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ON DOUBLE COMPLEX AL-ZUGHAIR TRANSFORM TECHNIQUE AND ITS PROPERTIES

Walaa Hussein Ahmed[1], Baneen Sadeq Mohammed Ali[2] and Emad A. Kuffi[3]

[1]Department of Mathematics, University of Kerbala,
College of Education for Pure Science, Kerbala, Iraq
e-mail: `lamia.walaa.h@uokerbala.edu.iq`

[2]Department of Mathematics, University of Kerbala,
College of Education for Pure Science, Kerbala, Iraq
e-mail: `baneen.s@uokerbala.edu.iq`

[3]Department of Mathematics, Mustansiriyah University,
College of Basic Education, Baghdad, Iraq
e-mail: `emad.kuffi@uomustansiriyah.edu.iq`

Abstract. In this paper introduced double complex Al-Zughair transform technique, where it's important properties and its capability to evaluate the solution of special equations from ordinary and partial differential equations with variable coefficients.

1. Introduction

Due to the importance of differential equations and their multiplicity in life, physical, and engineering applications and their branches, many integral transforms have emerged, which considered modified or developed cases of the Laplace transform, such as the Sumudu, ElZaki, Al Jafari, Sadik, SEE, complex Sadik, complex SEE, AEM, Al-Tememe transforms and others and all

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[0]Corresponding author: Emad A. Kuffi(`emad.kuffi@uomustansiriyah.edu.iq`).

the transforms that were mentioned in the same interval, while the complex Al-Zughair transform it varies with the transform interval as well as the transform kernel [1-14].

In this research paper, we will present the double complex Al-Zughair transform, its inverse and its properties along with illustrative examples that aim to find an exact solution to special types of differential equations with variable coefficients.

Definition 1.1. ([1]) Suppose that f is a function defined on (a, b). The integral transform technique for f whose symbol $\underline{\mathrm{F}}(p)$ such as:

$$\mathrm{F}(p) = \int_a^b k(p,t) f(t) dt,$$

where $k(p,t)$ is a function of two factors, p and t and it's called the kernel of the transformation and $\underline{a}, \underline{b}$ belong to set of real numbers such that above integral convergence.

Definition 1.2. ([1]) Suppose that f is a function defined on $[1, e]$, complex Al-Zughair integral transform technique is characterized via the following integral:

$$Z^C\{f(t)\} = \int_1^e \frac{(\ln(t))^{ip}}{t} f(t) dt = F(ip).$$

Such that definitional converges and p belong to the set of real numbers, p > -1, and $\frac{(\ln(t))^{ip}}{t}$ is the kernel of this transform, $i = \sqrt{-1}$.

2. Methodology

Definition 2.1. The double complex Al-Zughair integral transform is defined as:

$$\begin{aligned} z_2^c(f(x,t)) &= \int_1^e \int_1^e \frac{(\ln(x))^{ip}}{x} \frac{(\ln(t))^{is}}{t} f(x,t) dt dx \\ &= \int_1^e \int_1^e \frac{(\ln(x))^{ip} (\ln(t))^{is}}{xt} f(x,t) dt dx \\ &= F\left(p_i, s_i\right), \end{aligned}$$

$i \in C$, p and s are two parameters and $p > -1$, $s > -1$, $e = 2.7183$.

Definition 2.2. The inverse of z_2^c of $F\left(p_i, s_i\right)$ denoted by $z_2^{c^{-1}}$ and defined as:

$$z_2^{c^{-1}}\left(z_2^c(f(x,t)\right) = f(x,t)\ .$$

Theorem 2.3. (1) $z_2^c(k) = \frac{k}{(is+1)(ip+1)}$, *where k is a constant.*

(2) $z_2^c\left((\ln(x))^n(\ln(t))^m\right) = \frac{1}{(ip+(n+1))(is+(m+1))}$.
(3) $z_2^c((\ln\ln(x))(\ln\ln(t))) = \frac{-1}{(ip+1)^2}\cdot\frac{-1}{(is+1)^2}$.
(4)

$$\begin{aligned} &z_2^c\left((\ln\ln(x))^n(\ln\ln(t))^m\right) \\ &\quad = \int_1^e\int_1^e \frac{(\ln(x))^{ip}}{x}\frac{(\ln(t))^{is}}{t}(\ln\ln(x))^n(\ln\ln(t))^m \\ &\quad = \int_1^e \frac{(\ln(t))^{is}}{t}(\ln\ln(t))^m\left(\int_1^e \frac{(\ln(x))^{ip}}{x}(\ln\ln(x))^n dx\right)dt. \end{aligned}$$

(5) $z_2^c\left(\sin\left(a\ln(\ln(x))\sin(b\ln(\ln(t))\right)\right) = \frac{ab}{(a^2-(p-i)^2)(b^2-(s-i)^2)}$.
(6) $z_2^c\left(\cos\left(a\ln(\ln(x))\cos(b\ln(\ln(t))\right)\right) = \frac{(pi+1)}{((pi+1)^2+a^2)}\cdot\frac{(si+1)}{((si+1)^2+a^2)}$.
(7) $z_2^c\left(\sinh\left(a\ln(\ln(x))\sinh(b\ln(\ln(t))\right)\right) = \frac{ab}{((p+i)^2-a^2)((s+i)^2-b^2)}$.
(8) $z_2^c\left((\ln(x))^n(\ln(t))^m f(x,t)\right) = z_2^c(ip+n,\ is+m)$.

Proof. (1) By the definition,

$$\begin{aligned} z_2^c(k) &= \int_1^e\int_1^e \frac{(\ln(x))^{ip}}{x}\frac{(\ln(t))^{is}}{t}k\,dtdx \\ &= k\int_1^e\int_1^e \frac{(\ln(x))^{ip}}{x}\frac{(\ln(t))^{is}}{t}dtdx \\ &= k\int_1^e \frac{(\ln(x))^{ip}}{x}\left[\frac{(\ln(t))^{is+1}}{(is+1)}\right|_1^e dx \\ &= \frac{k}{(is+1)}\int_1^e \frac{(\ln(x))^{ip}}{x}dx \\ &= \frac{k}{(is+1)(ip+1)}. \end{aligned}$$

(2) By the definition,

$$\begin{aligned} z_2^c\left((\ln(x))^n(\ln(t))^m\right) &= \int_1^e\int_1^e \frac{(\ln(x))^{ip}}{x}\frac{(\ln(t))^{is}}{t}(\ln(x))^n(\ln(t))^m dtdx \\ &= \int_1^e\int_1^e \frac{(\ln(x))^{ip+n}}{x}\frac{(\ln(t))^{is+m}}{t}dtdx \\ &= \int_1^e \frac{(\ln(x))^{ip+n}}{x}\left(\int_1^e \frac{(\ln(t))^{is+m}}{t}dt\right)dx \\ &= \frac{1}{(ip+(n+1))(is+(m+1))}. \end{aligned}$$

(3) By the definition,

$$\begin{aligned}
&z_2^c((\ln\ln(x))(\ln\ln(t))) \\
&= \int_1^e \int_1^e \frac{(\ln(x))^{ip}}{x}\frac{(\ln(t))^{is}}{t}(\ln\ln(x))(\ln\ln(t))dtdx \\
&= \int_1^e \frac{(\ln(t))^{is}}{t}(\ln\ln(t))\left(\int_1^e \frac{(\ln(x))^{ip}}{x}(\ln\ln(x))dx\right)dt \\
&= \int_1^e \frac{(\ln(t))^{is}}{t}(\ln\ln(t))\left[\frac{(\ln(x))^{ip+1}}{ip+1}(\ln\ln(x))\Big|_1^e - \int_1^e \frac{(\ln(x))^{ip}}{x(ip+1)}dx\right]dt \\
&= \frac{-1}{(ip+1)^2}\int_1^e \frac{(\ln(t))^{is}}{t}(\ln\ln(t))dt \\
&= \frac{-1}{(ip+1)^2}\cdot\frac{-1}{(is+1)^2}.
\end{aligned}$$

(4) By the definition,

$$\begin{aligned}
&z_2^c\left((\ln\ln(x))^n(\ln\ln(t))^m\right) \\
&= \int_1^e \int_1^e \frac{(\ln(x))^{ip}}{x}\frac{(\ln(t))^{is}}{t}(\ln\ln(x))^n(\ln\ln(t))^m \\
&= \int_1^e \frac{(\ln(t))^{is}}{t}(\ln\ln(t))^m\left(\int_1^e \frac{(\ln(x))^{ip}}{x}(\ln\ln(x))^n dx\right)dt.
\end{aligned}$$

If $n=2$,

$$\begin{aligned}
&z_2^c\left((\ln\ln(x))^2(\ln\ln(t))^m\right) \\
&= \int_1^e \frac{(\ln(t))^{is}}{t}(\ln\ln(t))^m \\
&\quad\times\left[\frac{(\ln(x))^{ip+1}}{ip+1}(\ln\ln(x))^2\Big|_1^e - \int_1^e \frac{(\ln(x))^{ip}}{x(ip+1)}\cdot 2(\ln(\ln(x)))dx\right]dt \\
&= \int_1^e \frac{(\ln(t))^{is}}{t}(\ln\ln(t))^m\left[\frac{-2}{(ip+1)}\int_1^e \frac{(\ln(x))^{ip}}{x}\cdot(\ln(\ln(x)))dx\right]dt \\
&= \int_1^e \frac{(\ln(t))^{is}}{t}(\ln\ln(t))^m\left[\frac{-2}{(ip+1)}\frac{-1}{(ip+1)^2}\right]dt \\
&= \frac{2}{(ip+1)^3}\int_1^e \frac{(\ln(t))^{is}}{t}(\ln\ln(t))^m dt.
\end{aligned}$$

If $m=2, n=2$,

$$z_2^c\left((\ln\ln(x))^2(\ln\ln(t))^2\right) = \frac{2(-1)^2}{(ip+1)^3}\cdot\frac{2(-1)^2}{(is+1)^3}.$$

If $m = 3, n = 3$,

$$
\begin{aligned}
&z_2^c\left((\ln\ln(x))^3(\ln\ln(t))^3\right)\\
&= \int_1^e \frac{(\ln(t))^{is}}{t}(\ln\ln(t))^m\left[\frac{(\ln(x))^{ip+1}}{ip+1}(\ln\ln(x))^3\Big|_1^e\right.\\
&\quad\left. - \int_1^e \frac{(\ln(x))^{ip}}{(ip+1)x\ln(x)}\cdot 3(\ln(\ln(x)))^2 dx\right] dt\\
&= \int_1^e \frac{(\ln(t))^{is}}{t}(\ln\ln(t))^m\left[\frac{3}{(ip+1)}\int_1^e \frac{(\ln(x))^{ip}}{x}\cdot(\ln(\ln(x)))^2 dx\right] dt\\
&= \int_1^e \frac{(\ln(t))^{is}}{t}(\ln\ln(t))^m\left[\frac{6}{(ip+1)^4}\right] dt\\
&= \frac{6}{(ip+1)^4}\int_1^e \frac{(\ln(t))^{is}}{t}(\ln\ln(t))^m dt\\
&= \frac{36(-1)^3(-1)^3}{(ip+1)^4(is+1)^4}.
\end{aligned}
$$

We know that for all $m, n > 0$,

$$
z_2^c\left((\ln\ln(x))^n(\ln\ln(t))^m\right) = \frac{(-1)^n(-1)^m n! m!}{(ip+1)^{n+1}(is+1)^{m+1}}.
$$

(5) By the definition,

$$
\begin{aligned}
&z_2^c(\sin(a\ln(\ln(x))\sin(b\ln(\ln(t)))\\
&= \int_1^e\int_1^e \frac{(\ln(x))^{ip}}{x}\frac{(\ln(t))^{is}}{t}\sin(a\ln(\ln(x))\sin(b\ln(\ln(t))dtdx\\
&= \int_1^e \frac{(\ln(x))^{ip}}{x}\sin\left(a\ln(\ln(x))\int_1^e \frac{(\ln(t))^{is}}{t}\sin(b\ln(\ln(t))dt\right)dx\\
&= \frac{-a}{(a^2-(p-i)^2)}\cdot\frac{-b}{(b^2-(s-i)^2)}\\
&= \frac{ab}{(a^2-(p-i)^2)\,(b^2-(s-i)^2)}.
\end{aligned}
$$

(6) By the definition,

$$
\begin{aligned}
&z_2^c(\cos(a\ln(\ln(x))\cos(b\ln(\ln(t)))\\
&= \int_1^e\int_1^e \frac{(\ln(x))^{ip}}{x}\frac{(\ln(t))^{is}}{t}\cos(a\ln(\ln(x))\cos(b\ln(\ln(t))dtdx
\end{aligned}
$$

$$= \int_1^e \frac{(\ln(x))^{ip}}{x} \cos\left(a\ln(\ln(x)) \left(\int_1^e \frac{(\ln(t))^{is}}{t} \cos(b\ln(\ln(t)\right) dt\right) dx$$

$$= \frac{(pi+1)}{((pi+1)^2+a^2)} \cdot \frac{(si+1)}{((si+1)^2+a^2)}.$$

(7) By the definition,

$$z_2^c(\sinh(a\ln(\ln(x))\sinh(b\ln(\ln(t)))$$

$$= \int_1^e \int_1^e \frac{(\ln(x))^{ip}}{x} \frac{(\ln(t))^{is}}{t} \sinh(a\ln(\ln(x))\sin h(b\ln(\ln(t))dtdx$$

$$= \int_1^e \frac{(\ln(x))^{ip}}{x} \sinh\left(a\ln(\ln(x)) \int_1^e \frac{(\ln(t))^{is}}{t} \sin h(b\ln(\ln(t))dt\right) dx$$

$$= \frac{-a}{((p+i)^2-a^2)} \cdot \frac{-b}{((s+i)^2-b^2)}$$

$$= \frac{ab}{((p+i)^2-a^2)\,((s+i)^2-b^2)}.$$

(8) By the definition,

$$z_2^c\left((\ln(x))^n(\ln(t))^m f(x,t)\right)$$

$$= \int_1^e \int_1^e \frac{(\ln(x))^{ip}}{x} \frac{(\ln(t))^{is}}{t} (\ln(x))^n(\ln(t))^m f(x,t)dtdx$$

$$= \int_1^e \frac{(\ln(x))^{ip+n}}{x} \left(\int_1^e \frac{(\ln(t))^{is+m}}{t} f(x,t)dt\right) dx$$

$$= \int_1^e \frac{(\ln(x))^{ip+n}}{x} F(x, is+m)dx$$

$$= V(ip+n, is+m)$$

$$= z_2^c(ip+n, is+m).$$

□

Example 2.4.

(1) $z_2^c(4) = \frac{4}{(is+1)(ip+1)}$.
(2) $z_2^c\left((\ln(x))^2(\ln(t))^{-1}\right) = \frac{1}{is(ip+3)}$.
(3) $z_2^c\left((\ln\ln(x))^2(\ln\ln(t))^3\right) = \frac{-12}{(ip+1)^3(is+1)^4}$.
(4) $z_2^c\left(\sinh\left(\ln(\ln(x))\sinh(2\ln(\ln(t))\right)\right) = \frac{2}{((p+i)^2-1)((s+i)^2-4)}$.

Theorem 2.5. $z_2^c\left(\ln(x)\frac{\partial f(\ln x, \ln t)}{\partial x}\right) = z^c\left(f(1,\ln t) - (pi+1)z_2^c(f(\ln x, \ln t)\right).$

Proof. By the definition

$$
\begin{aligned}
&z_2^c\left(\ln(x)\frac{\partial f(\ln x,\ln t)}{\partial x}\right)\\
&\quad=\int_1^e\int_1^e\frac{(\ln(x))^{ip}}{x}\frac{(\ln(t))^{is}}{t}\ln(x)\frac{\partial f(\ln x,\ln t)}{\partial x}dxdt\\
&\quad=\int_1^e\int_1^e\frac{(\ln(x))^{ip+1}}{x}\frac{(\ln(t))^{is}}{t}\frac{\partial f(\ln x,\ln t)}{\partial x}dxdt\\
&\quad=\int_1^e\left[(\ln x)^{ip+1}f(\ln x,\ln t)\big|_1^e-(pi+1)\int_1^e f(\ln x,\ln t)\frac{(\ln x)^{pi}}{x}\right]\frac{(\ln(t))^{is}}{t}dt\\
&\quad=\int_1^e\left[f(1,\ln t)-(pi+1)\int_1^e f(\ln x,\ln t)\frac{(\ln x)^{pi}}{x}\right]\frac{(\ln(t))^{is}}{t}dt\\
&\quad=z^c\left(f(1,\ln t)-(pi+1)z_2^c(f(\ln x,\ln t)\right).
\end{aligned}
$$

□

Theorem 2.6.

$$
\begin{aligned}
z_2^c\left((\ln(x))^2\frac{\partial^2 f(\ln x,\ln t)}{\partial x^2}\right)&=\frac{\partial}{\partial x}z^c\left(f(1,\ln t)-(pi+2)z_2^c(f(1,\ln t)\right)\\
&\quad+(ip+1)(pi+2)z_2^c(f(\ln x,\ln t)).
\end{aligned}
$$

Proof. By the definition

$$
\begin{aligned}
&z_2^c\left((\ln(x))^2\frac{\partial^2 f(\ln x,\ln t)}{\partial x^2}\right)\\
&\quad=\int_1^e\int_1^e\frac{(\ln(x))^{ip}}{x}\frac{(\ln(t))^{is}}{t}(\ln(x))^2\frac{\partial^2 f(\ln x,\ln t)}{\partial x^2}dxdt\\
&\quad=\int_1^e\int_1^e\frac{(\ln(x))^{ip+2}}{x}\frac{(\ln(t))^{is}}{t}\frac{\partial^2 f(\ln x,\ln t)}{\partial x^2}dxdt\\
&\quad=\int_1^e\left[(\ln x)^{ip+2}\frac{\partial f(\ln x,\ln t)}{\partial x}\bigg|_1^e-(pi+2)\int_1^e\frac{\partial f(\ln x,\ln t)}{\partial x}\frac{(\ln x)^{pi+1}}{x}dx\bigg|_1^e\right]\\
&\qquad\times\frac{(\ln(t))^{is}}{t}dt\\
&\quad=\int_1^e\left[\frac{\partial f(1,\ln t)}{\partial x}-(pi+2)\int_1^e\frac{\partial f(\ln x,\ln t)}{\partial x}\frac{(\ln x)^{pi+1}}{x}dx\right]\frac{(\ln(t))^{is}}{t}dt\\
&\quad=\int_1^e\frac{\partial f(1,\ln t)}{\partial x}\frac{(\ln(t))^{is}}{t}dt\\
&\qquad-(pi+2)\int_1^e\int_1^e\frac{\partial f(\ln x,\ln t)}{\partial x}\frac{(\ln(t))^{is}}{t}\frac{(\ln x)^{pi+1}}{x}dxdt
\end{aligned}
$$

$$= \frac{\partial}{\partial x} z^c \left(f(1, \ln t) - (pi+2) z_2^c \left(\ln x \frac{\partial f(\ln x, \ln t)}{\partial x} \right) \right.$$
$$= \frac{\partial}{\partial x} z^c (f(1, \ln t) - (pi+2) z_2^c (f(1, \ln t))$$
$$+ (ip+1)(pi+2) z_2^c (f(\ln x, \ln t)).$$

□

Theorem 2.7.

$$z_2^c \left((\ln(x))^3 \frac{\partial^3 f(\ln x, \ln t)}{\partial x^3} \right)$$
$$= \frac{\partial^2}{\partial x^2} z^c \left(f(1, \ln t) - (pi+3) \frac{\partial}{\partial x} z^c (f(1, \ln t)) \right.$$
$$+ (ip+3)(pi+2) z_2^c (f(1, \ln t))$$
$$- (ip+3)(pi+2)(ip+1) z_2^c (f(\ln x, \ln t)).$$

Proof. By the definition

$$z_2^c \left((\ln(x))^3 \frac{\partial^3 f(\ln x, \ln t)}{\partial x^3} \right)$$
$$= \int_1^e \int_1^e \frac{(\ln(x))^{ip}}{x} \frac{(\ln(t))^{is}}{t} (\ln(x))^3 \frac{\partial^3 f(\ln x, \ln t)}{\partial x^3} dx dt$$
$$= \int_1^e \int_1^e \frac{(\ln(x))^{ip+3}}{x} \frac{(\ln(t))^{is}}{t} \frac{\partial^3 f(\ln x, \ln t)}{\partial x^3} dx dt$$
$$= \int_1^e \left[(\ln x)^{ip+3} \frac{\partial^2 f(\ln x, \ln t)}{\partial x^2} \bigg|_1^e - (pi+3) \int_1^e \frac{\partial^2 f(\ln x, \ln t)}{\partial x^2} \frac{(\ln x)^{pi+2}}{x} \bigg|_1^e dx \right]$$
$$\times \frac{(\ln(t))^{is}}{t} dt$$
$$= \int_1^e \left[\frac{\partial^2 f(1, \ln t)}{\partial x^2} - (pi+3) \int_1^e \frac{\partial^2 f(\ln x, \ln t)}{\partial x^2} \frac{(\ln x)^{pi+2}}{x} \right] \frac{(\ln(t))^{is}}{t} dt$$
$$= \int_1^e \frac{\partial^2 f(1, \ln t)}{\partial x^2} \frac{(\ln(t))^{is}}{t} dt$$
$$- (pi+3) \int_1^e \int_1^e \frac{\partial^2 f(\ln x, \ln t)}{\partial x^2} \frac{(\ln(t))^{is}}{t} \frac{(\ln x)^{pi+2}}{x} dx dt$$
$$= \frac{\partial^2}{\partial x^2} z^c (f(1, \ln t)) - (pi+3) z_2^c \left((\ln x)^2 \frac{\partial^2 f(\ln x, \ln t)}{\partial x^2} \right)$$
$$= \frac{\partial^2}{\partial x^2} z^c \Big(f(1, \ln t) - (pi+3) \frac{\partial}{\partial x} z^c (f(1, \ln t)) + (ip+3)(pi+2) z_2^c (f(1, \ln t)) \Big)$$
$$- (ip+3)(pi+2)(ip+1) z_2^c (f(\ln x, \ln t)).$$

In general, for $n > 0$,

$$
\begin{aligned}
&z_2^c\left((\ln(x))^n \frac{\partial^n f(\ln x, \ln t)}{\partial x^n}\right)\\
&\quad= \frac{\partial^n}{\partial x^n} z^c\Big(f(1, \ln t) - (pi+n)\frac{\partial^{n-2}}{\partial x^{n-2}} z^c(f(1,\ln t))\\
&\qquad + (ip+n)(pi+(n-1))\Big)\frac{\partial^{n-3}}{\partial x^{n-3}} z_2^c(f(1,\ln t))\\
&\qquad + \cdots + (-1)^n (ip+n)(ip+(n-1))\cdots \frac{\partial}{\partial x} z^c(f(1,\ln t))\\
&\qquad + (ip+n)(ip+(n-1)) z_2^c(f(1,\ln t)) - \cdots\\
&\qquad + (-1)^n (ip+n)(pi+(n-1))\cdots(ip+3)(ip+2)(ip+1) z_2^c(f(\ln x, \ln t)).
\end{aligned}
$$

□

Theorem 2.8.

$$
z_2^c\left(\ln t \frac{\partial f(\ln x, \ln t)}{\partial t}\right) = z^c(f(\ln x, 1)) - (si+1) z_2^c(f(\ln x, \ln t)).
$$

Proof. By the definition,

$$
\begin{aligned}
&z_2^c\left(\ln t \frac{\partial f(\ln x, \ln t)}{\partial t}\right)\\
&\quad= \int_1^e \int_1^e \frac{(\ln t)^{si+1}}{t} \frac{(\ln x)^{pi}}{x} \frac{\partial f(\ln x, \ln t)}{\partial t} dx dt\\
&\quad= \int_1^e \left[(\ln t)^{si+1} f(\ln x, \ln t)\Big|_1^e - (si+1)\int_1^e \frac{(\ln t)^{si}}{t} f(\ln x, \ln t) dt\right] \frac{(\ln x)^{pi}}{x} dx\\
&\quad= \int_1^e \left[f(\ln x, 1) - (si+1)\int_1^e \frac{(\ln t)^{si}}{t} f(\ln x, \ln t) dt\right] \frac{(\ln x)^{pi}}{x} dx\\
&\quad= \int_1^e f(\ln x, 1)\frac{(\ln x)^{pi}}{x} dx - (si+1)\int_1^e \frac{(\ln t)^{si}}{t}\frac{(\ln x)^{pi}}{x} f(\ln x, \ln t) dt dx\\
&\quad= z^c(f(\ln x, 1)) - (si+1) z_2^c(f(\ln x, \ln t)).
\end{aligned}
$$

□

Theorem 2.9.

$$
\begin{aligned}
z_2^c\left((\ln t)^2 \frac{\partial^2 f(\ln x, \ln t)}{\partial t^2}\right) &= \frac{\partial}{\partial t} z^c(f(\ln x, 1)) - (si+2) z^c(f(\ln x, 1))\\
&\quad + (si+2)(si+1) z_2^c(f(\ln x, \ln t)).
\end{aligned}
$$

Proof. By the definition,

$$\begin{aligned}
&z_2^c\left((\ln t)^2\frac{\partial^2 f(\ln x,\ln t)}{\partial t^2}\right)\\
&\quad=\int_1^e\int_1^e\frac{(\ln t)^{si+2}}{t}\frac{(\ln x)^{pi}}{x}\frac{\partial^2 f(\ln x,\ln t)}{\partial t^2}dxdt\\
&\quad=\int_1^e\left((\ln t)^{si+2}\frac{\partial f(\ln x,\ln t)}{\partial t}\Big|_1^e-(si+2)\int_1^e\frac{(\ln t)^{si+2}}{t}\frac{\partial f(\ln x,\ln t)}{\partial t}dt\right)\\
&\qquad\times\frac{(\ln x)^{pi}}{x}dx\\
&\quad=\frac{\partial}{\partial t}z^c(f(\ln x,1))-(si+2)z_2^c\left(\ln t\frac{\partial f(\ln x,\ln t)}{\partial t}\right)\\
&\quad=\frac{\partial}{\partial t}z^c(f(\ln x,1))-(si+2)z^c(f(\ln x,1))+(si+2)(si+1)z_2^c(f(\ln x,\ln t)).
\end{aligned}$$

In general,

$$\begin{aligned}
&z_2^c\left((\ln t)^n\frac{\partial^n f(\ln x,\ln t)}{\partial t^n}\right)\\
&\quad=\frac{\partial^{n-1}}{\partial t^{n-1}}z^c(f(\ln x,1))-(ip+n)\frac{\partial^{n-2}}{\partial t^{n-2}}z^c(f(\ln x,1))\\
&\qquad+(ip+n)(ip+(n-1))\frac{\partial^{n-3}}{\partial t^{n-3}}z^c(f(\ln x,1))+\cdots\\
&\qquad+(-1)^n(ip+n)(ip+(n-1))\dots(ip-1)\frac{\partial}{\partial t}z^c(f(\ln x,1)).
\end{aligned}$$

□

Theorem 2.10.

$$\begin{aligned}
z_2^c\left((\ln x)(\ln t)\frac{\partial^2 f(\ln x,\ln t)}{\partial x\partial t}\right)&=f(1,1)-(si+1)z^c(f(1,\ln t))\\
&\quad-(ip+1)z^c(f(\ln x,1))\\
&\quad+(ip+1)(si+1)z_2^c(f(\ln x,\ln t)).
\end{aligned}$$

Proof. By the definition,

$$\begin{aligned}
&z_2^c\left((\ln x)(\ln t)\frac{\partial^2 f(\ln x,\ln t)}{\partial x\partial t}\right)\\
&\quad=\int_1^e\int_1^e\frac{(\ln x)^{ip+1}}{x}\frac{(\ln t)^{is+1}}{t}\frac{\partial^2 f(\ln x,\ln t)}{\partial x\partial t}dxdt
\end{aligned}$$

$$= \int_1^e \left(\int_1^e \frac{(\ln x)^{ip+1}}{x} \frac{\partial^2 f(\ln x, \ln t)}{\partial x \partial t}\, dx\right) \frac{(\ln t)^{is+1}}{t} dt$$

$$= \int_1^e \left[(\ln x)^{ip+1} \frac{\partial f(\ln x, \ln t)}{\partial t}\Big|_1^e - (pi+1) \int_1^e \frac{\partial f(\ln x, \ln t)}{\partial x} \frac{(\ln x)^{pi}}{x} dx \Big|_1^e \right]$$

$$\times \frac{(\ln(t))^{is+1}}{t} dt$$

$$= \int_1^e \frac{\partial f(1, \ln t)}{\partial x} \frac{(\ln(t))^{is+1}}{t} dt$$

$$- (pi+1) \int_1^e \int_1^e \frac{\partial f(\ln x, \ln t)}{\partial t} \frac{(\ln x)^{pi}}{x} \frac{(\ln(t))^{is+1}}{t} dx dt$$

$$= \int_1^e \frac{(\ln x)^{pi}}{x} \left(\int_1^e \frac{\partial f(1, \ln t)}{\partial x} \frac{(\ln(t))^{is}}{t} dt \right) dx$$

$$= f(1,1) - (si+1) z^c(f(1, \ln t))$$

$$= f(1,1) - (si+1) z^c(f(1, \ln t)) - (pi+1) \int_1^e (\ln t)^{si+1} f(\ln x, \ln t) \frac{(\ln x)^{pi}}{x}\Big|_1^e$$

$$+ (pi+1)(si+1) \int_1^e \int_1^e f(\ln x, \ln t) \frac{(\ln(t))^{is} (\ln x)^{pi}}{t} dx dt$$

$$= f(1,1) - (si+1) z^c(f(1, \ln t)) - (ip+1) z^c(f(\ln x, 1))$$

$$+ (ip+1)(si+1) z_2^c(f(\ln x, \ln t)).$$

And also, we have

$$z_2^c \left((\ln x)(\ln t) \frac{\partial^2 f(\ln x, \ln t)}{\partial x \partial t} \right)$$

$$= \int_1^e \int_1^e \frac{(\ln x)^{ip+1}}{x} \frac{(\ln t)^{is+1}}{t} \frac{\partial^2 f(\ln x, \ln t)}{\partial x \partial t}\, dx\, dt$$

$$= \int_1^e \left(\int_1^e \frac{(\ln x)^{ip+1}}{x} \frac{\partial^2 f(\ln x, \ln t)}{\partial x \partial t}\, dx \right) \frac{(\ln t)^{is+1}}{t}\, dt$$

$$= \int_1^e \left[(\ln x)^{ip+1} \frac{\partial f(\ln x, \ln t)}{\partial t}\Big|_1^e - (ip+1) \int_1^e \frac{\partial f(\ln x, \ln t)}{\partial x} \frac{(\ln x)^{ip}}{x}\, dx \right]$$

$$\times \frac{(\ln t)^{is+1}}{t} dt$$

$$= \int_1^e \frac{\partial f(1, \ln t)}{\partial x} \frac{(\ln t)^{is+1}}{t} dt$$

$$- (ip+1) \int_1^e \int_1^e \frac{\partial f(\ln x, \ln t)}{\partial x} \frac{(\ln x)^{ip}}{x} \frac{(\ln t)^{is+1}}{t} dx dt$$

$$
\begin{aligned}
&= \int_1^e \frac{(\ln x)^{ip}}{x} \left(\int_1^e \frac{\partial f(1, \ln t)}{\partial x} \frac{(\ln t)^{is}}{t}\, dt \right) dx \\
&= f(1,1) - (si+1) z^c(f(1, \ln t)) \\
&= f(1,1) - (si+1) z^c(f(1, \ln t)) - (ip+1) z^c(f(\ln x, 1)) \\
&\quad + (ip+1)(si+1) z_2^c(f(\ln x, \ln t)).
\end{aligned}
$$

□

Theorem 2.11.

$$
\begin{aligned}
z_2^c \left((\ln x)(\ln t) \frac{\partial^2 f(\ln x, \ln t)}{\partial x \partial t} \right) &= f(1,1) - (pi+1) z^c(f(\ln x, 1)) \\
&\quad - (is+1) z^c(f(1, \ln t)) \\
&\quad + (ip+1)(si+1) z_2^c(f(\ln x, \ln t)).
\end{aligned}
$$

Proof. By the definition,

$$
\begin{aligned}
&z_2^c \left((\ln x)(\ln t) \frac{\partial^2 f(\ln x, \ln t)}{\partial x \partial t} \right) \\
&= \int_1^e \int_1^e \frac{(\ln(x))^{ip+1}}{x} \frac{(\ln(t))^{is+1}}{t} \frac{\partial^2 f(\ln x, \ln t)}{\partial x \partial t} dx dt \\
&= \int_1^e \left(\int_1^e \frac{(\ln(t))^{is+1}}{t} \frac{\partial^2 f(\ln x, \ln t)}{\partial x \partial t} dx \right) \frac{(\ln(x))^{ip+1}}{x} dt \\
&= \int_1^e \left[(\ln t)^{is+1} \frac{\partial f(\ln x, \ln t)}{\partial x} \Bigg|_1^e - (si+1) \int_1^e \frac{\partial f(\ln x, \ln t)}{\partial x} \frac{(\ln t)^{si}}{t} dt \Bigg|_1^e \right] \\
&\quad \times \frac{(\ln(x))^{ip+1}}{x} dx \\
&= \int_1^e \frac{\partial f(\ln x, 1)}{\partial x} \frac{(\ln(x))^{ip+1}}{x} dt \\
&\quad - (si+1) \int_1^e \int_1^e \frac{\partial f(\ln x, \ln t)}{\partial x} \frac{(\ln x)^{pi+1}}{x} \frac{(\ln(t))^{is}}{t} dx dt \\
&= \int_1^e \frac{(\ln x)^{pi+1}}{x} \frac{\partial f(\ln x, 1)}{\partial x} dt \\
&\quad - \int_1^e \int_1^e (si+1) \frac{(\ln(t))^{is}}{t} \frac{(\ln x)^{pi+1}}{x} \frac{\partial f(\ln x, \ln t)}{\partial x} dt dx \\
&= z^c \left(\frac{\partial}{\partial x} f(\ln x, 1) \right) - (si+1) \int_1^e \frac{(\ln(t))^{is}}{t} \Big[(\ln x)^{ip+1} f(\ln x, \ln t) \Big|_1^e \\
&\quad - (ip+1) \int_1^e \frac{(\ln x)^{pi}}{x} f(\ln x, \ln t) dx \Big] dt
\end{aligned}
$$

$$= (\ln x)^{ip+1} f(\ln x, 1)\big|_1^e - (ip+1)\int_1^e f(\ln x, 1)\frac{(\ln x)^{pi}}{x}dx$$
$$- (si+1)\int_1^e f(1, \ln t)\frac{(\ln t)^{si}}{t}dx + (si+1)(pi+1),$$
$$= f(1,1) - (pi+1)z^c(f(\ln x, 1)) - (is+1)z^c(f(1, \ln t))$$
$$+ (ip+1)(si+1)z_2^c(f(\ln x, \ln t)).$$

□

3. Applications

In this part, we introduced some important applications to the above theorems:

Example 3.1.

$$(\ln x)^2\frac{\partial^2 f(\ln x, \ln t)}{\partial x^2} = (\ln t)^2\frac{\partial^2 f(\ln x, \ln t)}{\partial t^2} + 2\ln t\frac{\partial f(\ln x, \ln t)}{\partial t} + f(\ln x, \ln t)$$

with $f(\ln x, 1) = 0$, $f(1, \ln t) = 0$, $f_t(\ln x, 1) = 0$, $f_x(1, \ln t) = 0$. In fact,

$$z_2^c\left((\ln x)^2\frac{\partial^2 f(\ln x, \ln t)}{\partial x^2}\right) = z_2^c\left((\ln t)^2\frac{\partial^2 f(\ln x, \ln t)}{\partial t^2}\right)$$
$$+ 2z_2^c\left(\ln t\frac{\partial f(\ln x, \ln t)}{\partial t}\right) + z_2^c(f(\ln x, \ln t)),$$

$$\frac{\partial}{\partial x}z^c(f(1, \ln t)) - (ip+2)z^c(f(1, \ln t)) + (ip+2)(ip+1)z_2^c(f(\ln x, \ln t))$$
$$= \frac{\partial}{\partial t}z^c\left(f(\ln x, 1) - (si+2)z^c(f(\ln x, 1))\right.$$
$$+ (si+2)(si+1)z_2^c(f(\ln x, \ln t)) + 2z^c(f(\ln x, 1))$$
$$- 2(si+1)z_2^c(f(\ln x, \ln t)) + z_2^c(f(\ln x, \ln t)).$$

Applying initial conditions, we obtain:

$$(si+2)(si+1)z_2^c(f(\ln x, \ln t)) - 2(si+1)z_2^c(f(\ln x, \ln t)) + z_2^c(f(\ln x, \ln t))$$
$$= (ip+2)(ip+1)z_2^c z_2^c(f(\ln x, \ln t)$$
$$= (ip+2)(ip+1)z_2^c(f(\ln x, \ln t)\left((si)^2 + si + 1\right)$$
$$\times \left(\left((si)^2 + si + 1\right) - (ip+2)(ip+1)\right) z_2^c f(\ln x, \ln t)$$
$$= (ip+2)(ip+1)z_2^c(f(\ln x, \ln t) z_2^c(f(\ln x, \ln t) = 0.$$

Taking $z_2^{c^{-1}}$ to both sides, we obtain

$$f(\ln x, \ln t) = 0.$$

Example 3.2.

$$(\ln x)\frac{\partial f(\ln x,\ln t)}{\partial x}=2$$

with $f(1,\ln t)=0,\quad z_2^c\left((\ln x)\frac{\partial f(\ln x,\ln t)}{\partial x}\right)=z_2^c(2).$

In fact, since

$$z^c(f(1,\ln t))-(pi+1)z_2^c(f(\ln x,\ln t))=\frac{2}{(si+1)(pi+1)},$$

$$z_2^c(f(\ln x,\ln t))=-\frac{2}{(si+1)(pi+1)^2}$$

and

$$f(\ln x,\ln t)=-z_2^{c^{-1}}\left(\frac{2}{(si+1)(pi+1)^2}\right)=2(\ln t)^0(\ln(\ln x)).$$

We can get the desired result.

4. Conclusion

In this paper, a new definition was made, which is the double integral transform to the complex Al-Zughair transform, and a theoretical construct was built for this definition with theorems and properties. The aim of this definition is to solve partial differential equations and their applications.

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