Nonlinear Functional Analysis and Applications
Vol. 30, No. 3 (2025), pp. 897-913
ISSN: 1229-1595(print), 2466-0973(online)
https://doi.org/10.22771/nfaa.2025.30.03.15
http://nfaa.kyungnam.ac.kr/journal-nfaa

NEW EXISTENCE RESULT ON FRACTIONAL IMPULSIVE DIFFERENTIAL EQUATIONS INVOLVES NON-LOCAL CONDITIONS VIA TOPOLOGICAL SEQUENCES

Swathi Dhandapani[1], Ananthi Kantheeban[2], Karthik Raja Umapathi[1], Kamal Shah[3], Bahaaeldin Abdalla[3] and Thabet Abdeljawad[3,4]

[1]Research Centre and PG Department of Mathematics, The Madura College, Madurai - 625 011, Tamilnadu, India
e-mail: swathikrishna25@gmail.com, karthikraja@yahoo.co.in

[2] Department of Mathematics, Sree Sowdambika College of Engineering, Aruppukotai-626 134, Tamilnadu, India
e-mail: ananthiganesh160522@gmail.com

[3]Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
e-mail: kshah@psu.edu.sa, babdallah@psu.edu.sa

[4]Department of Mathematics and Applied Mathematics School of Science and Technology, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa
e-mail: tabdeljawad@psu.edu.sa

Abstract. This article is related to examine some appropriate conditions for the existence and uniqueness of solutions to a class of impulsive fractional partial differential equations. First, we provide a formula for the general solution to the considered problem in Banach space that involves a nonlocal condition using the conventional Caputo derivative. Moreover, using fixed point methods, we establish sufficient and precise conditions for at least one solution. Also, we use topological degree techniques to reevaluate some of the operators for determining the solution's uniqueness and existence to the considered class of fractional impulsive problem with two variable function. For justification, we give an example also.

[0]Received March 21, 2025. Revised June 8, 2025. Accepted June 16, 2025.
[0]2020 Mathematics Subject Classification: 46E35, 49K20, 76D05, 35Q30.
[0]Keywords: Fractional calculus, impulsive, Caputo derivative, existence, topological theory.
[0]Corresponding author: T. Abdeljawad(tabdeljawad@psu.edu.sa).

1. Introduction

The development of fractional theory has sparked the interest of researchers. Fractional differential equations having wide range of applications in artificial neural network [18], biological science [1, 15, 16], ecological science like prey-predator system [6], COVID-19 [10]. For more applications of mentioned area, we refer to [4, 8, 9, 13, 14, 17, 21]. When it comes to studying a wide range of nonlinear analysis problems, topological degree methods have become the most essential technique.

Undoubtedly, the application of topological techniques is in close comparison to assessing whether or not fractional differential equation solutions have been found in recent decades.

The works of Agarwal demonstrate the increasing interest of researchers in fractional differential equations in Banach spaces [2, 3]. The nonlocal nature of the fractional order operators is the primary driver behind the increasing acceptance of the fractional calculus, which explains the inherent properties of different materials and processes. In most cases, fractional derivatives can explain many real world problems more comprehensively as compared to integer order derivative.

Researchers [12, 20] have deduced sufficient conditions for the existence of solutions to nonlocal impulsive fractional differential equations. The most vital measure for examining a wide range of nonlinear analysis issues is topological degree methods. For instance, Isia established the prerequisites for the existence of solutions to a few nonlinear integral equations using the aforementioned degree tools [7].

Wang and his colleagues [23] employed the priori estimate method known as topological degree theory to determine the terms needed to for the existence theory of solution to nonlinear differential equation of fractional order. It has gained more attention and assumed a central role in the thoughts of mathematicians and experts due to its importance in several domains. Zho and his co-authors [24] have produced a detailed work on topological degree theory.

Using Monch's fixed point theorem and the method of measures of noncompactness, Benchohra and Seba examined the existence of solutions to impulsive fractional differential equations in a Banach space [5]. Using approximating sequences, Karthik Raja et al. [19]. investigated the existence results on random nonlocal fractional differential equations.

Motivated by the above work, a class of fractional impulsive differential equations with the fractional Caputo derivative and a non-local condition is investigated in Banach space for the existence theory of solution. The considered problem is described by

$$\begin{cases} {}^{c}[\mathcal{D}]^{\tau}u(\mathtt{p},\mathtt{q}) = \mathfrak{g}(\mathtt{p},\mathtt{q},u(\mathtt{p},\mathtt{q})), \\ (\mathtt{p},\mathtt{q}) \in (\mathcal{J}/\{\mathtt{p}_0,\mathtt{p}_1,\mathtt{p}_2,\ldots,\mathtt{p}_m\},\mathcal{J}/\{\mathtt{q}_0,\mathtt{q}_1,\mathtt{q}_2,\ldots,\mathtt{q}_m\}), \mathcal{J}=[0,T], \\ \Delta u(\mathtt{p}_i,\mathtt{q}_i) = I_i(u(\mathtt{p}_i,\mathtt{q}_i)) \\ u(0,0) = u_0, \end{cases} \tag{1.1}$$

where $\mathfrak{g} : \mathcal{J}\times\mathcal{J}\times\mathcal{B}\to\mathcal{B}, u_0\in\mathcal{B}$ is a continuous map; $I_i : \mathcal{B}\times\mathcal{B}\to\mathcal{B}$ is a continuous map and $\mathtt{p}_i,\mathtt{q}_i$ satisfies $0=\mathtt{p}_0<\mathtt{p}_1<\mathtt{p}_2<\cdots<\mathtt{p}_m<\mathtt{p}_{m+1}=a\le T; 0=\mathtt{q}_0<\mathtt{q}_1<\mathtt{q}_2<\cdots<\mathtt{q}_m<\mathtt{q}_{m+1}=b\le T$.

We have organized this paper as follows: Introduction is given in section 1. Preliminaries are described in section 2. Main results are given in section 3. Example to justify results are provided in section 4. Last section is related to a brief conclusion.

2. Preliminaries

We define the Banach space $\mathbb{PC}(\mathcal{J}\times\mathcal{J},\mathcal{B}) = \{u : \mathcal{J}\times\mathcal{J}\to\mathcal{B} : u\in C((I_i,I_{i+1}],\mathcal{B}), i = 0,1,2,\ldots,m+1$ and there exist $u(I_i^-)$ and $u(I_i^+), i = 1,2,\ldots,m$ with $u(I_i^-) = u(\mathtt{p}_i,\mathtt{q}_i)\}$. The Banach space $\mathbb{PC}(\mathcal{J}\times\mathcal{J},\mathcal{B})$ has the norm $\|u\|_{\mathbb{PC}} = \sup\{\|u(\mathtt{p},\mathtt{q})\| : (\mathtt{p},\mathtt{q})\in\mathcal{J}\times\mathcal{J}\}$.

Definition 2.1. ([11]) The τ^{th} order fractional integral of a continuous function $\mathfrak{g}$ in the closed interval $[a,b]$ is defined as

$$\begin{aligned} [\mathtt{I}]^{\tau}\mathfrak{g}(\mathtt{p},\mathtt{q},u(\mathtt{p},\mathtt{q})) &= \frac{1}{(\tau_1-1)!(\tau_2-1)!}\int_0^{\mathtt{p}}\int_0^{\mathtt{q}}(\mathtt{p}-s)^{\tau_1-1} \\ &\quad \times(\mathtt{q}-t)^{\tau_2-1}\mathfrak{g}(s,t,u(s,t))dsdt. \end{aligned} \tag{2.1}$$

Definition 2.2. ([11]) The τ^{th} R-L fractional order derivative for continuous function $\mathfrak{g}$ in the closed interval $[a,b]$, is defined by

$$\begin{aligned} [\mathcal{D}]^{\tau}_{a+}\mathfrak{g}(\mathtt{p},\mathtt{q},u(\mathtt{p},\mathtt{q})) &= \frac{1}{(n-\tau_1)!(n-\tau_2)!}\Big(\frac{d}{dt}\Big)^n\int_a^{\mathtt{p}}\int_a^{\mathtt{q}}(\mathtt{p}-s)^{n-\tau_1-1} \\ &\quad \times(\mathtt{q}-t)^{n-\tau_2-1}\mathfrak{g}(s,t,u(s,t))dsdt, \end{aligned} \tag{2.2}$$

where $n=[\tau]+1$, $[\tau]$ is the figures part of τ.

Definition 2.3. ([11]) For a given uninterpreted function$\mathfrak{g}$ in the closed interval $[a,b]$, the fractional Caputo order derivative is given by

$$\begin{aligned} {}^{c}[\mathcal{D}]^{\tau}_{a+}\mathfrak{g}(\mathtt{p},\mathtt{q},u(\mathtt{p},\mathtt{q})) &= \frac{1}{\Gamma(n-\tau_1)\Gamma(n-\tau_2)}\int_a^{\mathtt{p}}\int_a^{\mathtt{q}}(\mathtt{p}-s)^{n-\tau_1-1} \\ &\quad \times(\mathtt{q}-t)^{n-\tau_2-1}\mathfrak{g}^n(s,t,u(s,t))dsdt, \end{aligned} \tag{2.3}$$

where $n = [\tau] + 1$.

Lemma 2.4. ([23]) *Assuming* $n - 1 < \tau \leq n$,

$I^{\tau}[{}^{c}\mathcal{D}^{\tau}u](\mathtt{p}, \mathtt{q}) = u(\mathtt{p}, \mathtt{q}) + \mu_0 + \mu_1(\mathtt{p} + \mathtt{q}) + \mu_2(\mathtt{p}^2 + \mathtt{q}^2) + \cdots + c_{n-1}(\mathtt{p}^{n-1} + \mathtt{q}^{n-1})$
for some $\mu_i \in \mathcal{B}, i = 0, 1, 2, \ldots, n-1, n = [\tau] + 1$.

Theorem 2.5. ([24]) *If* $\mathfrak{g} : \mathcal{X} \to \mathcal{X}$ *is a contraction mapping with a constant* $\kappa < 1$ *and* $\mathcal{X}$ *is a complete metric space, then* $\mathfrak{g}$ *has a unique fixed point.*

Theorem 2.6. ([24]) *If* $\mathbb{B}$ *is a bounded, closed, convex and nonempty subset of a Banach space* $\mathcal{B}$, *then itha at least one fixed point for* $\mathfrak{g}$ *in* $\mathbb{B}$ *if it is a complete continuous operator such that* $\mathfrak{g}(\mathbb{B}) \subset \mathbb{B}$.

Definition 2.7. ([5]) Let $\mathbb{B}$ be the bounded set of $\mathcal{B}$, where $\mathcal{B}$ be a Banach space, then the Kuratowski measure of noncompactness with $\tau : \mathbb{B} \to [0, \infty)$ is defined by

$$\tau(\mathbb{B}) = \inf \left\{ \epsilon > 0 : \mathbb{B} \subseteq \bigcup_{i=1}^{m} \mathcal{B}_i \text{ and diam } (\mathcal{B}_i) \leq \epsilon \right\}.$$

Definition 2.8. ([24]) Let $\mathfrak{g} : \Psi \to \mathcal{X}$ be a continuous bounded map and $\Psi \subset \mathcal{X}$. If there exists $k \geq 0$ such that $\mathfrak{g}$ is a $\tau-$ Lipschitz, then

$$\tau(\mathfrak{g}(\mathbb{B})) \leq \text{ k } \tau(\mathbb{B})$$

for all bounded subset $\mathbb{B} \subset \Psi$. We can refer to $\mathfrak{g}$ as a strict τ-contraction if $k < 1$. If $\mathfrak{g}$ is τ-condensing, we can say that, for all bounde subset $\mathbb{B} \subset \Psi$ with $\tau(\mathbb{B}) > 0$.

$$\tau(\mathfrak{g}(\mathbb{B})) < \tau(\mathbb{B}).$$

The contraction map τ is represented $SC_{\tau}(\Psi)$, and all τ-condensing is represented $C_{\tau}(\Psi)$.

Definition 2.9. ([24]) $\Psi \subset \mathcal{B}$, $\mathfrak{g} : \Psi \to \mathcal{B}$ is said to be k-Lipschitz, if there exists $k > 0$ such that

$$\|\mathfrak{g}\mathtt{p} - \mathfrak{g}\mathtt{q}\| \leq k\|\mathtt{p} - \mathtt{q}\|, \quad \forall\, \mathtt{p}, \mathtt{q} \subset \Psi,$$

$\mathfrak{g}$ is a contraction, if $k < 1$.

Proposition 2.10. ([24]) *If* $\mathbb{B} \subset C(\mathcal{J}, \mathcal{B})$ *equicontinuous and bounded, then* $(\mathtt{p}, \mathtt{q}) \longmapsto \tau(B(\mathtt{p}, \mathtt{q}))$ *is continuous on* $\mathcal{J} \times \mathcal{J}$, *and*

$$\begin{aligned} \tau(\mathbb{B}) &= \max \tau(\mathbb{B}(\mathtt{p}, \mathtt{q})), \tau\Big(\int_0^{\mathtt{p}} \int_0^{\mathtt{q}} \mathbb{B}(s, t) ds dt\Big) \\ &\leq \mathtt{q} \int_0^{\mathtt{p}} \int_0^{\mathtt{q}} \tau(\mathbb{B}(s, t)) ds dt, \quad \forall\, x, \mathtt{q} \in \mathcal{J}. \end{aligned}$$

Theorem 2.11. ([22]) *Let* $\mathbb{S} = \{u \in \mathcal{B} :$ *there exists* $\eta \in [0,1] \ni: u = \eta\mathcal{M}(u)\}$ *be the* $\tau-$*condensing of* $\mathcal{M} : \mathcal{B} \to \mathcal{B}$. *For* $\zeta > 0$ *and* $\mathbb{S} \subset \mathbb{B}_\zeta(0)$, *if* $\mathbb{S}$ *is a bounded set in* $\mathcal{B}$, *then*

$$D(I - \eta\mathcal{M}, \mathbb{B}_\zeta(0), 0) = 1, \quad \forall\, \varpi \in [0,1],$$

where is one degree function and $\mathcal{F} : \mathbb{T} \to \tau$.

The expression $\mathbb{T} = \{(I - \mathcal{M}, \Psi, p) : \Psi \subset \mathcal{B}$ *open and bounded,* $\mathcal{M} \in C_\tau(\overline{\Psi}), u \in \mathcal{B}\ (I - \mathcal{M})(\partial\Psi)\}$. *As a result,* $\mathcal{M}$ *has at least one point, and* $\mathbb{B}_\zeta(0)$ *contains the set of fixed points of* $\mathcal{M}$.

Proposition 2.12. ([24]) *If* $f, h : \Psi \to \mathcal{B}$ *are* $\tau-$*Lipschitz maps with the constant* κ,$\kappa^{'}$ *discretely, then* $\mathfrak{g}^{th} : \Psi \to \mathcal{B}$ *is* $\tau-$*Lipschitz with constant* $\kappa + \kappa^{'}$.

Proposition 2.13. ([24]) $\mathfrak{g}$ *is* $\tau-$*Lipschitz with zero constant if* $\mathfrak{g} : \Psi \to \mathcal{B}$ *is compact.*

Proposition 2.14. ([24]) f *is* $\tau-$*Lipschitz with the same constant* κ *if* $\mathfrak{g} : \Psi \to \mathcal{B}$ *is Lipschitz with a constant* κ.

3. Existence and uniqueness solution

The general solution of impulsive fractional differential equations is first defined as follows:

Definition 3.1. If the function $u \in \mathbb{PC}(\mathcal{J}, \mathcal{B})$ satisfies the equation

$$^{c}[\mathcal{D}]^{\tau}\mathfrak{g}(\mathtt{p}, \mathtt{q}, u(\mathtt{p}, \mathtt{q})) = u(\mathtt{p}, \mathtt{q}),$$

then it is considered the solution of problem (1.1). $\Delta u(\mathtt{p}_i, \mathtt{q}_i) = I_i u(\mathtt{p}_i, \mathtt{q}_i)$, $i = 1, 2, \ldots, m$, and $u(0,0) = u_0$ are the conditions for $(a.e)$ on $\mathcal{J}$.

Lemma 3.2. *The fractional integral equation*

$$\begin{aligned} u(\mathtt{p}, \mathtt{q}) = u_0 - \frac{1}{(\tau_1 - 1)!(\tau_2 - 1)!} &\sum_{\substack{0<\mathtt{p}_1<\mathtt{p} \\ 0<\mathtt{q}_1<\mathtt{q}}} \int_{\mathtt{p}_{i-1}}^{\mathtt{p}_i} \int_{\mathtt{q}_{i-1}}^{\mathtt{q}_i} (\mathtt{p}_i - s)^{\tau_1 - 1} \\ &\times (\mathtt{q}_i - t)^{\tau_2 - 1}\mathfrak{g}(s, t, u(s,t))dsdt \\ &+ \frac{1}{(\tau_1 - 1)!((\tau_2 - 1)!)} \int_0^{\mathtt{p}} \int_0^{\mathtt{q}} (\mathtt{p} - s)^{\tau_1 - 1} \\ &\times (\mathtt{q} - t)^{\tau_2 - 1}\mathfrak{g}(s, t, u(s,t))dsdt \\ &+ \sum_{\substack{0<\mathtt{p}_1<\mathtt{p} \\ 0<\mathtt{q}_1<\mathtt{q}}} I_i u(\mathtt{p}_i, \mathtt{q}_i),\ i = 1, 2, \ldots, m \end{aligned} \tag{3.1}$$

has a solution $u \in \mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B})$ *for* $\mathtt{p} \in (\mathtt{p}_i, \mathtt{p}_{i+1})$, $\mathtt{q} \in (\mathtt{q}_i, \mathtt{q}_{i+1})$, $i = 1, 2, \ldots, m$ *if and only if* u *is a solution to the fractional differential equations with impulsive condition* (1.1).

Proof. Assume that u is a solution of impulsive fractional differential equation solution (1.1). If $(\mathtt{p}, \mathtt{q}) \in [0, (\mathtt{p}_1) \times [0, \mathtt{q}_1)]$, then

$$ {}^{c}[\mathcal{D}]^{\tau} u(\mathtt{p}, \mathtt{q}) = \mathfrak{g}(\mathtt{p}, \mathtt{q}, u(\mathtt{p}, \mathtt{q})) \quad \text{with} \quad u(0,0) = u_0. $$

If $\mathtt{p}_1 \leq p < \mathtt{p}_2, \mathtt{q}_1 \leq q < \mathtt{q}_2$, then

$$ \begin{aligned} {}^{c}[\mathcal{D}]^{\tau} u(\mathtt{p}, \mathtt{q}) &= \mathfrak{g}(\mathtt{p}, \mathtt{q}, u(\mathtt{p}, \mathtt{q})), \mathtt{p} \in [\mathtt{p}_1, \mathtt{p}_2), \mathtt{q} \in [\mathtt{q}_1, \mathtt{q}_2) \text{ with} \\ \Delta u(\mathtt{p}_1, \mathtt{q}_1) &= u(\mathtt{p}_1^+, \mathtt{q}_1^+) - u(\mathtt{p}_2^-, \mathtt{q}_2^-) = I_1(u(\mathtt{p}_1, \mathtt{q}_1)). \end{aligned} \tag{3.2} $$

By integrating the formula (3.2) from $(\mathtt{p}_1, \mathtt{q}_1)$ to $(\mathtt{p}, \mathtt{q})$, we conclude that

$$ \begin{aligned} u(\mathtt{p}, \mathtt{q}) &= u(\mathtt{p}_1^+, \mathtt{q}_1^+) + \frac{1}{(\tau_1 - 1)!(\tau_2 - 1)!} \int_{\mathtt{p}_1}^{\mathtt{p}} \int_{\mathtt{q}_1}^{\mathtt{q}} (\mathtt{p} - s)^{\tau_1 - 1} \\ &\quad \times (\mathtt{q} - t)^{\tau_2 - 1} \mathfrak{g}(s, t, u(s,t)) ds dt. \end{aligned} $$

Consequently, it follows that

$$ \begin{aligned} u(\mathtt{p}, \mathtt{q}) &= u(\mathtt{p}_1, \mathtt{q}_1) + I_1(u(\mathtt{p}_1, \mathtt{q}_1)) + \frac{1}{(\tau_1 - 1)!(\tau_2 - 1)!} \int_{\mathtt{p}_1}^{\mathtt{p}} \int_{\mathtt{q}_1}^{\mathtt{q}} (\mathtt{p} - s)^{\tau_1 - 1} \\ &\quad \times (\mathtt{q} - t)^{\tau_2 - 1} \mathfrak{g}(s, t, u(s,t)) ds dt \\ &= u(0,0) + I_1(u(\mathtt{p}_1, \mathtt{q}_1)) - \frac{1}{(\tau_1 - 1)!(\tau_2 - 1)!} \int_0^{\mathtt{p}_1} \int_0^{\mathtt{q}_1} (\mathtt{p}_1 - s)^{\tau_1 - 1} \\ &\quad \times (\mathtt{q}_1 - t)^{\tau_2 - 1} \mathfrak{g}(s, t, u(s,t)) ds dt \\ &\quad + \frac{1}{(\tau_1 - 1)!(\tau_2 - 1)!} \int_0^{\mathtt{p}} \int_0^{\mathtt{q}} (\mathtt{p} - s)^{\tau_1 - 1} (\mathtt{q} - t)^{\tau_2 - 1} \mathfrak{g}(s, t, u(s,t)) ds dt. \end{aligned} $$

If $\mathtt{p}_2 \leq p < \mathtt{p}_3, \mathtt{q}_2 \leq y < \mathtt{q}_3$, then

$$ \begin{aligned} {}^{c}[\mathcal{D}]^{\tau} u(\mathtt{p}, \mathtt{q}) &= \mathfrak{g}(\mathtt{p}, \mathtt{q}, u(\mathtt{p}, \mathtt{q})), \mathtt{p} \in [\mathtt{p}_2, \mathtt{p}_3) \mathtt{q} \in [\mathtt{q}_2, \mathtt{q}_3) \text{ with} \\ u(\mathtt{p}_2^+, \mathtt{q}_2^+) &= u(\mathtt{p}_2^-, \mathtt{q}_2^-) + I_2(u(\mathtt{p}_2, \mathtt{q}_2)). \end{aligned} \tag{3.3} $$

Integrate (3.3) from $(\mathtt{p}_2, \mathtt{q}_2)$ to $(\mathtt{p}, \mathtt{q})$, we reach

$$ \begin{aligned} u(\mathtt{p}, \mathtt{q}) &= u(\mathtt{p}_2^+, \mathtt{q}_2^+) + \frac{1}{(\tau_1 - 1)!(\tau_2 - 1)!} \int_{\mathtt{p}_2}^{\mathtt{p}} \int_{\mathtt{q}_2}^{\mathtt{q}} (\mathtt{p} - s)^{\tau_1 - 1} \\ &\quad \times (\mathtt{q} - t)^{\tau_2 - 1} \mathfrak{g}(s, t, u(s,t)) ds dt. \end{aligned} $$

Thus, it follows that

$$\begin{aligned}
u(\mathtt{p},\mathtt{q}) &= u(\mathtt{p}_2^-,\mathtt{q}_2^-) + I_2(u(\mathtt{p}_2,\mathtt{q}_2)) + \frac{1}{(\tau_1-1)!(\tau_2-1)!}\int_{\mathtt{p}_2}^{\mathtt{p}}\int_{\mathtt{q}_2}^{\mathtt{q}}(\mathtt{p}-s)^{\tau_1-1} \\
&\quad \times (\mathtt{q}-t)^{\tau_2-1}\mathfrak{g}(s,t,u(s,t))dsdt \\
&= u(0,0) + I_1(u(\mathtt{p}_1,\mathtt{q}_1)) + I_2(u(\mathtt{p}_2,\mathtt{q}_2)) \\
&\quad - \frac{1}{(\tau_1-1)!(\tau_2-1)!}\int_0^{\mathtt{p}_1}\int_0^{\mathtt{q}_1}(\mathtt{p}_1-s)^{\tau_1-1}(\mathtt{q}_1-t)^{\tau_2-1}\mathfrak{g}(s,t,u(s,t))dsdt \\
&\quad - \frac{1}{(\tau_1-1)!(\tau_2-1)!}\int_{\mathtt{p}_1}^{\mathtt{p}_2}\int_{\mathtt{q}_1}^{\mathtt{q}_2}(\mathtt{p}_2-s)^{\tau_1-1}(\mathtt{q}_2-t)^{\tau_2-1}\mathfrak{g}(s,t,u(s,t))dsdt \\
&\quad + \frac{1}{(\tau_1-1)!(\tau_2-1)!}\int_0^{\mathtt{p}}\int_0^{\mathtt{q}}(\mathtt{p}-s)^{\tau_1-1}(\mathtt{q}-t)^{\tau_2-1}\mathfrak{g}(s,t,u(s,t))dsdt.
\end{aligned}$$

Thus, if $\mathtt{p}\in(\mathtt{p}_i,\mathtt{p}_{i+1}]$, $\mathtt{q}\in(\mathtt{q}_i,\mathtt{q}_{i+1}]$, we obtain

$$\begin{aligned}
u(\mathtt{p},\mathtt{q}) = u(0,0) &+ \sum_{\substack{0<\mathtt{p}_i<\mathtt{p}\\0<\mathtt{q}_i<\mathtt{q}}} I_i(u(\mathtt{p}_i,\mathtt{q}_i)) \\
&- \frac{1}{(\tau_1-1)!(\tau_2-1)!}\int_{\mathtt{p}_{i-1}}^{\mathtt{p}_i}\int_{\mathtt{q}_{i-1}}^{\mathtt{q}_i}(\mathtt{p}_i-s)^{\tau_1-1} \\
&\times (\mathtt{q}_i-t)^{\tau_2-1}\mathfrak{g}(s,t,u(s,t))dsdt \\
&+ \frac{1}{(\tau_1-1)!(\tau_2-1)!}\int_0^{\mathtt{p}}\int_0^{\mathtt{q}}(\mathtt{p}-s)^{\tau_1-1} \\
&\times (\mathtt{q}-t)^{\tau_2-1}\mathfrak{g}(s,t,u(s,t))dsdt
\end{aligned}$$

for $i=1,2,\ldots,m$.
By the condition $u(0,0)=u_0$, we get that

$$\begin{aligned}
u(\mathtt{p},\mathtt{q}) = u_0 &+ \sum_{\substack{0<\mathtt{p}_i<\mathtt{p}\\0<\mathtt{q}_i<\mathtt{q}}} I_i(u(\mathtt{p}_i,\mathtt{q}_i)) \\
&- \frac{1}{(\tau_1-1)!(\tau_2-1)!}\int_{\mathtt{p}_{i-1}}^{\mathtt{p}_i}\int_{\mathtt{q}_{i-1}}^{\mathtt{q}_i}(\mathtt{p}_i-s)^{\tau_1-1} \\
&\times (\mathtt{q}_i-t)^{\tau_2-1}\mathfrak{g}(s,t,u(s,t))dsdt \\
&+ \frac{1}{(\tau_1-1)!(\tau_2-1)!}\int_0^{\mathtt{p}}\int_0^{\mathtt{q}}(\mathtt{p}-s)^{\tau_1-1} \\
&\times (\mathtt{q}-t)^{\tau_2-1}\mathfrak{g}(s,t,u(s,t))dsdt
\end{aligned}$$

for $i=1,2,\ldots,m$.

On the other hand, suppose that u satisfies (3.1). $u(0,0)=u_0$ if $\mathtt{p}\in(0,\mathtt{p}_1]$, $\mathtt{q}\in(0,\mathtt{q}_1)$. Given that for $i=1,2,\ldots,m,\mathtt{p}\in(\mathtt{p}_i,\mathtt{p}_{i+1}],\mathtt{q}\in(\mathtt{q}_i,\mathtt{q}_{i+1}]$ by the knowledge that ${}^c[\mathcal{D}]^\tau$ is the left inverse of ${}^cI^\tau$, and we obtain by applying the principle that the Caputo derivative of constant equals zero. For every $\mathtt{p}, \mathtt{q}$ in the interval $[\mathtt{p}_i, \mathtt{p}_{i+1}], [\mathtt{q}_i, \mathtt{q}_{i+1}], {}^c[\mathcal{D}]^{\tau} = \mathfrak{g}(\mathtt{p}, \mathtt{q}, u(\mathtt{p}, \mathtt{q}))$, and $u(\mathtt{p}_i^+, \mathtt{q}_i^+) = u(\mathtt{p}_i^-, \mathtt{q}_i^-) + I_i u(\mathtt{p}_i, \mathtt{q}_i)$. The proof is now complete. □

Now, let us define the functions as follows:

$\mathcal{W}_1 : P(\mathcal{J}, \mathcal{J}, \mathcal{B}) \to P(\mathcal{J}, \mathcal{J}, \mathcal{B})$ given by

$$\mathcal{W}_1(u(\mathtt{p}, \mathtt{q})) = u_0 + \sum_{\substack{0<\mathtt{p}_i<\mathtt{p} \\ 0<\mathtt{q}_i<\mathtt{q}}} I_i(u(\mathtt{p}_i, \mathtt{q}_i))$$

for $\mathtt{p} \in (\mathtt{p}_i, \mathtt{p}_{i+1}]$, $\mathtt{q} \in (\mathtt{q}_i, \mathtt{q}_{i+1}]$, $i = 1, 2, \ldots, m$.

$\mathcal{W}_2 : P(\mathcal{J}, \mathcal{J}, \mathcal{B}) \to P(\mathcal{J}, \mathcal{J}, \mathcal{B})$ given by

$$\mathcal{W}_2(u(\mathtt{p}, \mathtt{q})) = \frac{1}{(\tau_1 - 1)!(\tau_2 - 1)!} \int_0^{\mathtt{p}} \int_0^{\mathtt{q}} (\mathtt{p} - s)^{\tau_1 - 1} \times (\mathtt{q} - t)^{\tau_2 - 1} \mathfrak{g}(\mathtt{p}, \mathtt{q}, u(\mathtt{p}, \mathtt{q})) dsdt$$

for $\mathtt{p} \in [0, \mathtt{p}_1)$, $\mathtt{q} \in [0, \mathtt{q}_1)$.

$\mathcal{W}_3 : P(\mathcal{J}, \mathcal{J}, \mathcal{B}) \to P(\mathcal{J}, \mathcal{J}, \mathcal{B})$ is given by

$$\mathcal{W}_3(u(\mathtt{p}, \mathtt{q})) = -\frac{1}{(\tau_1 - 1)!(\tau_2 - 1)!} \sum_{\substack{0<\mathtt{p}_i<\mathtt{p} \\ 0<\mathtt{q}_i<\mathtt{q}}} \int_{\mathtt{p}_{i-1}}^{\mathtt{p}_i} \int_{\mathtt{q}_{i-1}}^{\mathtt{q}_i} (\mathtt{p}_i - s)^{\tau_1 - 1} \times (\mathtt{q}_i - t)^{\tau_2 - 1} \mathfrak{g}(\mathtt{p}, \mathtt{q}, u(\mathtt{p}, \mathtt{q})) dsdt$$

for $\mathtt{p} \in (\mathtt{p}_i, T]$, $\mathtt{q} \in (\mathtt{q}_i, T]$, $i = 1, 2, \ldots, m$.

Expression (3.1) $\mathcal{M} : \mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B}) \to \mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B})$ indicates the operator, and $\mathcal{M}u$ is equivalent to $\mathcal{W}_1 u + \mathcal{W}_2 u + \mathcal{W}_3 u$. It is clear that $\mathcal{M}$ is mentioned completely. Next, the operator below can be used to express the fractional integral that was obtained from (3.1).

$$u = \mathcal{M}u = \mathcal{W}_1 u + \mathcal{W}_2 u + \mathcal{W}_3 u. \tag{3.4}$$

Thus, the presence of operator H in (3.4) to investigate the existence of solution the fractional IFDEs (1.1). The successive iterations are necessary in order to address the existence problem for an IFDE solution (1.1).

$\mathbb{H}_1$. $\mathfrak{g} : \mathcal{J} \times \mathcal{J} \times \mathcal{B}$ is continuous in combination.

$\mathbb{H}_2$. There exists a constant $\lambda_f > 0$ such that

$$\|\mathfrak{g}(\mathtt{p}, \mathtt{q}, u) - \mathfrak{g}(\mathtt{p}, \mathtt{q}, v)\|_{\mathbb{PC}} \leq \lambda_f \|u - v\|_{\mathbb{PC}}, \ \forall\, u, v \in \mathcal{B}, \mathtt{p}, \mathtt{q} \in \mathcal{J}.$$

$\mathbb{H}_3$. For $\lambda_1, \lambda_2 > 0, \beta \in [0, 1)$,

$$\|\mathfrak{g}(\mathtt{p}, \mathtt{q}, u)\|_{\mathbb{PC}} \leq \lambda_1 \|u\|_{\mathbb{PC}}^{\beta} + \lambda_2, \ \forall\, (\mathtt{p}, \mathtt{q}, u) \in \mathcal{J} \times \mathcal{J} \times \mathcal{B}.$$

$\mathbb{H}_4$. Constants $\Phi_I \in [0, \frac{1}{m})$ exist and λ_σ so that

$$\|I_i(u) - I_i(v)\| \leq \Phi_I \|u - v\|, \quad \forall\, u, v \in \mathcal{B}, i = 1, 2, \ldots, m.$$

$\mathbb{H}_5$. There exist $\Phi_1, \Phi_2, \lambda_3, \lambda_4 > 0, \beta_1, \beta_2 \in [0, 1)$ such that

$$\|I_i(u)\| \leq \Phi_1 \|u\|^{\beta_1} + \Phi_2, \quad \forall\, u \in \mathcal{B}, i = 1, 2, \ldots, m.$$

Lemma 3.3. *$\mathcal{W}_1 : \mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B}) \to \mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B})$ is a Lipschitz operator with constant $m\Phi_I$. $\mathcal{W}_1$ is therefore $\tau-$Lipschitz with the same constant $m\Phi_I$. Furthermore, $\mathcal{W}_1$ meets the requirements for growth listed below:*

$$\|\mathcal{W}_1 u\|_{\mathbb{PC}} \leq \|u_0\| + m\Phi_1 \|u\|^{\beta_1} + m\Phi_2. \tag{3.5}$$

Proof. For $\mathtt{p} \in (\mathtt{p}_i, \mathtt{p}_{i+1}]$, $\mathtt{q} \in (\mathtt{p}_i, \mathtt{p}_{i+1}]$, by applying the assumptions ($\mathbb{H}_4$), we get

$$\begin{aligned} \|\mathcal{W}_1 u - \mathcal{W}_1 v\|_{\mathbb{PC}} &= \left\| \sum_{\substack{0<\mathtt{p}_i<\mathtt{p} \\ 0<\mathtt{q}_i<\mathtt{q}}} I_i(u(\mathtt{p}_i, \mathtt{q}_i)) - \sum_{\substack{0<\mathtt{p}_i<\mathtt{p} \\ 0<\mathtt{q}_i<\mathtt{q}}} I_i(v(\mathtt{p}_i, \mathtt{q}_i)) \right\| \\ &\leq \sum_{\substack{0<\mathtt{p}_i<\mathtt{p} \\ 0<\mathtt{q}_i<\mathtt{q}}} \|I_i(u(\mathtt{p}_i, \mathtt{q}_i)) - I_i(v\mathtt{p}_i, \mathtt{q}_i))\| \\ &\leq m\Phi_I \|u - v\|_{\mathbb{PC}}. \end{aligned}$$

Thus, for $\mathtt{p} \in (\mathtt{p}_i, \mathtt{p}_{i+1}]$, $\mathtt{q} \in (\mathtt{q}_i, \mathtt{q}_{i+1}]$, $\mathcal{W}_1$ is Lipschitz with constant $m\Phi_I \in [0, 1)$. Proposition 2.14 indicates that $\mathcal{W}_1$ is $\tau-$Lipschitz with the same constant $m\Phi_I$ for $\mathtt{p} \in (\mathtt{p}_i, \mathtt{p}_{i+1}], \mathtt{q} \in (\mathtt{q}_i, \mathtt{q}_{i+1}]$. Relation (3.5) is an easy derivation from ($\mathbb{H}_5$) as follows:

$$\begin{aligned} \|\mathcal{W}_1(u(\mathtt{p}, \mathtt{q}))\|_{\mathbb{PC}} &\leq \|u_0\| + \left\| \sum_{\substack{0<\mathtt{p}_i<\mathtt{p} \\ 0<\mathtt{q}_i<\mathtt{q}}} I_i(u(\mathtt{p}_i, \mathtt{q}_i)) \right\| \\ &\leq \|u_0\| + m[\Phi_1 \|u\|^{\beta_1} + \Phi_2] \\ &\leq \|u_0\| + m\Phi_1 \|u\|^{\beta_1} + m\Phi_2. \end{aligned}$$

□

Lemma 3.4. *$\mathcal{W}_2, \mathcal{W}_3 : \mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B}) \to \mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B})$ are continuous operators and hence $\mathcal{W}_2 + \mathcal{W}_3$ is continuous. Additionally, $\mathcal{W}_2 + \mathcal{W}_3$ fulfills the subsequent requirement:*

$$\|\mathcal{W}_2 u\|_{\mathbb{PC}} + \|\mathcal{W}_3 u\|_{\mathbb{PC}} \leq \frac{(1+m)T^{\tau_1+\tau_2}(\lambda_1 \|u\|^{\beta}_{\mathbb{PC}} + \lambda_2)}{(\tau_1)!(\tau_2)!} \tag{3.6}$$

for all $u \in \mathbb{PC}(\mathcal{J} \times \mathcal{J}, \mathcal{B})$.

Proof. A bounded set $\mathbb{B}_k(k>0) \subseteq \mathbb{PC}(\mathcal{J},\mathcal{B}) \times \mathbb{PC}(\mathcal{J},\mathcal{B})$ has sequences $\{u_n\}$ such that $u_n \to u$ in $\mathbb{B}_k$. As n approaches ∞, we must now demonstrate that

$$\|\mathcal{W}_2 u_n - \mathcal{W}_2 u\|_{\mathbb{PC}} \to 0.$$

The continuity of $\mathfrak{g}$ makes it simple to see that

$$\mathfrak{g}(s,t,u_n(s,t)) \to \mathfrak{g}(s,t,u(s,t)) \text{ as } n \to \infty.$$

Applying $(\mathbb{H}_3)$ gives us for each $s,t \in \mathcal{J}$,

$$\begin{aligned}\|\mathfrak{g}(s,t,u_n(s,t)) - \mathfrak{g}(s,t,u(s,t))\|_{\mathbb{PC}} &\le \|\mathfrak{g}(s,t,u_n(s,t))\|_{\mathbb{PC}} + \|\mathfrak{g}(s,t,u(s,t))\|_{\mathbb{PC}} \\ &\le 2(\lambda_1\|u\|^{\beta}_{\mathbb{PC}} + \lambda_2).\end{aligned}$$

Then,

$$\begin{aligned}&(\mathtt{p}-s)^{\tau_1-1}(\mathtt{q}-t)^{\tau_2-1}\|\mathfrak{g}(s,t,u_n(s,t)) - \mathfrak{g}(s,t,u(s,t))\|_{\mathbb{PC}} \\ &\qquad\le 2(\mathtt{p}-s)^{\tau_1-1}(\mathtt{q}-t)^{\tau_2-1}(\lambda_1\|u\|^{\beta}_{\mathbb{PC}} + \lambda_2).\end{aligned}$$

Using the Lebesgue dominated convergence theorem and the function

$$(s),(t) \to 2(\mathtt{p}-s)^{\tau_1-1}(\mathtt{q}-t)^{\tau_2-1}(\lambda_1\|u\|^{\beta}_{\mathbb{PC}} + \lambda_2)$$

is integrative for $s \in [0,x]$, $t \in [0,y]$, $\mathtt{p},\mathtt{q} \in \mathcal{J}$ leads to

$$\int_0^{\mathtt{p}}\int_0^{\mathtt{q}}(\mathtt{p}-s)^{\tau_1-1}(\mathtt{q}-t)^{\tau_2-1}\|\mathfrak{g}(s,t,u_n(s,t)) - \mathfrak{g}(s,t,u(s,t))\|_{\mathbb{PC}}dsdt \to 0$$

as $n \to \infty$. Thus, for all $x,\mathtt{q} \in \mathcal{J}$,

$$\begin{aligned}\|\mathcal{W}_2(u_n(\mathtt{p},\mathtt{q})) - \mathcal{W}_2(u(\mathtt{p},\mathtt{q}))\|_{\mathbb{PC}} &\le \frac{1}{(\tau_1-1)!(\tau_2-1)!}\int_0^{\mathtt{p}}\int_0^{\mathtt{q}}(\mathtt{p}-s)^{\tau_1-1}(\mathtt{q}-t)^{\tau_2-1} \\ &\quad\times\|\mathfrak{g}(s,t,u_n(s,t)) - \mathfrak{g}(s,t,u(s,t))\|_{\mathbb{PC}}dsdt \\ &\to 0.\end{aligned}$$

Hence, $\mathcal{W}_2$ is continuous, since $\mathcal{W}_2 u_n \to \mathcal{W}_2 u$ *as* $n \to \infty$. We can obtain the continuity of operator $\mathcal{W}_3$ on $\mathbb{PC}(([0,\mathtt{p}_1],[0,\mathtt{q}_1],\mathcal{B}))$, $\mathbb{PC}((\mathtt{p}_i,\mathtt{p}_{i+1}],(\mathtt{q}_i,\mathtt{q}_{i+1}],\mathcal{B})$ and $\mathbb{PC}((\mathtt{p}_m,T],(\mathtt{q}_m,T],\mathcal{B})$. It is clear that $\mathcal{W}_2+\mathcal{W}_3$ is continuous. As a result,

$(\mathbb{H}_3)$ has the following simple consequences for relation (3.6):

$$\begin{aligned}
&\|\mathcal{W}_2 u(\mathtt{p},\mathtt{q})+\mathcal{W}_3 u(\mathtt{p},\mathtt{q})\|_{\mathbb{PC}}\\
&\quad\le \|\mathcal{W}_2(\mathtt{p},\mathtt{q})\|_{\mathbb{PC}}+\|\mathcal{W}_3(\mathtt{p},\mathtt{q})\|_{\mathbb{PC}}\\
&\quad\le \frac{1}{(\tau_1-1)!(\tau_2-1)!}\int_0^{\mathtt{p}}\int_0^{\mathtt{q}}(\mathtt{p}-s)^{\tau_1-1}(\mathtt{q}-t)^{\tau_2-1}\|\mathfrak{g}(s,t,u(s,t))\|_{\mathbb{PC}}dsdt\\
&\qquad+\frac{1}{(\tau_1-1)!(\tau_2-1)!}\sum_{\substack{0<\mathtt{p}_i<\mathtt{p}\\0<\mathtt{q}_i<\mathtt{q}}}\int_{\mathtt{p}_{i-1}}^{\mathtt{p}_i}\int_{\mathtt{q}_{i-1}}^{\mathtt{q}_i}(\mathtt{p}-s)^{\tau_1-1}(\mathtt{q}-t)^{\tau_2-1}\|\mathfrak{g}(s,t,u(s,t))\|_{\mathbb{PC}}dsdt\\
&\quad\le \frac{1}{(\tau_1-1)!(\tau_2-1)!}\int_0^{\mathtt{p}}\int_0^{\mathtt{q}}(\mathtt{p}-s)^{\tau_1-1}(\mathtt{q}-t)^{\tau_2-1}(\lambda_1\|u\|_{\mathbb{PC}}^{\beta}+\lambda_2)dsdt\\
&\qquad+\frac{1}{(\tau_1-1)!(\tau_2-1)!}\sum_{\substack{0<\mathtt{p}_i<\mathtt{p}\\0<\mathtt{q}_i<\mathtt{q}}}\int_{\mathtt{p}_{i-1}}^{\mathtt{p}_i}\int_{\mathtt{q}_{i-1}}^{\mathtt{q}_i}(\mathtt{p}-s)^{\tau_1-1}(\mathtt{q}-t)^{\tau_2-1}(\lambda_1\|u\|_{\mathbb{PC}}^{\beta}+\lambda_2)dsdt.
\end{aligned}$$

Then,

$$\|\mathcal{W}_2 u\|_{\mathbb{PC}}+\|\mathcal{W}_3 u\|_{\mathbb{PC}}\le \frac{(1+m)T^{\tau_1+\tau_2}(\lambda_1\|u\|_{\mathbb{PC}}^{\beta}+\lambda_2)}{(\tau_1)!(\tau_2)!},\ \ \forall\, u\in\mathbb{PC}(\mathcal{J}\times\mathcal{J},\mathcal{B}).$$

We need to make the following assumption in order to talk about the compactness of $\mathcal{W}_2,\mathcal{W}_3$.

$\mathbb{H}_6$. For any $\rho>0$, there exists a constant $\varpi_\rho>0$ such that

$$\vartheta(\mathfrak{g}(s,t,\mathbb{B}))\le\varpi_\rho\vartheta(\mathbb{B})$$

for all $s,t\in\mathcal{J}$, $\mathbb{B}\subset\mathbb{B}_\rho=\{u\in\mathbb{PC}(\mathcal{J},\mathcal{J},\mathbb{B}):\|u\|_{\mathbb{PC}}\le\rho\}$ and

$$\frac{4\varpi_\rho T^{\tau_1+\tau_2}}{(\tau_1)!(\tau_2)!}<1.$$

□

Lemma 3.5. $\mathcal{W}_2,\mathcal{W}_3:\mathbb{PC}(\mathcal{J},\mathcal{J},\mathcal{B})\to\mathbb{PC}(\mathcal{J},\mathcal{J},\mathcal{B})$ *are compact operators. Accordingly,* $\mathcal{W}_2$ *and* $\mathcal{W}_3$ *have zero constant and are* $\tau-$*Lipschitz. Furthermore, the constant of* $\mathcal{W}_2+\mathcal{W}_3$ *is zero and* $\tau-$*Lipschitz.*

Proof. A bounded subset $\mathbb{B}\subset\mathbb{PC}(\mathcal{J},\mathcal{J},\mathcal{B})$ is considered. To show that $\mathcal{W}_2,\mathcal{W}_3$ is compact, we must show that $\mathcal{W}_2(\mathbb{B})$ and $\mathcal{W}_3(\mathbb{B})$ are relatively compact in $\mathbb{PC}(\mathcal{J},\mathcal{J},\mathcal{B})$. Let $\{u_n\}$ be a sequence on $\mathbb{B}\subset\mathbb{B}_\rho$ for each $u_n\in\mathbb{B}$. As per relation (3.6), we possess

$$\|\mathcal{W}_2 u\|_{\mathbb{PC}}+\|\mathcal{W}_3 u\|_{\mathbb{PC}}\le\frac{(1+m)T^{\tau_1+\tau_2}(\lambda_1\|u\|_{\mathbb{PC}}^{\beta}+\lambda_2)}{(\tau_1)!(\tau_2)!},$$

$\mathcal{W}_2(\mathbb{B})$ and $\mathcal{W}_3(\mathbb{B})$ are bounded in $\mathbb{B}_\rho$ for each $u_n \in \mathbb{B}$. We now obtain, for $0 \leq \mathtt{p}_1 < \mathtt{p}_2 < T$, $0 \leq \mathtt{q}_1 < \mathtt{q}_2 < T$,

$$\begin{aligned}
&\|\mathcal{W}_2 u(\mathtt{p}_2, \mathtt{q}_2) - \mathcal{W}_2 u(\mathtt{p}_1, \mathtt{q}_1)\|_{\mathbb{PC}} \\
&\quad = \Big\| \frac{1}{(\tau_1-1)!(\tau_2-1)!} \int_0^{\mathtt{p}_2} \int_0^{\mathtt{q}_2} (\mathtt{p}_2 - s)^{\tau_1-1} (\mathtt{q}_2 - t)^{\tau_2-1} \mathfrak{g}(s,t,u(s,t)) ds dt \\
&\qquad - \frac{1}{(\tau_1-1)!(\tau_2-1)!} \int_0^{\mathtt{p}_1} \int_0^{\mathtt{q}_1} (\mathtt{p}_1 - s)^{\tau_1-1} (\mathtt{q}_1 - t)^{\tau_2-1} \mathfrak{g}(s,t,u(s,t)) ds dt \Big\|_{\mathbb{PC}} \\
&\quad \leq \frac{1}{(\tau_1-1)!(\tau_2-1)!} \int_0^{\mathtt{p}_1} \int_0^{\mathtt{q}_1} [(\mathtt{p}_2 - s)^{\tau_1-1} - (\mathtt{p}_1 - s)^{\tau_1-1}] \\
&\qquad \times [(\mathtt{q}_2 - t)^{\tau_2-1} - (\mathtt{q}_1 - t)^{\tau_2-1}] \|\mathfrak{g}(s,t,u(s,t))\|_{\mathbb{PC}} ds dt \\
&\qquad + \frac{1}{(\tau_1-1)!(\tau_2-1)!} \int_{\mathtt{p}_1}^{\mathtt{p}_2} \int_{\mathtt{q}_1}^{\mathtt{q}_2} (\mathtt{p}_2 - s)^{\tau_1-1} (\mathtt{q}_2 - t)^{\tau_2-1} \|\mathfrak{g}(s,t,u(s,t))\| ds dt \\
&\quad \leq \frac{(\lambda_1 \|u\|_{\mathbb{PC}}^\beta + \lambda_2)}{(\tau_1-1)!(\tau_2-1)!} \Bigg(\frac{[\mathtt{p}_2^{\tau_1} + (\mathtt{p}_2 - \mathtt{p}_1)^{\tau_1} - \mathtt{p}_1^{\tau_1} + (\mathtt{p}_2 - \mathtt{p}_1)^{\tau_1}]}{\tau_1} \\
&\qquad \times \frac{[\mathtt{q}_2^{\tau_2} + (\mathtt{q}_2 - \mathtt{q}_1)^{\tau_2} - \mathtt{q}_1^{\tau_2} + (\mathtt{q}_2 - \mathtt{q}_1)^{\tau_2}]}{\tau_2} \Bigg).
\end{aligned}$$

Hence, we have

$$\|\mathcal{W}_2 u(\mathtt{p}_2, \mathtt{q}_2) - \mathcal{W}_2 u(\mathtt{p}_1, \mathtt{q}_1)\|_{\mathbb{PC}} \leq \frac{4(\lambda_1 \|u\|_{\mathbb{PC}}^\beta + \lambda_2)}{(\tau_1)!(\tau_2)!} (\mathtt{p}_2 - \mathtt{p}_1)^{\tau_1} (\mathtt{q}_2 - \mathtt{q}_1)^{\tau_2}.$$

The inequality above tends to zero on the right hand side as $\mathtt{p}_2 \to \mathtt{p}_1$, $\mathtt{q}_2 \to \mathtt{q}_1$. Hence, $\{\mathcal{W}_2 u_n\}$ is equicontinuous as a result.

Let $\mathbb{B}(\mathtt{p}, \mathtt{q})$ be a bounded set $\{u_n(\mathtt{p}, \mathtt{q}) : u_n(\mathtt{p}, \mathtt{q}) = \frac{1}{(\tau_1-1)!(\tau_2-1)!} \int_0^{\mathtt{p}} \int_0^{\mathtt{q}} (\mathtt{p} - s)^{\tau_1-1} (\mathtt{q} - t)^{\tau_1-1} \mathfrak{g}(s,t,u_n(s,t)) ds dt\} \subset \mathbb{B}_\rho$. Proposition 2.12 states that $(\mathtt{p}, \mathtt{q}) \to \vartheta(\mathbb{B}(\mathtt{p}, \mathtt{q}))$ is continuous on $\mathcal{J}$, since $\mathbb{B}(\mathtt{p}, \mathtt{q}) \subset \mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B})$ is bounded and equicontinuous. For $s \in [0, \mathtt{p}]$, $t \in [0, \mathtt{q}]$, $\mathtt{p}, \mathtt{q} \in \mathcal{J}$, we can use $(\mathbb{H}_6)$ to obtain

$$\begin{aligned}
&\vartheta(\mathbb{B}(\mathtt{p}, \mathtt{q})) \\
&\quad \leq \vartheta\Big(\frac{1}{(\tau_1-1)!(\tau_2-1)!} \int_0^{\mathtt{p}} \int_0^{\mathtt{q}} (\mathtt{p} - s)^{\tau_1-1} (\mathtt{q} - t)^{\tau_2-1} \mathfrak{g}(s,t,u_n(s,t)) ds dt \Big) \\
&\quad \leq \frac{4}{(\tau_1-1)!(\tau_2-1)!} \int_0^{\mathtt{p}} \int_0^{\mathtt{q}} (\mathtt{p} - s)^{\tau_1-1} (\mathtt{q} - t)^{\tau_2-1} \vartheta(\mathfrak{g}(s,t,u_n(s,t))) ds dt \\
&\quad \leq \frac{4\varpi_\rho}{(\tau_1-1)!(\tau_2-1)!} \int_0^{\mathtt{p}} \int_0^{\mathtt{q}} (\mathtt{p} - s)^{\tau_1-1} (\mathtt{q} - t)^{\tau_2-1} \vartheta(\mathbb{B}) ds dt \\
&\quad \leq \frac{4\varpi_\rho T^{\tau_1+\tau_2}}{(\tau_1)!(\tau_2)!} \vartheta(\mathbb{B}) < \vartheta(\mathbb{B}).
\end{aligned}$$

Given that $\frac{4\varpi_\rho T^{\tau_1+\tau_2}}{(\tau_1)!(\tau_2)!} < 1$, we can infer that $\vartheta(\mathbb{B}) = 0$. Since $\mathcal{W}_2(\mathbb{B}) \subset \mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B})$ is a relatively compact set, $\mathcal{W}_2$ is $\tau-$Lipschitz with zero constant according to Proposition 2.13. By repeating the operation of the operator $\mathcal{W}_2$ on $C([0, \mathtt{p}_1], [0, \mathtt{q}_1, \mathcal{B}])$, we can ascertain the compactness of $\mathcal{W}_3$ on $C((\mathtt{p}_i, \mathtt{p}_{i+1}], (\mathtt{q}_i, \mathtt{q}_{i+1}], \mathcal{B})$ and $\mathcal{W}_3^m$ on $C((\mathtt{p}_m, T], (\mathtt{q}_m, T], \mathcal{B})$. $\mathcal{W}_3$ is $\tau-$Lipschitz with zero constant, according to Proposition 2.13. □

Theorem 3.6. *Assume that* $(\mathbb{H}_1) - (\mathbb{H}_6)$. *Then at least one solution* $u \in \mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B})$ *exists to the class of IFDEs* (1.1) *and the set of solutions is bounded.*

Proof. Let the operators $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{M} : \mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B}) \to \mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B})$. They have boundaries and are uninterrupted. Furthermore, $\mathcal{W}_2$ and $\mathcal{W}_3$ is $\tau-$Lipschitz with zero constant, whereas $\mathcal{W}_I$ is $\tau-$Lipschitz with constant $m\Phi_1$. $\mathcal{M}$ is $\tau-$Lipschitz $m\Phi_I \in [0, 1)$, as demonstrated by Proposition 2.12. It is also a strict $\tau-$contraction with constant $m\Phi_I$. Consider

$$Q = \{u \in \mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B}) : \text{there exist } \eta \in [0, 1] \text{ such that } u = \eta\mathcal{M}u\}.$$

Next, we need to show that $Q \in \mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B})$ is bounded. Now u is in Q and η is in $[0, 1]$ so that $u = \eta\mathcal{M}u$. This is inferred from (3.5) and (3.6)

$$\begin{aligned}\|u\|_{\mathbb{PC}} &\leq \|\eta\mathcal{M}u\|_{\mathbb{PC}} \leq \eta(\|\mathcal{W}_1 u\|_{\mathbb{PC}} + \|\mathcal{W}_2 u\|_{\mathbb{PC}} + \|\mathcal{W}_3 u\|_{\mathbb{PC}}) \\ &\leq \|u_0\| + m\Phi_1\|u\|_1^\beta + m\Phi_2 + \frac{(1+m)T^{\tau_1+\tau_2}(\lambda_1\|u\|_{\mathbb{PC}}^\beta + \lambda_2)}{(\tau_1)!(\tau_2)!}.\end{aligned}$$

The previous inequality indicates that Q is bounded in $\mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B})$, along with $\beta < 1$, $\beta_1 < 1$ and $\beta_2 < 1$. From Theorem 2.11, we can therefore deduce that the set $\mathcal{M}$ of fixed points are bounded in $\mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B})$ and that H has a minimum of one fixed point. □

Remark 3.7. Now

- **(a):** The results of Theorem 3.6 are still true if an expansion condition $(\mathbb{H}_3)$ is constructed for $\beta = 1$. This is because $\frac{(1+m)T^{\tau_1+\tau_2}\lambda_1}{(\tau_1)!(\tau_2)!} < 1$.
- **(b):** The results of Theorem 3.6 are still true if the expansion condition $(\mathbb{H}_4)$ is written for $\beta_1 = \beta_2 = 1$. This is because $m\Phi_1 < 1$.
- **(c):** The results of Theorem 3.6 are still true and $(\mathbb{H}_3)$ and $(\mathbb{H}_4)$ created for $\beta = \beta_1 = \beta_2 = 1$. This is because $m\Phi_1 + \frac{(1+m)T^{\tau_1+\tau_2}\lambda_1}{(\tau_1)!(\tau_2)!} < 1$.

Theorem 3.8. *Let us assume that* $(\mathbb{H}_1) - (\mathbb{H}_6)$ *hold. If*

$$m\Phi_1 + \frac{(1+m)\lambda_f T^{\tau_1+\tau_2}}{(\tau_1)!(\tau_2)!} < 1, \tag{3.7}$$

then there is only one solution for the IFDEs (1.1), *where* $u \in \mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B})$.

Proof. Let $u, v \in \mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B})$, $(\mathbb{H}_2)$ and $(\mathbb{H}_4)$, which make it simple to demonstrate that $\mathcal{M}$ is a contraction operator on $\mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B})$.

$$
\begin{aligned}
&\|\mathcal{M}u(\mathtt{p},\mathtt{q}) - \mathcal{M}v(\mathtt{p},\mathtt{q})\|_{\mathbb{PC}} \\
&\quad \le \sum_{\substack{0<\mathtt{p}_i<\mathtt{p}\\0<\mathtt{q}_i<\mathtt{q}}} \|I_i(u(\mathtt{p}_i,\mathtt{q}_i)) - I_i(v\mathtt{p}_i,\mathtt{q}_i))\| \\
&\qquad + \frac{1}{(\tau_1-1)!(\tau_2-1)!} \sum_{\substack{0<\mathtt{p}_i<\mathtt{p}\\0<\mathtt{q}_i<\mathtt{q}}} \int_{\mathtt{p}_{i-1}}^{\mathtt{p}_i} \int_{\mathtt{q}_{i-1}}^{\mathtt{q}_i} (\mathtt{p}_i - s)^{\tau_1-1}(\mathtt{q}_i - t)^{\tau_2-1} \\
&\qquad \times \|\mathfrak{g}(s,t,u) - \mathfrak{g}(s,t,v)\|_{\mathbb{PC}} ds dt \\
&\qquad + \frac{1}{(\tau_1-1)!(\tau_2-1)!} \int_0^{\mathtt{p}} \int_0^{\mathtt{q}} (\mathtt{p} - s)^{\tau_1-1}(\mathtt{q} - t)^{\tau_2-1} \\
&\qquad \times \|\mathfrak{g}(s,t,u) - \mathfrak{g}(s,t,v)\|_{\mathbb{PC}} ds dt \\
&\quad \le \sum_{\substack{0<\mathtt{p}_i<\mathtt{p}\\0<\mathtt{q}_i<\mathtt{q}}} \Phi_1 \|u - v\| \\
&\qquad + \frac{1}{(\tau_1-1)!(\tau_2-1)!} \sum_{\substack{0<\mathtt{p}_i<\mathtt{p}\\0<\mathtt{q}_i<\mathtt{q}}} \int_{\mathtt{p}_{i-1}}^{\mathtt{p}_i} \int_{\mathtt{q}_{i-1}}^{\mathtt{q}_i} (\mathtt{p}_i - s)^{\tau_1-1}(\mathtt{q}_i - t)^{\tau_2-1} \lambda_f \|u - v\|_{\mathbb{PC}} ds dt \\
&\qquad + \frac{1}{(\tau_1-1)!(\tau_2-1)!} \int_0^{\mathtt{p}} \int_0^{\mathtt{q}} (\mathtt{p} - s)^{\tau_1-1}(\mathtt{q} - t)^{\tau_2-1} \lambda_f \|u - v\|_{\mathbb{PC}} ds dt,
\end{aligned}
$$

$i = 1, 2, \ldots, m$. Hence, we have

$$
\|\mathcal{M}u(\mathtt{p},\mathtt{q}) - \mathcal{M}v(\mathtt{p},\mathtt{q})\|_{\mathbb{PC}} \le \left[m\Phi_1 + \frac{(1+m)\lambda_f T^{\tau_1+\tau_2}}{(\tau_1)!(\tau_2)!} \right] \|u - v\|_{\mathbb{PC}}.
$$

Therefore, $\mathcal{M}$ is a contraction operator on $\mathbb{PC}(\mathcal{J}, \mathcal{J}, \mathcal{B})$ along with a contraction constant $m\Phi_1 + \frac{(1+m)\lambda_f T^{\tau_1+\tau_2}}{(\tau_1)!(\tau_2)!}$ as demonstrated by the condition $m\Phi_1 + \frac{(1+m)\lambda_f T^{\tau_1+\tau_2}}{(\tau_1)!(\tau_2)!} < 1$. So equation (1.1) has a unique solution in view of fixed point theorem. □

4. Example

To demonstrate the results established in section 3, we give an example.

Example 4.1. Consider the problem described by

$$
\begin{cases}
{}^c[\mathcal{D}]^{\frac{1}{4}} u(\mathtt{p},\mathtt{q}) = \frac{p+q}{\sqrt{7}} \sin(\frac{1}{5} u(\mathtt{p},\mathtt{q})), & \mathtt{p},\mathtt{q} \in (0,2)/1, \\
\Delta u(1,1) = \frac{2}{5\sqrt{7}}, \\
u(0,0) = 0.
\end{cases}
\tag{4.1}
$$

Let $\mathfrak{g}(\mathtt{p},\mathtt{q},u) = \frac{3}{\sqrt{7}}\sin\Big(\frac{1}{15}u(\mathtt{p},\mathtt{q})\Big)$. Also for $\mathtt{p},\mathtt{q} \in (0,2)/1$, we now define $I_i(u(\mathtt{p}_i,\mathtt{q}_i)) = \frac{2}{15\sqrt{37}}, i = 1$. Then the satisfaction of all the assumptions in Theorem 3.3 can be easily observed.

$$\begin{aligned}\|\mathfrak{g}(\mathtt{p},\mathtt{q},u) - \mathfrak{g}(\mathtt{p},\mathtt{q},v)\| &\leq \frac{3}{\sqrt{7}}\|\sin\Big(\frac{1}{5}u(\mathtt{p},\mathtt{q})\Big) - \sin\Big(\frac{1}{5}v(\mathtt{p},\mathtt{q})\Big)\| \\ &\leq \frac{3}{5\sqrt{7}}\|u-v\|\end{aligned}$$

and

$$|I(u) - I(v)| = \frac{2}{5\sqrt{7}}.$$

We will now verify that condition (3.7) is met with $T = \frac{1}{2}$, $m = 1$, $\mathtt{p} = \mathtt{q} = \frac{1}{4}$ and

$$m\Phi_1 + \frac{(1+m)\lambda_f T^{\mathtt{p}+\mathtt{q}}}{(\mathtt{p})!(\mathtt{q})!} = \frac{2}{5\sqrt{7}} + \frac{2(\frac{3}{5\sqrt{7}})(\frac{1}{2})^{\frac{1}{4}+\frac{1}{4}}}{\Gamma(\frac{1}{4}+1)\Gamma(\frac{1}{4}+1)} = 0.5415567214 < 1.$$

Therefore, the fractional impulsive problem can be resolved using our results.

5. Conclusion

By using topological sequences, we have established some sufficient conditions for the existence theory of solution to fractional impulsive differential equations involving nonlocal conditions. We have presented a formula for solving an impulsive fractional problem in a Banach space that involved a generalization of the classical Caputo derivative with a nonlocal condition. Using fixed point techniques, certain prerequisites for the existence of solutions have been established. We deduced some results based on topological degree techniques to determine the existence and uniqueness of solution to impulsive fractional order problem under our consideration. For justification, an example has been given to demonstrate our findings.

Acknowledgment: Authors would like to thank Prince Sultan University for the support through the TAS research lab.

References

[1] K.A. Abro, A. Atangana and J.F. Gómez-Aguilar, *A comparative analysis of plasma dilution based on fractional integro-differential equation: an application to biological science*, Int. J. Model. Simulation, **43**(1) (2022), 1-10.

[2] R.P. Agarwal, M. Benchohra and S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta. Appl. Math., **109** (2010), 973-1033.

[3] R.P. Agarwal, Y. Zhou and Y. He, *Existence of fractional neutral functional differential equations*, Comput. Math. Appl., **59** (2010), 1095-1100.

[4] A. Alkhazzan, J. Wang, Y. Nie, S.M.A. Shah, D.K. Almutairi, H Khan and J. Alzabut, *Lyapunov-based analysis and worm extinction in wireless networks using stochastic SVEIR model*, Alexand. Eng. J., **118** (2025), 337-353.

[5] M. Benchohra and D. Seba, *Impulsive fractional differential equations in Banach Spaces*, Elect. J. Qualitative Theory Dif. Equ., **8** (2009), 1-14.

[6] B. Ghanbari, *A new model for investigating the transmission of infectious diseases in a prey-predator system using a non-singular fractional derivative*, Math. Meth. Appl. Sci., **46**(7) (2023), 8106-8125.

[7] F. Isaia, *On a nonlinear integral equation without compactness*, Acta Mathematica Universitatis Comenianae. New Series, **75**(2) (2006), 233-240.

[8] H. Khan, S. Ahmed, J. Alzabut and A. Taher Azar,*A generalized coupled system of fractional differential equations with application to finite time sliding mode control for Leukemia therapy*, Chaos, Solitons Fractals, **174** (2023), 113901.

[9] H. Khan, A.H. Rajpar, J. Alzabut, M. Aslam, S. Etemad and S. Rezapour, *On a Fractal-fractional-based modeling for influenza and its analytical results*, Qual. Theory Dyn. Syst., **23**:70 (2024), 21 pages.

[10] P. Kumar and V.S. Ertürk, *The analysis of a time delay fractional COVID-19 model via Caputo type fractional derivative*, Math. Meth. Appl. Sci., **46**(7) (2023), 7618-7631.

[11] K.S. Miller and B. Ross, *An introduction to the fractional calculus and differential equations*, New York, John Wiley, 1993.

[12] C. Monickpriya, U. Karthik Raja, D. Swathi and V. Pandiyammal, *Results on the existence of impulsive neutral fractional differential equations with infite delay,* J. Algebr. Stat., **13**(3) (2022), 1285-1295.

[13] S.M.A. Shah, Y. Nie, A. Alkhazzan, C. Tunc and D. Anwarud, *A stochastic hepatitis B model with media coverage and Lévy noise,* J. Taibah Univ. Sci., **18** (2024), 2414523.

[14] S.M.A.Shah, Y. Nie, D. Anwarud and A. Alkhazzan, *Stochastic Optimal Control Analysis for HBV Epidemic Model with Vaccination*, Symmetry, **16**:10 (2024).

[15] S.M.A. Shah, Y. Nie, D. Anwarud, A. Alkhazzan and B. Younas, *Stochastic modeling and analysis of Hepatitis and Tuberculosis co-infection dynamics,* Chinese Phys. B., **33** (2024), 110203.

[16] S.M.A. Shah, Y. Nie, A. Din and A. Alkhazzan, *Dynamics of hepatitis b virus transmission with a Lévy process and vaccination effects*, Mathematics, **12**(11) (2024), 1645.

[17] S.M.A. Shah, H. Tahir, A. Khan, W.A. khan and A. Arshad, *Stochastic Model on the Transmission of Worms in Wireless Sensor Network,* Journal of Mathematical Techniques in Modeling, **1**(1) (2024), 75-88.

[18] A.M. Shloof, N. Senu, A. Ahmadian, M. Pakdaman and S. Salahshour, *A new iterative technique for solving fractal-fractional differential equations based on artificial neural network in the new generalized Caputo sense*, Engineering with Computers, **39**(1) (2023), 505-515.

[19] D. Swathi, K. Ananthi, U. Karthik Raja and S.N. Kottakkaran, *New existence results on random nonlocal fractional differential equation using approximating sequences*, Partial Dif. Equ. Appl. Math., **13** (2025), 101–106.

[20] D. Swathi, K. Ananthi, U. Karthik Raja and C. Monick Priya, *New Results on Existence of Coupled Caputo Fractional Derivative System with nonlocal integral conditions*, Telematique, **2022**(2022), 2089-2099.

[21] H. Tajadodi, Z.A. Khan, J.F. Gómez-Aguilar, A. Khan and H. Khan, *Exact solutions of conformable fractional differential equations*, Results Phys., **22** (2021), 103916.

[22] J. Wang, Y. Zhou and M. Medve, *Qualitative analysis for nonlinear fractional differential equations via topological degree method*, Topol. Methods, Nonlinear Anal., **40**(2) (2012), 245-271.

[23] J. Wang, Y. Zhou and W. Wei, *Study in fractional differential equations by means of topological degree methods*, Numer. Funct. Anal. Optim., **33**(2) (2012), 216-238.

[24] Y. Zhou, J. Wang and L. Zhang, *Basic Theory of Fractional Differential Equations*, Worldscientific, 2016.