}}. \end{aligned}$$

Hence, $\|\zeta_x^z\| \leqslant A\|x\|\|z\|_{\mathcal{Z}}$. Then there exists a unique element $v \in \mathcal{H}$ such that $\zeta_x^z(h) = \langle h, v \rangle_{\mathcal{H}}$ (Riesz representation theorem), and this element called $b$-dual product of $z$ and $x$ and denoted $\langle z/x \rangle$ (see [15]).

Let $\{y_k\}_{k \in \mathbb{N}} \subset \mathcal{B}$ be a $b$-orthonormal basis in $\mathcal{Z}$ and $\{e_k\}_{k \in \mathbb{N}}$ the orthonormal basis of a separable Hilbert space $\mathcal{H}$ and the bilinear mapping $b$ has a dense range such that :

$$\sum_{k \in \mathbb{N}} \|y_k\| < \infty.$$

Let $\mathcal{Y} \subset \mathcal{B}$ such that:

$$\mathcal{Y} = \overline{span\{y_k / k \in \mathbb{N}\}}.$$

**Proposition 2.4.** ([15]) *Let $S \in \mathcal{L}(\mathcal{Y}, \mathcal{Y})$, $\{h_k\}_{k \in \mathbb{N}} \subset \mathcal{H}$ and $T : \mathcal{Z} \to \mathcal{Z}$ be an operator such that: for each $z = \sum_{k \in \mathbb{N}} b(h_k, y_k) \in \mathcal{Z}$,*

$$T^*(z) = \sum_{k \in \mathbb{N}} b(h_k, U y_k).$$

*Then $T^*$ is bounded linear operator.*

**Theorem 2.5.** ([15]) *Let* $\{y_k\}_{k\in\mathbb{N}} \subset \mathcal{B}$ *be a* $b$*-orthonormal basis in* $\mathcal{Z}$*. Let* $S \in \mathcal{L}(\mathcal{Y},\mathcal{Y})$*. Then there exists only one* $T \in \mathcal{L}(\mathcal{Z})$ *such that, for all* $z \in \mathcal{Z}$*,*

$$\langle Tz/y_k\rangle = \langle z/Sy_k\rangle,\ \forall\, k \in \mathbb{N}. \tag{2.2}$$

**Corollary 2.6.** ([15]) *Let* $\{y_k\}_{k\in\mathbb{N}} \subset \mathcal{B}$ *be a* $b$*-orthonormal basis in* $\mathcal{Z}$*. Let* $S \in \mathcal{L}(\mathcal{Y},\mathcal{Y})$*. Then there exists only one* $T \in \mathcal{L}(\mathcal{Z})$ *such that*

$$\langle Tz/y\rangle = \langle z/Sy\rangle,\ \forall\, y \in \mathcal{Y},\ \forall\, z \in \mathcal{Z}. \tag{2.3}$$

In which case $T$ called the $b$-adjoint of $S$ denoted by $S^b$.

Let $K \in \mathcal{L}(\mathcal{Z})$ and let $K^*$ be its adjoint, we define a $b$-frame and $K-b$-frame as follows:

**Definition 2.7.** ([15]) A sequence $\{\eta_k\}_{k\in\mathbb{I}} \subset B$ is called a $b$-frame for $Z$, if there exist constants $0 < \mathcal{A}_\eta \leq \mathcal{B}_\eta < \infty$ such that

$$\mathcal{A}_\eta\|z\|^2 \leq \sum_{k\in\mathbb{I}} \|< z/\eta_k >\|_H^2 \leq \mathcal{B}_\eta\|z\|^2, \quad \forall\, z \in \mathcal{Z}.$$

**Definition 2.8.** ([15]) A sequence $\{\eta_k\}_{k\in\mathbb{I}} \subset B$ is called a $K$-$b$-frame for $Z$, if there exist constants $0 < \mathcal{A}_\eta \leq \mathcal{B}_\eta < \infty$ such that

$$\mathcal{A}_\eta\|K^*z\|^2 \leq \sum_{k\in\mathbb{I}} \|< z/\eta_k >\|_H^2 \leq \mathcal{B}_\eta\|z\|^2, \quad \forall\, z \in \mathcal{Z}.$$

## 3. Woven $b$-frames and woven $K$-$b$-frames

### 3.1. Woven $b$-frames, woven $K$-$b$-frames and some examples.

For a fixed $n \in \mathbb{N}$, we define: $[n] = \{1, 2, ..., n\}$. We present several novel construction of weaving frame, we will explore woven $b$-frames and we begin by giving the following definitions, and some examples:

**Definition 3.1.** Let $\Lambda = \{\{\eta_{kl}\}_{k\in\mathbb{I}} : l \in [n]\}$ be a family of $b$-frames for $\mathcal{Z}$. If there exist universal constants $\mathcal{A}_\eta$ and $\mathcal{B}_\eta$ such that for every partition $\{\sigma_l\}_{l\in[n]}$, the family $\Lambda_l = \{\eta_{kl}\}_{k\in\sigma_l}$ is a $b$-frames for $\mathcal{Z}$ with bounds $\mathcal{A}_\eta$ and $\mathcal{B}_\eta$, then $\Lambda$ is said woven $b$-frames and for every $l \in [n]$ the $b$-frames $\Lambda_l$ are called weaving $b$-frames. The constants $\mathcal{A}_\eta$ and $\mathcal{B}_\eta$ are called the lower and upper woven $b$-frames bounds. If $\mathcal{A}_\eta = \mathcal{B}_\eta$, then $\Lambda = \{\{\eta_{kl}\}_{k\in\mathbb{I}} : l \in [n]\}$ is called a tight woven $b$-frames. If for every $l \in [n]$ the $b$-frames $\Lambda_l$ is weaving $b$-Besselian sequence, then the family $\Lambda = \{\{\eta_{kl}\}_{k\in\mathbb{I}} : l \in [n]\}$ is said to be woven $b$-Besselian sequence.

Let $K \in \mathcal{L}(\mathcal{Z})$ and let $K^*$ be its adjoint, we define a weaving $K$-$b$-frames and woven $K$-$b$-frames as follows:

**Definition 3.2.** Let $\Lambda = \{\{\eta_{kl}\}_{k\in\mathbb{I}} : l \in [n]\}$ be a family of $K$-$b$-frames for $\mathcal{Z}$. If there exist universal constants $A_\Lambda$ and $B_\Lambda$ such that for every partition $\{\sigma_l\}_{l\in[n]}$, the family $\Lambda_l = \{\eta_{kl}\}_{k\in\sigma_l}$ is a $K$-$b$-frame for $\mathcal{Z}$ with bounds $A_\Lambda$ and $B_\Lambda$, then $\Lambda$ is said woven $K$-$b$-frames and for every $l \in [n]$ the $K$-$b$-frames $\Lambda_l$ are called weaving $K$-$b$-frames. The constants $A_\Lambda$ and $B_\Lambda$ are called the lower and upper woven $K$-$b$-frames bounds. If $A_\Lambda = B_\Lambda$, then $\Lambda = \{\{\eta_{kl}\}_{k\in\mathbb{I}} : l \in [n]\}$ is called a tight woven $K$-$b$-frames and if for every $l \in [n]$ the $K$-$b$-frames $\Lambda_l$ is weaving $K$-$b$-Besselian sequences. Then the family $\Lambda = \{\{\eta_{kl}\}_{k\in\mathbb{I}} : l \in [n]\}$ is said to be woven $K$-$b$-Besselian sequences.

**Example 3.3.** Let $\mathcal{H} = \mathbb{R}^3$ with canonical base $(e_1, e_2, e_3)$, and $\mathcal{B} = \mathbb{R}^2$ with canonical base $(\alpha_1, \alpha_2)$ and $\mathcal{Z} = \mathbb{R}^4$ such that $\{\rho_i\}_{1\le i\le 4}$ are their canonical base, and let the bilinear mapping $b : \mathcal{H} \times \mathcal{B} \longrightarrow \mathcal{Z}$ defined by:

$$b(e_1,\alpha_1) = \rho_1, \qquad b(e_2,\alpha_1) = \rho_3, \qquad b(e_3,\alpha_1) = \rho_1,$$
$$b(e_1,\alpha_2) = \rho_2, \qquad b(e_2,\alpha_2) = \rho_4, \qquad b(e_3,\alpha_2) = \rho_2.$$

Then, for $h = \Sigma_{k=1}^{3} h_k e_k$ and $x = \Sigma_{k=1}^{2} x_k \alpha_k$, we have

$$b(h,x) = (h_1 + h_3)x_1\rho_1 + (h_1 + h_3)x_2\rho_2 + h_2x_1\rho_3 + h_2x_2\rho_4$$

and

$$\| b(h,x) \|_Z^2 = (h_1^2 + h_2^2 + h_3^2)(x_1^2 + x_2^2) + 2h_1h_3(x_1^2 + x_2^2) \le 2 \| h \|^2 \| x \|^2 .$$

Furthermore, we have

$$< z/x >= (x_1z_1 + x_2z_2)e_1 + (x_1z_3 + x_2z_4)e_2 + (x_1z_1 + x_2z_2)e_3$$

and

$$\|< z/x >\|_H^2 = 2(x_1z_1 + x_2z_2)^2 + (x_1z_3 + x_2z_4)^2.$$

Let

$$H = \{h_k\}_{k=1}^{3} = \{\alpha_1 + \alpha_2; \alpha_1 - \alpha_2; \alpha_1\}$$

and

$$G = \{g_k\}_{k=1}^{3} = \{2\alpha_2 + \alpha_1; 3\alpha_1 + \alpha_2; \alpha_2\}.$$

Then $H$ is a $b$-frame with lower and upper bounds 2 and 6, respectively. Similarly, $G$ is a $b$-frame with bounds 1 and 30. The frames $H$ and $G$ constitute a woven $b$-frame. For example, if we assume that $\sigma_1 = \{1, 2\}$, we have

$$\| z \|^2 \le \sum_{k\in\sigma_1} \|< z/h_k >\|_H^2 + \sum_{k\in\sigma_1^c} \|< z/g_k >\|_H^2 \le 6 \| z \|^2, \quad \forall z \in \mathcal{Z}.$$

So $\{h_k\}_{k\in\sigma_1} \cup \{g_k\}_{k\in\sigma_1^c}$ is $b$-frames with bounds $\mathbb{A}_1 = 2$ and $\mathbb{B}_1 = 6$.

Now, if we take

$$\mathbb{A} = \min\{\mathbb{A}_k; 1 \le k \le 8\} \quad \text{and} \quad \mathbb{B} = \max\{\mathbb{B}_k; 1 \le k \le 8\}.$$

Then $H$ and $G$ constitute a woven $b$-frames for $\mathcal{Z}$ with universal bounds $\mathbb{A}$ and $\mathbb{B}$.

According to the following, there is an automatic universal upper $b$-frame bound for every weaving.

**Theorem 3.4.** *If $\{\eta_{kl}\}_{k\in\mathbb{I}}$ is woven b-Besselian sequence for $\mathcal{Z}$ with bound $B_l$ for all $l \in [n]$, then $\{\eta_{kl}\}_{k\in\sigma_l,\, l\in[n]}$ is weaving b-besselian sequence with bound $\sum_{l=1}^{n} B_l$.*

*Proof.* Let $\sigma_l,\ l \in [n]$ any partition of $\mathbb{I}$ we have:

$$\sum_{l=1}^{n}\sum_{k\in\sigma_l} \| < z/\eta_{kl} > \|_H^2 \leq \sum_{l=1}^{n}\sum_{k\in\mathbb{I}} \| < z/\eta_{kl} > \|_H^2 \leq \sum_{l=1}^{n} B_l \|z\|, \qquad z \in \mathcal{Z}.$$

yielding the desired bound. □

3.2. **Operators for weaving and woven $b$-frames.** Now we introduce $b$-analysis, $b$-synthesis and $b$-frames operators of weaving and woven $b$-frames and it will be shown that if each family of subspaces $\{l^2(H)_l\}_{l\in[n]}$ of $l^2(H)$ we have the following space:

$$l^2(H)_l = \Big\{\{\theta_{kl}\}_{k\in\sigma_l}/\theta_{kl} \in H,\ \sigma_l \subset \mathbb{I},\ \sum_{k\in\sigma_l} \| \theta_k \|_H^2 < \infty\Big\},\ \forall\, l \in [n].$$

We define the space

$$\Theta = \left(\sum_{l\in[n]} \bigoplus (l^2(H))_l\right)_{l^2} = \Big\{\{\theta_{kl}\}_{k\in\mathbb{I},l\in[n]}/\{\theta_{kl}\}_{k\in\mathbb{I}} \in (l^2(H))_l,\ \forall\, l \in [n]\Big\}$$

is a Hilbert space for the inner product see [10] :

$$\Big\langle \{\theta_{kl}\}_{k\in\mathbb{I},l\in[n]}, \{\theta'_{kl}\}_{k\in\mathbb{I},l\in[m]} \Big\rangle = \sum_{k\in\mathbb{I},l\in[n]} < \theta_{kl}, \theta'_{kl} >_H .$$

**Theorem 3.5.** *Let $\Lambda = \{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]} \subset B$. The family $\Lambda$ is a woven b-Besselian sequence if and only if the operator*

$$T_\Lambda : \Theta \longrightarrow \mathcal{Z}$$

*defined by*

$$T_\Lambda \left(\{\theta_{kl}\}_{k\in\mathbb{U},l\in[n]}\right) = \sum_{k\in\mathbb{I},l\in[n]} b\left(\theta_{kl}, \eta_{kl}\right)$$

*is well-defined, linear and bounded.*

*Proof.* Suppose $\{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ be a woven $b$-Besselian. So a fixed $l\in[n]$ and $\sigma_l\subset\mathbb{I}$ the family $\{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ is a $b$-Bessel sequence with $b$-Bessel bound $B_l$. If we denoted the $b$-synthesis operator of $\{\eta_{kl}\}_{k\in\sigma_l}$ by $T_{\sigma_l}$, therefore for every $\{\theta_{kl}\}_{k\in\mathbb{I},l\in[n]}\in\Theta$, we have

$$N=\| T_\Lambda(\{\theta_{kl}\})\|^2=\|\sum_{k\in\mathbb{I},l\in[n]} b(\theta_{kl},\eta_{kl})\|^2 .$$

The serie $\sum_{k\in\mathbb{I}} b(\theta_{kl},\eta_{kl})$ converge for every $\theta_{kl}\in\Theta$ , $l\in[n]$. Hence

$$\begin{aligned}
N &=\| T_\Lambda(\{\theta_{kl}\})\|^2\\
&=\|\sum_{k\in\mathbb{I},l\in[n]} b(\theta_{kl},\eta_{kl})\|^2\\
&=\|\sum_{k\in\mathbb{I}} b(\theta_{k1},\eta_{k1})+\sum_{k\in\mathbb{I}} b(\theta_{k2},\eta_{k2})+\cdots+\sum_{k\in\mathbb{I}} b(\theta_{kn},\eta_{kn})\|^2\\
&\le 2(\|\sum_{k\in\mathbb{I}} b(\theta_{k1},\eta_{k1})\|^2+\|\sum_{k\in\mathbb{I}} b(\theta_{k2},\eta_{k2})\|^2+\cdots+\|\sum_{k\in\mathbb{I}} b(\theta_{kn},\eta_{kn})\|^2)\\
&\le 2(\| T_{\sigma_1}(\{\theta_{k1}\}_{k\in\sigma_1})\|^2+\| T_{\sigma_2}(\{\theta_{k2}\}_{k\in\sigma_2})\|^2+\cdots+\| T_{\sigma_n}(\{\theta_{kn}\}_{k\in\sigma_n})\|^2)\\
&\le 2(B_1\|(\{\theta_{k1}\}_{k\in\sigma_1})\|^2+B_2\|(\{\theta_{k2}\}_{k\in\sigma_2})\|^2+\cdots+B_n\|(\{\theta_{kn}\}_{k\in\sigma_n})\|^2)\\
&\le 2(\sum_{l=1}^{l=n} B_l\|(\{\theta_{kl}\}_{k\in\sigma_l})\|^2)\\
&\le 2(nB\|(\{\theta_{kl}\}_{k\in\mathbb{I},l\in[n]})\|^2)
\end{aligned}$$

with

$$B=\max\{B_l,\ 1\le l\le n\}.$$

Hence $T_\Lambda$ is bounded and well-defined and linear.

Conversely, suppose $T_\Lambda$ well-defined, linear and bounded with bound B. We have for every $z\in\mathcal{Z}$,

$$\begin{aligned}
\Big\langle T_\Lambda(\{\theta_{kl}\}_{k\in\mathbb{I},\,l\in[n]}),z\Big\rangle_{\mathcal{Z}} &= \Big\langle\sum_{k\in\mathbb{I},l\in[n]} b(\theta_{kl},\eta_{kl}),z\Big\rangle_{\mathcal{Z}}\\
&= \sum_{k\in\mathbb{I},l\in[n]} < b(\theta_{kl},\eta_{kl}),z>_{\mathcal{Z}}\\
&= \big\langle\{\theta_{kl}\}_{k\in\mathbb{I},l\in[n]},\{<z/\eta_{kl}>\}_{k\in\mathbb{I},\,l\in[n]}\big\rangle_{\Theta}.
\end{aligned}$$

Therefore, we have

$$T^*_\Lambda(z)=\{<z/\eta_{kl}>\}_{k\in\mathbb{I},\,l\in[n]}.$$

And
$$\sum_{k\in\mathbb{I},l\in[n]} \| < z/\eta_{kl} > \|_H^2 = \|T_\Lambda^*(z)\|_\Theta^2 \le \|T_\Lambda^*\|^2\|z\|_\mathcal{Z}^2 \le \|T_\Lambda\|^2\|z\|_\mathcal{Z}^2 \le B\|z\|_\mathcal{Z}^2.$$

Hence, the family $\Lambda = \{\eta_{kl}\}_{k\in\mathbb{I},\, l\in[n]}$ is woven $b$-besselian sequences. □

**Corollary 3.6.** *Let $\Lambda = \{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ be a family in $\mathcal{Z}$ and for all $\{\theta_{kl}\}_{k\in\mathbb{I},l\in[n]} \in \Theta$, the series $\sum_{k\in\mathbb{I},l\in[n]} b(\theta_{kl}, \eta_{kl})$ is convergent. Then the sequence $\Lambda$ is woven $b$-Besselian.*

*The woven $b$-Besselian condition:*
$$\sum_{k\in\mathbb{I},l\in[n]} \|< z/\eta_{kl} >\|_H^2 \le B \parallel z \parallel_\mathcal{Z}^2, \qquad \forall\, z \in \mathcal{Z}.$$

Like $b$-frames and their extensions (see [12]), woven $b$-frames can be characterized in terms of their associated woven $b$-frame operator.

**Definition 3.7.** Let $\Lambda = \{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ be a woven $b$-Besselian sequence. So for every partition $\{\sigma_l\}_{l\in[n]}$ of $I$, the family $\Lambda_l = \{\eta_{kl}\}_{k\in\sigma_l}$ for $l \in [n]$ is a $b$-Besselian sequence. Thus, we can define:

(1) The $b$-analysis operator of $\Lambda_l$ by :
$$W_{\sigma_l} : \mathcal{Z} \to l^2(\mathcal{H})_l$$
with $W_{\sigma_l}(z) = \{< z/\eta_{kl} >\}_{k\in\sigma_l}$, $\forall\, l \in [n]$, $z \in \mathcal{Z}$ and $\mathcal{R}(W_{\sigma_l}) \subseteq l^2(\mathcal{H})_l \subseteq l^2(\mathcal{H})$.

(2) The adjoint of $W_{\sigma_l}$ is called the $b$-synthesis operator and we denote by $W_{\sigma_l}^*$ also for every $l \in [n]$, we have $W_{\sigma_l}^* : l^2(\mathcal{H})_l \to \mathcal{Z}$ defined by
$$W_{\sigma_l}^*(\{\theta_{kl}\}_k) = \sum_{k\in\sigma_l} b(\theta_{kl}, \eta_{kl}), \quad \forall\, \{\theta_{kl}\}_{k\in\sigma_l} \in l^2(\mathcal{H})_l.$$

(3) The $b$-frames operator of a weaving $b$-Besselian sequence: For every $z \in \mathcal{Z}$ and $l \in [n]$,
$$\begin{aligned} S_{\sigma_l}(z) &= W_{\sigma_l}^* W_{\sigma_l}(z) \\ &= W_{\sigma_l}^*(\{< z/\eta_{kl} >\}_{k\in\sigma_l}) \\ &= \sum_{k\in\sigma_l} b(< z/\eta_{kl} >, \eta_{kl}), \quad \forall\, z \in \mathcal{Z}. \end{aligned}$$

Similarly, we define the woven $b$-analysis and the woven $b$-synthesis operators and the woven $b$-frames operator for the woven $b$-Besselian sequence $\Lambda = \{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ :

(4) $W_\Lambda : \mathcal{Z} \to \Theta$ defined by:

$$W_\Lambda(z) = \{< z/\eta_{kl} >\}_{k\in\mathbb{I},l\in[n]}, \quad \forall\, l \in [n],\ z \in \mathcal{Z}$$

is called the woven $b-$analysis operator of $\Lambda$.

(5) $W_\Lambda^* : \Theta \to \mathcal{Z}$ defined by:

$$W_\Lambda^*(\{\theta_{kl}\}_{k\in\mathbb{I},l\in[n]}) = \sum_{k\in\mathbb{I},l\in[n]} b(\theta_{kl}, \eta_{kl})$$

is called the woven $b-$synthesis operator of $\Lambda$.

(6) $S_\Lambda : \mathcal{Z} \to \mathcal{Z}$ defined by:

$$\begin{aligned} S_\Lambda(z) &= W_\Lambda^* W(z) \\ &= W_\Lambda^*(\{< z/\eta_{kl} >\}_{k\in\mathbb{I},l\in[n]}) \\ &= \sum_{k\in\mathbb{I},l\in[n]} b(< z/\eta_{kl} >, \eta_{kl}), \quad \forall\, z \in \mathcal{Z} \end{aligned}$$

is called the woven $b$-frames operator of $\Lambda$.

Note that the Operator $S_\Lambda$ is positive and self-adjoint. Indeed, let $\Lambda = \{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ be a woven $b$-Besselian. Then, we have

$$< S_\Lambda(z), z >_{\mathcal{Z}} = \Big\langle \sum_{k\in\mathbb{I},l\in[n]} b(< z/\eta_{kl} >, \eta_{kl}), z \Big\rangle = \sum_{k\in\mathbb{I},l\in[n]} \| < z/\eta_{kl} > \|_{\mathcal{H}}^2,$$

which means that $S_\Lambda$ is positive. Moreover, we have

$$S_\Lambda = W_\Lambda^* W_\Lambda \ \text{ and } \ S_\Lambda^* = (W_\Lambda^* W_\Lambda)^* = S_\Lambda,$$

which shows that $S_\Lambda$ is self-adjoint.

**Theorem 3.8.** *Let $\Lambda = \{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ be a finite family of b-Besselian sequences in Z. Then the following statements are equivalent:*

(1) *$\Lambda$ is woven b-frames with universal woven b-frame bounds $A_\Lambda$ and $B_\Lambda$.*

(2) *$S_\Lambda$ verifie $A_\Lambda I_{\mathcal{Z}} \le S_\Lambda \le B_\Lambda I_{\mathcal{Z}}$.*

*Proof.* (1) $\Rightarrow$ (2): We have for every $z \in \mathcal{Z}$,

$$< S_\Lambda(z), z >_{\mathcal{Z}} = \Big\langle \sum_{k\in\mathbb{I},l\in[n]} b(< z/\eta_{kl} >, \eta_{kl}), z \Big\rangle = \sum_{k\in\mathbb{I},l\in[n]} \| < z/\eta_{kl} > \|_{\mathcal{H}}^2.$$

Then

$$A_\Lambda \| z \|_{\mathcal{Z}}^2 \le < S_\Lambda(z), z >_{\mathcal{Z}} \le B_\Lambda \| z \|_{\mathcal{Z}}^2,\ \forall\, z \in \mathcal{Z}.$$

So, we have

$$A_\Lambda I_{\mathcal{Z}} \le S_\Lambda \le B_\Lambda I_{\mathcal{Z}}.$$

(2) $\Rightarrow$ (1): We have

$$\| S_\Lambda \| = \| W_\Lambda^* W_\Lambda \| = \| W_\Lambda \|^2 .$$

Now, for all $z \in \mathcal{Z}$, we have

$$\sum_{k\in\mathbb{I},l\in[n]} \| < z/\eta_{kl} > \|_{\mathcal{H}}^2 = \| W_\Lambda(z) \|^2 \leq \| S_\Lambda \| \| z \|^2 \leq B_\Lambda \| z \|_{\mathcal{Z}}^2 .$$

For the lower bound, for all $z \in \mathcal{Z}$, we have

$$\begin{aligned}\sum_{k\in\mathbb{I},l\in[n]} \| < z/\eta_{kl} > \|_{\mathcal{H}}^2 &= < S_\Lambda(z), z >_{\mathcal{Z}} \\ &\geq A_\Lambda \| z \|^2 .\end{aligned}$$

□

## 4. Perturbation of weaving $b$-frame sequences

In this section we study the stability and preservation of woven $b$-frames. Suppose the $b$-adjoint operator exists for every operator in the later results.

**Theorem 4.1.** *Let $\{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ be a woven $b$-frames for $\mathcal{Z}$ with universal bounds $C_\eta$ and $D_\eta$ and let $E$ be a continuous bounded linear operator of $\mathcal{Z}$. Then $\{E\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ is woven $E^{b^*}$-$b$ frames for $\mathcal{Z}$ with universal bounds $C'_\eta = C_\eta$ and $D'_\eta = D_\eta \| E^b \|^2$, where $E^{b^*}$ designs the adjoint operator of $E^b$ in $Z$, and $E^b$ the $b$-adjoint of $E$.*

*Proof.* Let $\{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ be a woven $b$-frames for $\mathcal{Z}$ with universal bounds $C_\eta$ and $D_\eta$. Then

$$C_\eta \| z \|_{\mathcal{Z}}^2 \leq \sum_{k\in\mathbb{I},l\in[n]} \|< z/\eta_{kl} >\|_H^2 \leq D_\eta \| z \|_Z^2, \quad \forall z \in \mathcal{Z}.$$

We have

$$\sum_{k\in\mathbb{I},l\in[n]} \|< z/E\eta_{kl} >\|_{\mathcal{H}}^2 = \sum_{k\in\mathbb{I},l\in[n]} \|< E^b z/\eta_{kl} >\|_H^2, \quad \forall z \in \mathcal{Z}.$$

Hence

$$C_\eta \| (E^{b^*})^* z \|_{\mathcal{Z}}^2 \leq \sum_{k\in\mathbb{I},l\in[n]} \|< z/E\eta_{kl} >\|_{\mathcal{H}}^2 \leq D_\eta \| E^b z \|_{\mathcal{Z}}^2, \quad \forall z \in \mathcal{Z}.$$

So, we obtain

$$C \| (E^{b^*})^* z \|_{\mathcal{Z}}^2 \leq \sum_{k\in\mathbb{I},l\in[n]} \|< z/E\eta_{kl} >\|_{\mathcal{H}}^2 \leq D \| E^b \|^2 \| z \|_{\mathcal{Z}}^2, \quad \forall z \in \mathcal{Z}.$$

Hence, the family $\{E\eta_{kl}\}_{k\in I,l\in[n]}$ is woven $E^{b^*} - b$ frames for $\mathcal{Z}$ with universal bounds $C'_\eta = C_\eta$ and $D'_\eta = D_\eta \| E^b \|^2$. □

**Corollary 4.2.** *Let $\Lambda = \{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ be a woven $b$-frame with universal bounds $A_\Lambda$ and $B_\Lambda$ for $\mathcal{Z}$ and $S_\Lambda$ the woven $b$-frames operator of $\Lambda$. Then for every partition $\{\sigma_l\}_{l\in[n]}$ of $I$, we have the sequence $\{S_{\sigma_l}\eta_{kl}\}_{k\in\sigma_l}$ for every $l\in[n]$ is a weaving $b$-frames, that is, $\{S_\Lambda\eta_{kl}\}_{k\in\sigma_l,l\in[n]}$ is woven $b$-frames for $\mathcal{Z}$.*

*Proof.* Let's put $E = S_{\sigma_l}$ and $E = S_\Lambda$ in Theorem 4.1 follow. □

**Theorem 4.3.** (Transitivity of weaving $b$-frames) *Let $\Phi = \{\varphi_k\}_{k\in\mathbb{I}}$, $\Psi = \{\psi_k\}_{k\in\mathbb{I}}$ and $\Gamma = \{\gamma_k\}_{k\in\mathbb{I}}$ be $b$-frames for $\mathcal{Z}$. If $\Phi$ and $\Psi$ are woven $b$-frames with a universal lower bound $A_{\Phi\Psi}$, and $\Psi$ is woven with $\Gamma$ by a universal lower bound $A_{\Psi\Gamma}$ such that $A_{\Phi\Psi} + A_{\Psi\Gamma} - B_\Psi > 0$ and $B_\Psi$ the upper bound of the $b$-frames $\Psi$. Then the families $\Phi$ and $\Gamma$ are woven for $\mathcal{Z}$.*

*Proof.* Suppose that the families $\Phi$ and $\Psi$ are woven $b$-frames with a universal bounds $A_{\Phi\Psi}$ and $B_{\Phi\Psi}$, the families $\Psi$ and $\Gamma$ are woven $b$-frames with a universal bounds $A_{\Psi\Gamma}$ and $B_{\Psi\Gamma}$ and $B_\Psi$ the upper bound of the $b$-frames $\Psi$. Then for every $\sigma \subset I$ and $z \in \mathcal{Z}$, we have

$$\begin{aligned}(A_{\Phi\Psi} + A_{\Psi\Gamma})\|z\|^2 &\leq \sum_{k\in\sigma} \| < z/\varphi_k > \|^2 + \sum_{k\in\sigma^c} \| < z/\psi_k > \|^2 \\ &\quad + \sum_{k\in\sigma} \| < z/\psi_k > \|^2 + \sum_{k\in\sigma^c} \| < z/\gamma_k > \|^2 \\ &\leq (B_{\Phi\Psi} + B_{\Psi\Gamma})\|z\|^2.\end{aligned}$$

That is

$$\begin{aligned}(A_{\Phi\Psi} + A_{\Psi\Gamma})\|z\|^2 &\leq \sum_{k\in\sigma} \| < z/\varphi_k > \|^2 + \sum_{k\in\sigma^c} \| < z/\gamma_k > \|^2 \\ &\quad + \sum_{k\in\mathbb{I}} \| < z/\psi_k > \|^2 \\ &\leq (B_{\Phi\Psi} + B_{\Psi\Gamma})\|z\|^2.\end{aligned}$$

As $\Psi$ is $b$-frames, $\sum_{k\in\mathbb{I}} \| < z/\psi_k > \|^2 \leq B_\Psi\|z\|^2$, so we obtain

$$\begin{aligned}(A_{\Phi\Psi} + A_{\Psi\Gamma} - B_\Psi)\|z\|^2 &\leq (\sum_{k\in\sigma} \| < z/\varphi_k > \|^2 + \sum_{k\in\sigma^c} \| < z/\gamma_k > \|^2) \\ &\leq (B_{\Phi\Psi} + B_{\Psi\Gamma})\|z\|^2.\end{aligned}$$

Hence, $\{\varphi_k\}_{k\in\mathbb{I}}$ and $\{\gamma_k\}_{i\in\mathbb{I}}$ are woven $b$-frames. □

Note that $T_\phi$ be the $b$-synthesis operator for the $b$-frames $\phi = \{\varphi_k\}_{k\in\mathbb{I}} \subset \mathcal{B}$ and $T_\psi$ be the $b$-synthesis operator for the $b$-frames $\psi = \{\psi_k\}_{k\in\mathbb{I}} \subset \mathcal{B}$ such that for every $\{\theta_k\}_{k\in\mathbb{I}} \in l^2(H)$, we have

$$T_\phi(\theta_k) = \sum_{k\in\mathbb{I}} b(\theta_k, \varphi_k),$$
$$T_\psi(\theta_k) = \sum_{k\in\mathbb{I}} b(\theta_k, \psi_k),$$
$$T_\phi^\sigma(\theta_k) = \sum_{k\in\sigma} b(\theta_k, \varphi_k),$$
$$T_\psi^\sigma(\theta_k) = \sum_{k\in\sigma} b(\theta_k, \psi_k).$$

**Theorem 4.4.** *Let $\phi = \{\varphi_k\}_{k\in\mathbb{I}} \subset B$ and $\psi = \{\psi_k\}_{k\in\mathbb{I}} \subset B$ be b-frames for $\mathcal{Z}$ with b-frames bounds $A_\phi$, $B_\phi$, $A_\psi$ and $B_\psi$ such that*

$$\sqrt{B_\phi} + \sqrt{B_\psi} \leq \frac{A_\phi}{2\alpha}$$

*with $0 < \alpha < 1$ and*

$$\|T_\phi^\sigma - T_\psi^\sigma\| \leq \|T_\phi - T_\psi\| \leq \alpha, \tag{4.1}$$

*where $T_\phi$(respectively $T_\psi$) is the b-synthesis operator for the b-frames $\{\varphi_k\}_{k\in I}$ (respectively $\{\psi_k\}_{k\in I}$). Then for every $\sigma \subset I$, the family $\{\varphi_k\}_{k\in\sigma} \cup \{\psi_k\}_{k\in\sigma^c}$ is a b-frames for $\mathcal{Z}$ with b-frames bounds $B_\phi + B_\psi$ and $\frac{A_\phi}{2}$. Thus, $\phi$ and $\psi$ are woven b-frames.*

*Proof.* Let $T_\phi$ (respectively, $T_\psi$) be the $b$-synthesis operator for the $b$-frames $\{\varphi_k\}_{k\in I}$ (respectively, $\{\psi_k\}_{k\in I}$) and $\sigma \subset I$. For each $\sigma \subset I$, we put

$$M = \|\sum_{k\in\sigma} b(< z/\varphi_k >, \varphi_k) - \sum_{k\in\sigma} b(< z/\psi_k >, \psi_k)\|_{\mathcal{Z}}.$$

Then, we have

$$\begin{aligned}
M &= \|\sum_{k\in\sigma} b(< z/\varphi_k >, \varphi_k) - \sum_{k\in\sigma} b(< z/\psi_k >, \psi_k)\|_{\mathcal{Z}} \\
&= \|T_\phi^\sigma (T_\phi^\sigma)^* z - T_\psi^\sigma (T_\psi^\sigma)^* z\| \\
&\leq \|(T_\phi^\sigma T_\phi^{\sigma *} - T_\phi^\sigma T_\psi^{\sigma *}) z\| + \|(T_\phi^\sigma T_\psi^{\sigma *} - T_\psi^\sigma T_\psi^{\sigma *}) z\| \\
&\leq \|T_\phi^\sigma\| \|T_\phi^{\sigma *} - T_\psi^{\sigma *}\| \|z\| + \|T_\phi^\sigma - T_\psi^\sigma\| \|T_\psi^{\sigma *}\| \|z\| \\
&\leq \alpha(\|T_\phi\| + \|T_\psi\|)\|z\| \\
&\leq \alpha(\sqrt{B_\phi} + \sqrt{B_\psi})\|z\| \\
&\leq \frac{A_\phi}{2}\|z\|.
\end{aligned}$$

Therefore, for every $z \in \mathcal{Z}$, we put

$$R = \| \sum_{k\in\sigma} b(< z/\varphi_k >, \varphi_k) + \sum_{k\in\sigma^c} b(< z/\psi_k >, \psi_k)\|.$$

Then,

$$\begin{aligned} R &= \| \sum_{k\in\mathbb{I}} b(< z/\psi_k >, \psi_k) + \sum_{k\in\sigma} b(< z/\varphi_k >, \varphi_k) - \sum_{k\in\sigma} b(< z/\psi_k >, \psi_k)\| \\ &\geq \| \sum_{k\in\mathbb{I}} b(< z/\psi_k >, \psi_k)\| - \| \sum_{k\in\sigma} b(< z/\varphi_k >, \varphi_k) - \sum_{k\in\sigma} b(< z/\psi_k >, \psi_k)\|_{\mathcal{Z}} \\ &\geq A_\phi \|z\| - \frac{A_\phi}{2}\|z\| \\ &= \frac{A_\phi}{2}\|z\|. \end{aligned}$$

So, the lower $b$-frames bounds is $\frac{A_\phi}{2}$ and the upper bounds is $B_\phi + B_\psi$. Thus $\phi$ and $\psi$ are woven $b$-frames. □

**Theorem 4.5.** *Let $\Lambda = \{\eta_k\}_{k\in\mathbb{I}}$ be a $b$-frames for $\mathcal{Z}$ with $b$-frames bounds $A_\Lambda$, $B_\Lambda$ and let $E$ be a bounded operator such that*

$$\| I_z - E^b \|^2 \leq \frac{A_\Lambda}{B_\Lambda}.$$

*Then $\{\eta_k\}_{k\in\mathbb{I}}$ and $\{E\eta_k\}_{k\in\mathbb{I}}$ are woven $b$-frames with universal lower bound $(\sqrt{A_\Lambda} - \sqrt{B_\Lambda} \| I_z - E^b \|)^2$.*

*Proof.* For every $\sigma \subset I$ and for every $z \in \mathcal{Z}$, we have

$$\begin{aligned} L &= (\sum_{k\in\sigma} \|< z/\eta_k >\|^2 + \sum_{k\in\sigma^c} \|< z/E\eta_k >\|^2)^{\frac{1}{2}} \\ L &= (\sum_{k\in\sigma} \|< z/\eta_k >\|^2 + \sum_{k\in\sigma^c} \|< z/\eta_k > - < z/\eta_k > + < z/E\eta_k >\|^2)^{\frac{1}{2}} \\ &= (\sum_{k\in\sigma} \|< z/\eta_k >\|^2 + \sum_{k\in\sigma^c} \|< z/\eta_k > - < z - E^b z/\eta_k >)^{\frac{1}{2}} \\ &= (\sum_{k\in\sigma} \|< z/\eta_k >\|^2 + \sum_{k\in\sigma^c} \|< z/\eta_k > - < (I_{\mathcal{Z}} - E^b)z/\eta_k >\|^2)^{\frac{1}{2}} \\ &\geq (\sum_{k\in\mathbb{I}} \|< z/\eta_k >\|^2)^{\frac{1}{2}} - (\sum_{k\in\sigma^c} \|< (I_{\mathcal{Z}} - E^b)z/\eta_k >\|^2)^{\frac{1}{2}} \\ &\geq \sqrt{A_\Lambda} \| z \| - \sqrt{B_\Lambda} \| (I_{\mathcal{Z}} - E^b) \| \| z \| \\ &\geq (\sqrt{A_\Lambda} - \sqrt{B_\Lambda} \| (I_{\mathcal{Z}} - E^b) \|) \| z \| . \end{aligned}$$

Since $(\sqrt{A_\Lambda} - \sqrt{B_\Lambda} \parallel (I_{\mathcal{Z}} - E^b) \parallel \geq 0$, so $\{\eta_k\}_{k\in\sigma} \cup \{E\eta_k\}_{k\in\sigma^c}$ is a $b$-frames. Hence, $\{\eta_k\}_{k\in\mathbb{I}}$ and $\{E\eta_k\}_{k\in\mathbb{I}}$ are woven $b$-frames with the universal lower bound $(\sqrt{A_\Lambda} - \sqrt{B_\Lambda} \parallel I_{\mathcal{Z}} - E^b \parallel)^2$. □

**Theorem 4.6.** *Let $\{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ be a woven $K$-$b$-frames for $\mathcal{Z}$ with universal bounds $C$ and $D$ and let $P \in \mathcal{L}(\mathcal{Z})$ such that $\mathcal{R}(P) \subseteq \mathcal{R}(K)$. Then $\{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ is a woven $P$-$b$-frames for $\mathcal{Z}$.*

*Proof.* Let $\{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ be a woven $K$-$b$-frames for $\mathcal{Z}$ with universal bounds $C$ and $D$, then

$$C \parallel K^*z \parallel_{\mathcal{Z}}^2 \leq \sum_{k\in\mathbb{I},l\in[n]} \parallel < z/\eta_{kl} > \parallel_{\mathcal{H}}^2 \leq D \parallel z \parallel_{\mathcal{Z}}^2, \quad \forall\, z \in \mathcal{Z}.$$

Let $P \in \mathcal{L}(\mathcal{Z})$ such that $\mathcal{R}(P) \subseteq \mathcal{R}(K)$. Then by Theorem 2.2, there exists $\lambda > 0$ such that $PP^* \leq \lambda^2 KK^*$, so we have

$$< PP^*z, z >_{\mathcal{Z}} \leq \lambda^2 < KK^*z, z >_{\mathcal{Z}}, \ \forall\, z \in \mathcal{Z},$$

which implies

$$< P^*z, P^*z >_{\mathcal{Z}} \leq \lambda^2 < K^*z, K^*z >_{\mathcal{Z}},$$

hence

$$\parallel P^*z \parallel_{\mathcal{Z}}^2 \leq \lambda^2 \parallel K^*z \parallel_{\mathcal{Z}}^2 .$$

Therefore,

$$\frac{C}{\lambda^2} \parallel P^*z \parallel_{\mathcal{Z}}^2 \leq C \parallel K^*z \parallel_{\mathcal{Z}}^2 \leq \sum_{k\in\mathbb{I},l\in[n]} \parallel < z/\eta_{kl} > \parallel_{H}^2 \leq D \parallel z \parallel_{\mathcal{Z}}^2 .$$

Thus, $\{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ is a woven $P$-$b$-frames for $\mathcal{Z}$ with universal bounds $C' = \dfrac{C}{\lambda^2}$ and $D' = D$. □

**Theorem 4.7.** *Let $K \in \mathcal{L}(\mathcal{Z})$ and $\{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ be a woven $K$-$b$-frames for $\mathcal{Z}$ with universal bounds $C$ and $D$ and let $U$ be a continous bounded linear operator. Then $\{U\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ is woven $(U^{b^*}K)$-$b$ frames for $\mathcal{Z}$ with universal bounds $C' = C$ and $D' = D \parallel U^b \parallel^2$, where $U^{b^*}$ designs the adjoint operator of $U^b$ in $\mathcal{Z}$, and $U^b$ the $b$-adjoint of $U$.*

*Proof.* Let $K \in \mathcal{L}(\mathcal{Z})$ and $\{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ be a woven $K$-$b$-frames for $\mathcal{Z}$ with universal bounds $C$ and $D$. Then

$$C \parallel K^*z \parallel_{Z}^2 \leq \sum_{k\in\mathbb{I},l\in[n]} \parallel < z/\eta_{kl} > \parallel_{\mathcal{H}}^2 \leq D \parallel z \parallel_{\mathcal{Z}}^2, \quad \forall\, z \in \mathcal{Z}.$$

Since,

$$\sum_{k\in\mathbb{I},l\in[n]} \|< z/U\eta_{kl} >\|^2_{\mathcal{H}} = \sum_{k\in\mathbb{I},l\in[n]} \|< U^b z/\eta_{kl} >\|^2_{\mathcal{H}}, \quad \forall\, z \in \mathcal{Z},$$

we have

$$C \parallel K^* U^b z \parallel^2_{\mathcal{Z}} \leq \sum_{k\in\mathbb{I},l\in[n]} \|< z/U\eta_{kl} >\|^2_{\mathcal{H}} \leq D \parallel U^b z \parallel^2_{\mathcal{Z}}, \quad \forall\, z \in \mathcal{Z}.$$

So, we obtain

$$C \parallel (U^{b^*} K)^* z \parallel^2_{\mathcal{Z}} \leq \sum_{k\in\mathbb{I},l\in[n]} \|< z/U\eta_{kl} >\|^2_{\mathcal{H}} \leq D \parallel U^b \parallel^2 \parallel z \parallel^2_{\mathcal{Z}}, \quad \forall\, z \in \mathcal{Z}.$$

Hence, the family $\{U\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ is woven $(U^{b^*}K)$-$b$-frames for $\mathcal{Z}$ with universal bounds $C' = C$ and $D' = D \parallel U^b \parallel^2$ . □

**Theorem 4.8.** *Let $\{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ be a woven $K$-$b$-frames for $\mathcal{Z}$ with universal bounds $C$ and $D$ and let $U$ be a bounded linear operator such that $U^b$ exist and it has a closed range with $U^b K = K U^b$. If $\mathcal{R}(K^*) \subset \mathcal{R}(U^b)$, then the family $\{U\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ is woven $K$-$b$-frames for $\mathcal{Z}$ with universal bounds $C \parallel U^{b^\dagger} \parallel^{-2}$ and $D \parallel U^b \parallel^2$ .*

*Proof.* Let $\{\sigma_l\}_{l\in[n]}$ be any partition of $I$. Then, for every $z \in \mathcal{Z}$, we obtain

$$\begin{aligned} \sum_{l\in[n]} \sum_{k\in\sigma_l} \|< z/U\eta_{kl} >\|^2_{\mathcal{H}} &= \sum_{l\in[n]} \sum_{k\in\sigma_l} \|< U^b z/\eta_{kl} >\|^2_{\mathcal{H}} \\ &\leq D \parallel U^b \parallel^2 \parallel z \parallel^2_{Z}, \quad \forall\, z \in \mathcal{Z}. \end{aligned}$$

Since $U^b K = K U^b$, $K^* U^{b^*} = U^{b^*} K^*$. By Lemma 2.1 and Theorem 2.1, we obtain

$$\begin{aligned} \parallel K^* z \parallel^2_{\mathcal{Z}} &= \parallel U^b U^{b^\dagger} K^* z \parallel^2_{\mathcal{Z}} \\ &= \parallel U^{b^{\dagger^*}} U^{b^*} K^* z \parallel^2_{\mathcal{Z}} \\ &= \parallel U^{b^{\dagger^*}} K^* U^{b^*} z \parallel^2_{\mathcal{Z}} \\ &\leq \parallel U^{b^\dagger} \parallel^2 \parallel K^* U^{b^*} z \parallel^2_{\mathcal{Z}}, \quad \forall\, z \in \mathcal{Z} \end{aligned}$$

and

$$\begin{aligned} \sum_{l\in[n]} \sum_{k\in\sigma_l} \|< z/U\eta_{kl} >\|^2_{\mathcal{H}} &= \sum_{l\in[n]} \sum_{k\in\sigma_l} \|< U^b z/\eta_{kl} >\|^2_{\mathcal{H}} \\ &\geq C \parallel K^* U^{b^*} z \parallel^2_{\mathcal{Z}} \\ &\geq C \parallel U^{b^\dagger} \parallel^{-2} \parallel K^* z \parallel^2_{\mathcal{Z}} . \end{aligned}$$

Hence, we have

$$C \parallel U^{b^{\dagger}} \parallel^{-2} \parallel K^* z \parallel_{\mathcal{Z}}^2 \leq \sum_{l\in[n]} \sum_{k\in\sigma_l} \parallel < z/U\eta_{kl} > \parallel_{\mathcal{H}}^2 \leq D \parallel U^b \parallel^2 \parallel z \parallel_{\mathcal{Z}}^2, \ \ \forall z \in \mathcal{Z}.$$

Then, the family $\{U\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ is woven $K$-$b$-frames for $\mathcal{Z}$ with universal bounds $C \parallel U^{b^{\dagger}} \parallel^{-2}$ and $D \parallel U^b \parallel^2$ . □

**Theorem 4.9.** *Suppose $K \in \mathcal{L}(\mathcal{Z})$ such that $K^b$ exist and has closed range, if $\{\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ is a woven $K$-$b$-frames for $\mathcal{Z}$ with universal bounds $C$ and $D$, then the family $\{K\eta_{kl}\}_{k\in\mathbb{I},l\in[n]}$ is woven $K^{b^*}$-$b$-frames for $\mathcal{Z}$ with universal bounds $C \parallel K^{b^{\dagger}} \parallel^{-2}$ and $D \parallel K^b \parallel^2$ .*

*Proof.* Suppose that $K^b$ has closed range and for each $z \in \mathcal{R}(K^b)$. Then, we have

$$z = K^b K^{b^{\dagger}} z = (K^b K^{b^{\dagger}})^* z = K^{b^{\dagger^*}} K^{b^*} z.$$

So, we have

$$\parallel z \parallel^2 = \parallel K^{b^{\dagger^*}} K^{b^*} z \parallel^2 \leq \parallel K^{b^{\dagger}} \parallel^2 \parallel K^{b^*} z \parallel^2 .$$

Let $\{\sigma_l\}_{l\in[n]}$ be any partition of $I$. Then, for every $z \in \mathcal{Z}$, we have

$$C \parallel K^* z \parallel^2 \leq \sum_{l\in[n]} \sum_{k\in\sigma_l} \parallel < z/\eta_{kl} > \parallel_{\mathcal{H}}^2 \leq D \parallel z \parallel_{\mathcal{Z}}^2, \quad \forall z \in \mathcal{Z}.$$

Hence, for each $z \in \mathcal{Z}$, we have

$$\begin{aligned} C \parallel K^{b^{\dagger}} \parallel^{-2} \parallel K^b z \parallel_{\mathcal{Z}}^2 &\leq C \parallel K^{b^*} K^b z \parallel^2 \\ &\leq \sum_{l\in[n]} \sum_{k\in\sigma_l} \parallel < K^b z/\eta_{kl} > \parallel_{\mathcal{H}}^2 \\ &\leq D \parallel K^b \parallel^2 \parallel z \parallel_{\mathcal{Z}}^2 . \end{aligned}$$

□

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