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$\mathcal{G}$-CYCLICITY FOR CERTAIN CLASSES OF OPERATORS IN BANACH SPACES

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Abstract. In this paper, we present new theoretical results concerning $\mathcal{G}$-cyclicity for certain classical linear operators. Specifically, we characterize all bilateral weighted shift operators that exhibit $\mathcal{G}$-cyclic behavior and establish a framework that differentiates $\mathcal{G}$-cyclicity from other forms such as supercyclicity, hypercyclicity, and diskcyclicity. Moreover, we demonstrate that some classical operators, including the composition and multiplication operators on the Hardy space, fail to be $\mathcal{G}$-cyclic under general conditions. These findings not only generalize existing criteria but also enrich the understanding of operator dynamics within function spaces.

1. Introduction

A bounded linear operator T on a separable Banach space $\mathcal{X}$ is hypercyclic if there exists a vector $x \in \mathcal{X}$ such that the orbit $Orb(T,x) = \{T^n x : n \geq 0\}$ is dense in $\mathcal{X}$. Such a vector x is called hypercyclic for T. The first example was constructed by Rolewicz in 1969 [10], showing that if B is the backward shift on $\ell^p(\mathbb{N})$, then λB is hypercyclic if and only if $|\lambda| > 1$.

Following this, Hilden and Wallen [6] introduced supercyclicity, where an operator T is supercyclic if there exists a vector x such that the projective

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orbit $\mathbb{C}Orb(T,x) = \{\alpha T^n x : \alpha \in \mathbb{C}, n \geq 0\}$ is dense in $\mathcal{X}$. Also, an operator T is diskcyclic if the disk orbit $\mathbb{D}Orb(T,x) = \{\alpha T^n x : \alpha \in D, n \in \mathbb{N}\}$ is dense in $\mathcal{X}$ [2]. Further generalizing this, Zeana [12] introduced $\mathcal{G}$-cyclicity. Let $\mathcal{G}$ be a multiplicative semigroup with identity in $\mathbb{C}$; then T is $\mathcal{G}$-cyclic if there exists $x \in \mathcal{X}$ such that $\mathcal{G}Orb(T,x) = \{\alpha T^n x : \alpha \in \mathcal{G}, n \in \mathbb{N}\}$ is dense in $\mathcal{X}$. For more information about these operators, the reader may refer to [1, 3, 7, 11].

We recall the following facts from [9].

Definition 1.1. ([9]) Let $T \in \mathcal{B}(\mathcal{X})$. Then T is called $\mathcal{G}$-transitive if for each pair of nonempty open sets U_1, U_2 of $\mathcal{X}$, there exists an $n \in \mathbb{N}$ and $\alpha \in \mathcal{G}\backslash\{0\}$ such that $T^{-n}\frac{1}{\alpha}U_1 \cap U_2 \neq \phi$ or equivalently, $T^n \alpha U_2 \cap U_1 \neq \phi$.

Proposition 1.2. ([9]) *An operator is $\mathcal{G}$-transitive if and only if it is $\mathcal{G}$-cyclic.*

Proposition 1.3. ([9]) *Let $T \in \mathcal{B}(\mathcal{X})$. Then the following statements are equivalent:*

(1) *T is $\mathcal{G}$-transitive.*
(2) *For all $x, y \in \mathcal{X}$, there exist sequences $\{x_k\}_{k\in\mathbb{N}} \subset \mathcal{X}$, $\{\alpha_k\}_{k\in\mathbb{N}} \subset \mathcal{G}\backslash\{0\}$ and an increasing sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$ such that $x_k \to x$ and $T^{n_k}\alpha_k x_k \to y$ as $k \to \infty$.*
(3) *For all $x, y \in \mathcal{X}$ and each neighborhood W for zero in $\mathcal{X}$, there exist $z \in \mathcal{X}$, $n \in \mathbb{N}$ and a non-zero $\alpha \in \mathcal{G}$ such that $x - z \in W$ and $T^n \alpha z - y \in W$.*

The motivation for studying $\mathcal{G}$-cyclicity stems from its ability to unify and generalize classical cyclic phenomena of operators. While hypercyclicity and supercyclicity have been extensively studied, $\mathcal{G}$-cyclicity introduces a more flexible framework by incorporating arbitrary multiplicative semigroups. This allows for a richer and more nuanced classification of operator behavior, particularly in functional spaces where classical cyclicity may not hold. The results presented here contribute to this understanding by characterizing $\mathcal{G}$-cyclicity for classes of operators that are central in analysis and operator theory.

The following diagram shows the relation among hypercyclic, diskcyclic, supercyclic and $\mathcal{G}$-cyclic operators.

$$\begin{array}{ccc} \text{Hypercyclicty} & \Rightarrow & \text{Diskcyclicity} \\ \Downarrow & & \Downarrow \\ \mathcal{G}\text{-cyclicty} & \Rightarrow & \text{Supercyclicty} \end{array}$$

Some of the reverse directions of the above diagram are not exist in the literature. Therefore, we give necessary and sufficient conditions for weighted shift operators to be $\mathcal{G}$-cyclic. We use these characterizations, to show that there exists supercyclic operator which is not $\mathcal{G}$-cyclic, and a $\mathcal{G}$-cyclic operator which is not hypercyclic. Also, we show that neither diskcyclicity implies $\mathcal{G}$-cyclicity nor $\mathcal{G}$-cyclicity implies diskcyclicity.

Moreover, we prove that the composition operator C_ϕ on the Hardy space cannot be $\mathcal{G}$-cyclic when the map ϕ has a fixed point within the unit disk $\mathcal{G}$. Furthermore, we show that the multiplication operator M_ϕ on the Hardy space is never $\mathcal{G}$-cyclic, regardless of the choice of ϕ. Finally, we prove that the derivative operator, when acting on the space of entire functions is $\mathcal{G}$-cyclic. These results play a deep understanding of operator behavior within functional spaces and provide new insights into the dynamics of these operators.

2. Main results

Throughout this paper, we assume that G is a multiplicative semigroup with identity in $\mathbb{C}$. Unless otherwise stated, all scalars in $\mathcal{G}$ are assumed to be nonzero. We begin with a general criterion for G-cyclicity which can be immediately followed by [8, Theorem 2.2].

Proposition 2.1. ($\mathcal{G}$-cyclic criterion) *Let T be an operator on a separable Banach space. Then T is said to satisfy the $\mathcal{G}$-cyclic criterion if there exist two dense subsets D_1, D_2 in $\mathcal{X}$, an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers, a sequence $\{\alpha_k\}_{k\in\mathbb{N}} \in G\backslash\{0\}$ and a sequence of maps $S_{n_k} : D_2 \to \mathcal{X}$ such that*

(1) $\alpha_k T^{n_k} x \to 0$ *for all* $x \in D_1$,
(2) $\dfrac{1}{\alpha_k} S_{n_k} y \to 0$ *for all* $y \in D_2$,
(3) $T^{n_k} S_{n_k} y \to y$ *for each* $y \in D_2$.

If T satisfies the $\mathcal{G}$-cyclic criterion, then T is $\mathcal{G}$-transitive.

Let $\{e_n : n \in \mathbb{Z}\}$ be the Schauder basis for $\mathbb{Z}$. We derive necessary and sufficient conditions for bilateral forward and backward weighted shift operators to be $\mathcal{G}$-cyclic, using variations of the above criterion.

Theorem 2.2. *Let T be a bilateral forward weighted shift in the Hilbert space $\ell^2(\mathbb{Z})$ with a positive weight sequence $\{w_n\}_{n\in\mathbb{Z}}$. Then T is $\mathcal{G}$-cyclic if and only if for a given $q \in \mathbb{N}$, there exist a sequence $\{\alpha_n\} \subset \mathcal{G}$ such that for all $|j| \le q$,*

$$\lim_{n\to\infty} \prod_{k=j}^{j+n-1} |\alpha_n| w_k = 0 \, and \, \lim_{n\to\infty} \prod_{k=1-j}^{n-j} \frac{1}{|\alpha_n| w_{-k}} = 0.$$

Proof. Let T be an $\mathcal{G}$-cyclic operator, $q \in \mathbb{N}$ and $y = z = \sum_{|j|\le q} e_j$. By Proposition 1.3, for every large number $n > 2q$, there exists a vector $x^{(n)} \in$

$\ell^2(\mathbb{Z})$, $\alpha_n \in \mathcal{G}$ and a small positive number ε_n such that

$$\left\| x^{(n)} - \sum_{|j|\le q} e_j \right\| < \varepsilon_n \tag{2.1}$$

and

$$\left\| \alpha_n T^n x^{(n)} - \sum_{|j|\le q} e_j \right\| < \varepsilon_n. \tag{2.2}$$

Inequality (2.1) implies that $|x_j^{(n)}| > 1-\varepsilon_n$ if $|j| \le q$ and $|x_j^{(n)}| < \varepsilon_n$ otherwise. Since $n > 2q$, inequality (2.2) implies that for $|j| \le q$

$$|x_j^{(n)}|\,\|\alpha_n T^n e_j\| = |x_j^{(n)}| \left(|\alpha_n| \prod_{k=0}^{n-1} w_{k+j} \right) < \varepsilon_n.$$

It follows that

$$\left(\prod_{k=0}^{n-1} |\alpha_n| w_{k+j} \right) < \frac{\varepsilon_n}{|x_j^{(n)}|} < \frac{\varepsilon_n}{1-\varepsilon_n} = \delta.$$

Then

$$\lim_{n\to\infty} \left(\prod_{k=0}^{n-1} |\alpha_n| w_{k+j} \right) = 0$$

and so

$$\lim_{n\to\infty} \prod_{k=j}^{j+n-1} |\alpha_n| w_k = 0. \tag{2.3}$$

Also inequality (2.2) implies that $\left\| x_{j-n}^{(n)}(\alpha_n T^n e_{j-n}) - e_j \right\| < \varepsilon_n$ for $|j| \le q$. Thus,

$$\left| x_{j-n}^{(n)} \right| \left| \alpha_n \prod_{k=0}^{n-1} w_{j-n+k} - 1 \right| = \left| x_{j-n}^{(n)} \right| \left| \alpha_n \prod_{k=1}^{n} w_{j-k} - 1 \right| < \varepsilon_n.$$

Therefore,

$$\left(|\alpha_n| \prod_{k=1}^{n} w_{j-k} \right) > \frac{1-\varepsilon_n}{\left| x_{j-n}^{(n)} \right|} > \frac{1-\varepsilon_n}{\varepsilon_n}.$$

It follows that

$$\left(\prod_{k=1}^{n} \frac{1}{|\alpha_n| w_{j-k}} \right) < \frac{\varepsilon_n}{1-\varepsilon_n}$$

and so

$$\lim_{n\to\infty}\prod_{k=1-j}^{n-j}\frac{1}{|\alpha_n|w_{-k}}=0. \tag{2.4}$$

The proof follows by equation (2.3) and (2.4).

Conversely, let $q\in\mathbb{N}$, then there exist a sequence $\{\alpha_n\}\subset\mathcal{G}$ such that for all $|j|\le q$,

$$\lim_{n\to\infty}\prod_{k=j}^{j+n-1}|\alpha_n|w_k=0 \text{ and } \lim_{n\to\infty}\prod_{k=1-j}^{n-j}\frac{1}{|\alpha_n|w_{-k}}=0.$$

Let $D=span\{e_j:|j|\le q\}$, and let $x=\sum_{|j|\le q}x_je_j$ and $y=\sum_{|j|\le q}y_je_j$ be two vectors in D. Then,

$$\|\alpha_nT^nx\|\le\left\{\prod_{k=j}^{j+n-1}|\alpha_n|w_k:|j|\le q\right\}\|x\|$$

and

$$\left\|\frac{1}{\alpha_n}S^ny\right\|\le\left\{\prod_{k=1-j}^{n-j}\frac{1}{|\alpha_n|w_{-k}}:|j|\le q\right\}\|y\|.$$

Therefore, $\lim_{n\to\infty}\|\alpha_nT^nx\|=\lim_{n\to\infty}\left\|\frac{1}{\alpha_n}S^ny\right\|=0$. Also, it is clear that $T^nS^ny=y$ for all $y\in D$. It follows that T satisfies $\mathcal{G}$-cyclic criterion and so T is $\mathcal{G}$-cyclic. □

Theorem 2.3. *Let T be an invertible bilateral forward weighted shift in the Hilbert space $\ell^2(\mathbb{Z})$ with a positive weight sequence $\{w_n\}_{n\in\mathbb{Z}}$. Then T is $\mathcal{G}$-cyclic if and only if there exist an increasing sequence of positive integers $n_k\to\infty$ and a sequence $\{\alpha_k\}\subset\mathcal{G}$ such that*

$$\lim_{k\to\infty}|\alpha_k|\prod_{j=1}^{n_k}w_j=0 \quad \textit{and} \quad \lim_{k\to\infty}\frac{1}{|\alpha_k|}\prod_{j=1}^{n_k}\frac{1}{w_{-j}}=0. \tag{2.5}$$

Proof. To prove the "if" part, we will verify the $\mathcal{G}$-cyclic criterion with $D=D_1=D_2$ is a dense subset of $\ell^2(\mathbb{Z})$ consisting of sequences that only have a finite number of non-zero entries. Let $x\in D$. Then it is enough to show that $x=e_i,\ i\in\mathbb{Z}$. In addition, by [4, Lemma 3.3.] it suffices to show that $\alpha_kT^{n_k}e_1\to0$ and $\frac{1}{\alpha_k}S^{n_k}e_0\to0$. However, that is clear because

$$\|\alpha_kT^{n_k}e_1\|=|\alpha_k|\prod_{j=1}^{n_k}w_j\to0 \quad\text{and}\quad \left\|\frac{1}{\alpha_k}S^{n_k}e_0\right\|=\frac{1}{|\alpha_k|}\prod_{j=1}^{n_k}\frac{1}{w_j}\to0$$

as $k \to \infty$. Moreover it is clear that $T^{n_k} S^{n_k} x = x$. Therefore, the conditions of $\mathcal{G}$-cyclic criterion are satisfied. The proof of the "only if" part follows from Theorem 2.2. □

It is well known that a weighted shift operator T is invertible if and only if there exists $b > 0$ such that $|w_n| \geq b$ for all $n \in \mathbb{Z}$. The next theorem shows that the above theorem still holds by assuming that there exists $b > 0$ such that $w_n \geq b$ for all $n < 0$.

Theorem 2.4. *Let T be a bilateral forward weighted shift in the Hilbert space $\ell^2(\mathbb{Z})$ with a positive weight sequence $\{w_n\}_{n\in\mathbb{Z}}$, $w_n \geq b > 0$ for all $n < 0$. Then T is $\mathcal{G}$-cyclic if and only if there exist an increasing sequence of positive integers $n_k \to \infty$ and a sequence $\{\alpha_k\} \subset \mathcal{G}$ such that*

$$\lim_{k\to\infty} |\alpha_k| \prod_{j=1}^{n_k} w_j = 0 \quad \textit{and} \quad \lim_{k\to\infty} \frac{1}{|\alpha_k|} \prod_{j=1}^{n_k} \frac{1}{w_{-j}} = 0. \tag{2.6}$$

Proof. For "if" part, we will verify Theorem 2.2. Let $\varepsilon > 0$, $q \in \mathbb{N}$ and let $\delta > 0$, by hypothesis there exists an arbitrary large $n_0 \in \{n_k\}$ and $\alpha_0 \in \mathcal{G}$ such that

$$\prod_{j=1}^{n_0} |\alpha_0| w_j < \delta \text{ and } \prod_{j=1}^{n_0} \frac{1}{|\alpha_0| w_{-j}} < \delta.$$

Let $n = n_0 + q + 1$ (which ensure that $j + n - 1 \geq n_0$ for all $j; |j| \leq q$). Now let $j \in Z$, $|j| \leq q$. Then

$$\begin{aligned} |\alpha_n| \prod_{i=j}^{n+j-1} w_i &= |\alpha_n| \left(\prod_{i=1}^{j-1} \frac{1}{w_i}\right) \left(\prod_{i=1}^{j-1} w_i\right) \left(\prod_{i=j}^{n_0} w_i\right) \left(\prod_{i=n_0+1}^{n+j-1} w_i\right) \\ &= \left(\prod_{i=1}^{j-1} \frac{1}{w_i}\right) \left(|\alpha_0| \prod_{i=1}^{n_0} w_i\right) \left(\frac{|\alpha_n|}{|\alpha_0|} \prod_{i=n_0+1}^{n+j-1} w_i\right) \\ &\leq C \left(\prod_{i=1}^{n_0} |\alpha_0| w_j\right) d \left\|T^{2q}\right\| \\ &\leq C\delta d \left\|T^{2q}\right\|, \end{aligned}$$

where $C = \prod_{i=1}^{j-1} \frac{1}{w_i}$ is a constant depending only on q, $\frac{|\alpha_n|}{|\alpha_0|} = d$ is constant and $\prod_{i=n_0+1}^{n+j-1} w_i \leq \left\|T^{2q}\right\|$. So, if $\delta < \frac{\varepsilon}{Cd\|T^{2q}\|}$, then

$$\prod_{i=j}^{n+j-1} |\alpha_n| w_i \leq \varepsilon$$

for all $|j| \leq q$.

Also, with the same choice of n (which ensure that $n-j > n_0+1$) and $|j| \leq q$, we have

$$\begin{aligned}\prod_{i=1-j}^{n-j} \frac{1}{|\alpha_n| w_{-i}} &= \left(\frac{|\alpha_0|}{|\alpha_n|} \prod_{i=1}^{-j} w_{-i}\right)\left(\frac{1}{|\alpha_0|} \prod_{i=1}^{-j} \frac{1}{w_{-i}}\right)\left(\prod_{i=1-j}^{n_0} \frac{1}{w_{-i}}\right)\left(\prod_{i=n_0+1}^{n-j} \frac{1}{w_{-i}}\right) \\ &= \left(\prod_{i=1}^{-j} w_{-i}\right)\left(\prod_{i=1}^{n_0} \frac{1}{|\alpha_0| w_{-i}}\right)\left(\frac{|\alpha_0|}{|\alpha_n|} \prod_{i=n_0+1}^{n-j} \frac{1}{w_{-i}}\right) \\ &\leq L\delta h \left(\frac{1}{b}\right)^{2q},\end{aligned}$$

where b is a lower bound for the negative weights, $L = \prod_{i=1}^{-j} w_{-i}$ is a constant depending only on q and $\frac{|\alpha_0|}{|\alpha_n|} = h$ is constant. Hence, if $\delta < \frac{b^{2q}\varepsilon}{Ld}$, then

$$\prod_{i=1-j}^{n-j} \frac{1}{|\alpha_n| w_{-i}} \leq \varepsilon$$

for all $|j| \leq q$. The converse side follows immediately from Theorem 2.2. □

By the same way we can characterize the $\mathcal{G}$-cyclic backward weighted shifts.

Proposition 2.5. *Let T be an invertible bilateral backward weighted shift in the Hilbert space $\ell^2(\mathbb{Z})$ with a positive weight sequence $\{w_n\}_{n\in\mathbb{Z}}$. Then T is $\mathcal{G}$-cyclic if and only if there exist an increasing sequence of positive integers $n_k \to \infty$ and a sequence $\{\alpha_k\} \subset \mathcal{G}$ such that*

$$\lim_{k\to\infty} \frac{1}{|\alpha_k|} \prod_{j=1}^{n_k} w_j = 0 \quad \textit{and} \quad \lim_{k\to\infty} |\alpha_k| \prod_{j=1}^{n_k} \frac{1}{w_{-j}} = 0. \tag{2.7}$$

Proposition 2.6. *Let T be a bilateral backward weighted shift in the Hilbert space $\ell^2(\mathbb{Z})$ with a positive weight sequence $\{w_n\}_{n\in\mathbb{Z}}$, $w_n \geq b > 0$ for all $n < 0$. Then T is $\mathcal{G}$-cyclic if and only if there exist an increasing sequence of positive integers $n_k \to \infty$ and a sequence $\{\alpha_k\} \subset \mathcal{G}$ such that*

$$\lim_{k\to\infty} \frac{1}{|\alpha_k|} \prod_{j=1}^{n_k} w_j = 0 \quad \textit{and} \quad \lim_{k\to\infty} |\alpha_k| \prod_{j=1}^{n_k} \frac{1}{w_{-j}} = 0. \tag{2.8}$$

Now, we use these characterizations to show that not every supercyclic operator is $\mathcal{G}$-cyclic.

Example 2.7. Let F be a bilateral forward weighted shift operator on $\ell^2(\mathbb{Z})$ with

$$w_n = \left\{ \begin{array}{ccc} \frac{1}{n+1} & \text{for} & n \geq 0 \\ \frac{1}{3} & \text{for} & n < 0 \end{array} \right\}.$$

Then F is supercyclic but not $\mathcal{G}$-cyclic. In fact, by [4, Theorem 4.1], an invertible bilateral forward weighted shift operator F is supercyclic if and only if there exists an increasing sequence $\{n_i\}$ of positive integers such that

$$\lim_{i\to\infty} \prod_{k=1}^{n_i} w_k \prod_{k=1}^{n_i} \frac{1}{w_{-k}} = 0.$$

If we take $n_i = n$ the whole natural numbers, then we get

$$\lim_{n\to\infty} \prod_{k=1}^{n} w_k \prod_{k=1}^{n} \frac{1}{w_{-k}} = \lim_{n\to\infty} \frac{1}{n!} 3^n = 0.$$

Therefore, F is supercyclic.

On the other hand, let z_0 be fixed point in the unit circle $\mathbb{T}$ and $\mathcal{G} = \{z_0^n : n \geq 0\}$ be a multiplication semigroup with identity in $\mathbb{C}$. Then, for any sequence $\{\alpha_i\} \subset \mathcal{G}$ and any increasing sequence of positive integers n_i if

$$\lim_{i\to\infty} |\alpha_i| \prod_{k=1}^{n_i} w_k = \lim_{i\to\infty} \frac{|\alpha_i|}{(n_i+1)!} = 0,$$

then

$$\lim_{i\to\infty} \frac{1}{|\alpha_i|} \prod_{k=1}^{n_i} \frac{1}{w_{-k}} = \lim_{i\to\infty} \frac{3^{n_i}}{\alpha_i} \neq 0,$$

since $\{\alpha_i\} \subset \mathcal{G}$ is a bounded sequence. Hence, F is not $\mathcal{G}$-cyclic.

The following example shows that not every $\mathcal{G}$-cyclic operator is hypercyclic.

Example 2.8. Let $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be a bilateral forward weighted shift with the weight sequence

$$w_n = \begin{cases} \frac{1}{4} & \text{if } n \geq 0, \\ \frac{1}{2} & \text{if } n < 0. \end{cases}$$

Then T is $\mathcal{G}$-cyclic but not hypercyclic. In fact, first, we apply [4, Theorem 4.1] to show that T is not hypercyclic. For all increasing sequence n_r of positive integers, we have

$$\lim_{r\to\infty} \prod_{k=1}^{n_r} \frac{1}{w_{-k}} = \lim_{r\to\infty} \prod_{k=1}^{n_r} 2 = \lim_{r\to\infty} 2^{n_r} = \infty. \tag{2.9}$$

Hence, T is not hypercyclic.

On the other hand, let $\mathcal{G} = \mathbb{N}$ be the semigroup of all natural numbers, and let $\{3^n\}$ be a sequence in $\mathcal{G}$. By taking $n_k = n$ in Theorem 2.4, we have

$$\lim_{n\to\infty} |\alpha_n| \prod_{j=1}^{n} w_j = \lim_{n\to\infty} 3^n \prod_{j=1}^{n} \frac{1}{4} = \lim_{n\to\infty} \prod_{j=1}^{n} \frac{3^n}{4^n} = 0$$

and

$$\lim_{n\to\infty} \frac{1}{|\alpha_n|} \prod_{j=1}^{n} \frac{1}{w_{-j}} = \lim_{n\to\infty} \frac{1}{3^n} \prod_{j=1}^{n} 2 = \lim_{n\to\infty} \frac{2^n}{3^n} = 0.$$

Then T is $\mathcal{G}$-cyclic.

Note that the above example also shows that $\mathcal{G}$-cyclicity doesn't always imply diskcyclicity by applying Equation 2.9 and [2, Corollary 2.15].

The following example shows that there exists a diskcyclic operator which is not $\mathcal{G}$-cyclic for some semigroup $\mathcal{G}$.

Example 2.9. Let $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the bilateral forward weighted shift with the weight sequence

$$w_n = \begin{cases} 2 & \text{if } n \geq 0, \\ 3 & \text{if } n < 0. \end{cases}$$

Then T is diskcyclic but not G-cyclic. In fact, by [2, Example 2.20], T is diskcyclic. on the other hand, let $\mathcal{G} = \mathbb{N}$ be the semigroup of all natural numbers, then for any increasing sequence of natural numbers n_i and any sequence $\{\alpha_i\} \in \mathcal{G}$

$$\lim_{i\to\infty} |\alpha_i| \prod_{k=1}^{n_i} w_k = \lim_{i\to\infty} |\alpha_i| \prod_{k=1}^{n_i} 2 = \lim_{i\to\infty} |\alpha_i| \, 2^{n_i} \neq 0.$$

Thus by Theorem 2.4, T is not $\mathcal{G}$-cyclic.

Now, let U be the open unit disk in $\mathbb{C}$ and $H^2(U)$ be the Hardy space, consisting of holomorphic functions $f : U \to \mathbb{C}$ with $\|f\|_{H^2} < \infty$. The composition operator C_ϕ acting on $H^2(U)$ is defined as:

$$C_\phi(f) = f \circ \phi \quad \text{for all } f \in H^2(U),$$

where ϕ is a holomorphic self-map of U.

Theorem 2.10. *Let ϕ be a holomorphic self-map of U with a fixed point in U. Then C_ϕ is not $\mathcal{G}$-cyclic on $H^2(U)$ for any bounded semigroup $\mathcal{G}$.*

Proof. Let ϕ be a holomorphic self-map of U with a fixed point $x_0 \in U$, that is, $\phi(x_0) = x_0$. For all $x \in U$, we have

$$C_\phi^n(f)(x) = f(\phi^n(x)). \tag{2.10}$$

Since ϕ is contraction, then $d(\phi(x), \phi(y)) \leq d(x, y)$ for all $x, y \in U$, where d is the hyperbolic metric on U. So

$$\begin{aligned} d(\phi^n(x), x_0) &= d(\phi^n(x), \phi^n(x_0)) \\ &\leq d(\phi^{n-1}(x), \phi^{n-1}(x_0)) \\ &\vdots \\ &\leq d(x, x_0). \end{aligned}$$

It follows that $\{d(\phi^n(x), x_0)\}$ is non-increasing sequence and bounded below by 0 and so $d(\phi^n(x), x_0) \to 0$. Since the hyperbolic metric and the usual metric define the same topology on U then $|\phi^n(x) - x_0| \to 0$ and so $\phi^n(x) \to x_0$ then by Eq. (2.10), we get

$$C_\phi^n(f)(x) \to f(x_0) \quad \text{for all } x \in U.$$

Moreover, since $\mathcal{G}$ is bounded, then $\mathcal{G}Orb(C_\phi, f) = \left\{\alpha C_\phi^n(f) : n \geq 0, \alpha \in \mathcal{G}\right\}$ is bounded and hence cannot be dense in $H^2(U)$. □

Let consider the space $H^\infty(U)$ of all bounded analytic functions on U, and let $\phi \in H^\infty(U)$. The multiplication operator M_ϕ on $H^2(U)$ is defined as:

$$M_\phi(f) = \phi f \quad \text{for all } f \in H^2(U).$$

Theorem 2.11. *The multiplication operator M_ϕ is not $\mathcal{G}$-cyclic on the Hardy space $H^2(U)$.*

Proof. Let $f \in H^2(U)$, then the orbit of f under M_ϕ is defined as follows:

$$\begin{aligned} \mathcal{G}Orb(M_\phi, f) &= \left\{\alpha M_\phi^n(f) : n \geq 0, \alpha \in \mathcal{G}\right\} \\ &= \left\{\alpha \phi^n f : n \geq 0, \alpha \in \mathcal{G}\right\}, \end{aligned}$$

which is subset of the closed subspace spanned by $\left\{f, \phi f, \phi^2 f, \phi^3 f, \ldots\right\}$ in $H^2(U)$ and so cannot be dense in $H^2(U)$. □

Now, consider the space $H(\mathbb{C})$ of all entire functions on $\mathbb{C}$ endowed with the topology of uniform convergence on compact sets. The derivative operator T on $H(\mathbb{C})$ is defined as

$$T(f) = f'.$$

Theorem 2.12. *The derivative operator T on $H(\mathbb{C})$ is $\mathcal{G}$-cyclic.*

Proof. We will apply $\mathcal{G}$-cyclic criterion with respect to the sequence $\{n\} \subset \mathbb{N}$, a bounded sequence $\alpha_n \in \mathcal{G}$ and the same dense sets $D = D_1 = D_2$ of all complex polynomials. Define a map S on $H(\mathbb{C})$ by $S(f)(z) = \int_0^z f(x)dx$. Let $p \in D$. Then $\alpha_n T^n p = 0$ eventually and hence condition (1) of Proposition 2.1 is satisfied. For condition (2), let $k \in \mathbb{N}$ be fixed such that $z^k \in D$ and $p \in D$. Since $\frac{1}{\alpha_n} S^n(z^k) = \frac{k!}{\alpha_n (k+n)!} z^{k+n}$ which converges to 0 as $n \to \infty$ and so by linearity $\frac{1}{\alpha_n} S^n(p) \to 0$. Moreover, $T^n S^n p = p$ for all $p \in D$ and hence condition (3) is satisfied. □

These results extend and refine earlier findings on operator cyclicity. For instance, while Zeana [12] and Naoum and Zeana [9] introduced $\mathcal{G}$-cyclicity, our work provides explicit characterizations for bilateral shifts and shows new separations between classes (e.g., supercyclic but not $\mathcal{G}$-cyclic). Moreover, our treatment of the composition and multiplication operators complements the analysis in [4] and [5], where related classes were investigated under hypercyclic or supercyclic frameworks. Thus, the present paper offers new perspectives and sharper distinctions within the landscape of linear dynamics.

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