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ITERATION SCHEME FOR TWO NONLINEAR MAPPINGS IN CAT(0) SPACES

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Abstract. This paper develops an explicit two step iteration scheme for a coupled pair of nonlinear mappings in $CAT(0)$ spaces. The framework treats a single-valued asymptotically nonexpansive mapping t and a multivalued asymptotically nonexpansive mapping T that are coupled through a nonexpansive mapping. Under standard bounded control parameters and a summable asymptotic modulus, we prove Δ-convergence of the generated sequence to a common fixed point of t and T. The analysis exploits the intrinsic nonpositive curvature of $CAT(0)$ spaces via quasi-Fejér estimates and a demiclosedness principle for (asymptotically) nonexpansive mappings, while the metric selection provides a stable interface between the single and multivalued components. The resulting theory yields a geometry-aware and computationally transparent algorithmic template for hybrid fixed point problems, with model examples illustrating scope and applicability.

1. Introduction

$CAT(0)$ spaces are uniquely geodesic metric spaces, meaning that any two points are connected by a single geodesic segment. A key concept in defining this property is the notion of a geodesic triangle, which consists of three vertices and certain geodesic segments that form its edges. The primary interest

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lies in comparing the shape of this geodesic triangle to that of a Euclidean triangle of the same size.

If the distance between any two points on a given geodesic triangle is less than or equal to the corresponding distance in the Euclidean comparison triangle, then the space is classified as a $CAT(0)$ space. This means that every geodesic triangle in such a space is at least as thin as its Euclidean counterpart.

This intrinsic property is known as the $CAT(0)$ inequality, which characterizes non-positive curvature. Consequently, the geometry of $CAT(0)$ spaces is not linear but follows a curved structure. It is well known that all pre-Hilbert spaces, including hyperbolic spaces and R-trees, are notable examples of $CAT(0)$ spaces.

The original study of non-positively curved spaces dates back to Alexandrov in the 1950s. However, the modern framework of $CAT(0)$ spaces, as widely used today, was developed later by Ballmann, Gromov and Schroeder [1]. Initially, $CAT(0)$ spaces played a significant role in geometric group theory, as highlighted by Bridson and Haefliger [2]. More recently, they have been explored in the context of optimization problems, nonlinear analysis, and fixed point theory. These studies have incorporated methods and techniques originally developed for Banach spaces (specifically, uniformly convex spaces) into the broader class of $CAT(0)$ spaces.

For instance, one key application is the uniqueness of the circumcenter under specific conditions. Additionally, $CAT(0)$ spaces possess a property known as Δ-convergence, introduced by Dhompongsa and Panyanak [5], which shares characteristics with weak convergence in normed spaces. A complete $CAT(0)$ space is often referred to as a Hadamard space.

Beyond specific conditions, it is also necessary to consider metrics, curvature, and fixed point problems related to various classes of operators. The Banach contraction principle has been used to establish the existence, uniqueness, and numerical approximation of fixed points via Picard iteration for contractive mappings.

However, in dealing with nonlinear equations, variational inequalities, and equilibrium problems, nonexpansive mappings are often preferred over contractive mappings. As a result, fixed point approximations have been extended to more general classes of operators such as: nonexpansive mappings [13], quasi-nonexpansive mappings [6], asymptotically nonexpansive mappings [10], Suzuki-type mappings under condition (C) [14].

This evolution of mapping techniques has led to the development of iterative methods for numerically computing fixed points, particularly for pairs of nonlinear hybrid mappings in $CAT(0)$ spaces.

This work lies at the interface of fixed point theory in $CAT(0)$ spaces and iterative methods for hybrid (single and multivalued) operators. In geodesic settings, Δ-convergence (Kirk and Panyanak [9]) is the appropriate surrogate of weak convergence, and demiclosedness principles for asymptotically nonexpansive mappings in $CAT(0)$ spaces (see, e.g., [12]) underpin many convergence arguments for single-valued maps. For multivalued nonexpansive mappings, existence and structural results are known in metric/Banach frameworks (e.g., [3]). However, results that explicitly couple a single-valued asymptotically nonexpansive mapping with a multivalued one within a unified iteration on $CAT(0)$ spaces remain limited.

We establish Δ-convergence of an explicit two step iteration scheme to a common fixed point of a pair (t, T), where t is single-valued asymptotically nonexpansive and T is multivalued asymptotically nonexpansive on a $CAT(0)$ space.

Our analysis requires only the mild summability $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ for the asymptotic nonexpansivity moduli together with bounded step parameters, and it relies on a nonexpansive mapping that couples the two components. As far as we are aware, this appears to be the first result that proves Δ-convergence of such an explicit mixed scheme to a common fixed point under these assumptions.

2. Preliminaries

In this paper, $\mathbb{R}^+$ and $\mathbb{N}$ symbolize the set of all non-negative real numbers and the set of positive integers, respectively.

Definition 2.1. Let (X, d) be a $CAT(0)$ space and let G be a nonempty subset of the $CAT(0)$ space. A mapping $t : G \to G$ is called:

(1) a nonexpansive mapping if $d(tv, tw) \leq d(v, w)$ for all $v, w \in G$,
(2) a quasi-nonexpansive mapping if $Fix(t) = \{w \in G : tw = w\} \neq \emptyset$ and $d(tw, p) \leq d(w, p)$ for all $w \in G$ and $p \in Fix(t)$,
(3) an asymptotically nonexpansive mapping if there exists a sequence $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$ for all $n \geq 1$, and $d(t^n v, t^n w) \leq k_n d(v, w)$ for all $v, w \in G$.

Let (X, d) be a geodesic space and let 2^G denote the family of all nonempty subsets of G. Let $FB(G)$ be the set of all nonempty bounded closed subsets of G, and let $KC(G)$ be the set of all nonempty compact convex subsets of G. A subset of X is said to be proximal if, for every $w \in X$, there exists $k \in G$ such that

$$d(v, k) = \text{dist}(v, G) = \inf\{d(v, w) : w \in G\}.$$

Let $PB(G)$ be the set of all nonempty bounded proximal subsets of G. Let H be the Hausdorff metric induced by d, defined as

$$H(A,B)=\max\{\sup_{v\in A}\operatorname{dist}(v,B),\sup_{w\in B}\operatorname{dist}(w,A)\},\quad A,B\in FB(X),$$

where $\operatorname{dist}(v,B)=\inf\{d(v,w):w\in B\}$ represents the distance from the point v to the subset B.

A multivalued mapping $T:G\to FB(G)$ is called nonexpansive, if

$$H(Tv,Tw)\le d(v,w)\quad \text{for all } v,w\in G.$$

A multivalued mapping $T:G\to FB(G)$ is said to satisfy Condition (E), if there exists $\mu\ge 1$ such that for all $v,w\in G$,

$$\operatorname{dist}(v,Tv)\le \mu\operatorname{dist}(v,Tw)+d(v,w).$$

Let $T:G\to PB(G)$ be a multivalued mapping, and define the selection mapping P_T for each w by

$$P_T(v):=\{w\in Tv:d(v,w)=\operatorname{dist}(v,Tv)\}.$$

The following lemma presents fundamental properties of CAT(0) spaces, which will be used in the proofs of the main theorems in this study.

Lemma 2.2. ([5]) *Let (X,d) be a CAT(0) space. Then the following properties hold:*

(1) *For all $v,w\in X$ and for every $\alpha\in[0,1]$, there exists a unique point $z\in[v,w]$ such that*

$$d(v,z)=\alpha d(v,w)\quad\text{and}\quad d(w,z)=(1-\alpha)d(v,w).$$

This unique point is denoted by $(1-\alpha)v\oplus\alpha w$.

(2) *For all $v,w,z\in X$ and $\alpha\in[0,1]$, the following convex inequality holds:*

$$d((1-\alpha)v\oplus\alpha w,z)\le(1-\alpha)d(v,z)+\alpha d(w,z).$$

(3) *For all $v,w,z\in X$ and $\alpha\in[0,1]$, we have*

$$d((1-\alpha)v\oplus\alpha w,z)^2\le(1-\alpha)d(v,z)^2+\alpha d(w,z)^2-\alpha(1-\alpha)d(v,w)^2.$$

(4) *For any bounded sequence $\{w_n\}$ in a CAT(0) space, the unique asymptotic center $A(\{w_n\})$ is given by*

$$A(\{w_n\})=\{w\in X:r(w,\{w_n\})=r(\{w_n\})\},$$

where

$$r(w,\{w_n\})=\limsup_{n\to\infty}d(w,w_n),\quad r(\{w_n\})=\inf\{r(w,\{w_n\}):w\in X\}.$$

(5) *If G is a closed convex subset of a CAT(0) space and $\{w_n\}$ is a bounded sequence in G, then the asymptotic center of $\{w_n\}$ is contained in G.*

The uniqueness result for the asymptotic center presented in Lemma 2.2 (4) suggests a new form of convergence that resembles weak convergence in normed spaces. A sequence $\{w_n\}$ in a $CAT(0)$ space is said to be Δ-convergent to $w \in X$ if w is the unique asymptotic center of every subsequence $\{z_n\}$ of $\{w_n\}$.

Lemma 2.3. ([9]) *Let (X, d) be a CAT(0) space. Then the following properties hold:*

(1) *Every bounded sequence in a CAT(0) space has a Δ-convergent subsequence.*
(2) *Every CAT(0) space satisfies the Opial property, which states that for any sequence $\{w_n\} \subset X$ that Δ-converges to w and for any $z \neq w$,*

$$\limsup_{n\to\infty} d(v, w_n) < \limsup_{n\to\infty} d(w_n, z).$$

Lemma 2.4. ([8, 9]) *Let $\{v_n\}$ be a sequence in a complete CAT(0) space X that Δ-converges to $v \in X$, and let $\{w_n\}$ be a sequence in X.*

(1) *If $\lim_{n\to\infty} d(v_n, w_n) = 0$, then $\{v_n\}$ is Δ-convergent to v.*
(2) *If $d(v, w) \leq \limsup_{n\to\infty} d(v_n, w)$ for all $w \in X$, then $v = w$.*

Lemma 2.5. ([3]) *Let $\{x_n\}$ be a sequence in a complete CAT(0) space X with a unique asymptotic center, and let $T : X \to FB(X)$ be a multivalued nonexpansive mapping. If the sequence $\{z_n\}$ with $z_n \in T(x_n)$ satisfies Δ-$\lim_{n\to\infty} z_n = z$, then $z \in T(z)$, that is, z is a fixed point of T.*

Lemma 2.6. ([3]) *If G is a closed convex subset of a complete CAT(0) space and if $\{w_n\}$ is a bounded sequence in G, then the asymptotic center of $\{w_n\}$ is in G.*

Lemma 2.7. ([11]) *Let (X, d) be a complete CAT(0) space, and let $x^* \in X$. Suppose $\{\alpha_n\}$ is a sequence in $[a, b]$ for some $a, b \in (0, 1)$, and $\{w_n\}, \{z_n\}$ are sequences in X. If*

$$\limsup_{n\to\infty} d(w_n, x^*) \leq r, \ \limsup_{n\to\infty} d(z_n, x^*) \leq r$$

and

$$\limsup_{n\to\infty} d((1-\alpha_n)w_n \oplus \alpha_n z_n, x^*) \leq r$$

for some $r \geq 0$, then

$$\lim_{n\to\infty} d(w_n, z_n) = 0.$$

Lemma 2.8. ([15]) *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1 - b_n)a_n$$

for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists. Furthermore, if a subsequence of $\{a_n\}$ converges to 0, then $\lim_{n\to\infty} a_n = 0$.

Theorem 2.9. ([12]) *Let G be a nonempty, bounded, closed and convex subset of a complete $CAT(0)$ space and let $t : G \to G$ be an asymptotically nonexpansive mapping. Then t has a fixed point.*

Corollary 2.10. ([4]) *Let G be a nonempty, bounded, closed and convex subset of a complete $CAT(0)$ space, and let $t : G \to G$ be an asymptotically nonexpansive mapping. Suppose that $\{w_n\}$ is a bounded sequence in G such that*

$$\lim_{n\to\infty} d(tw_n, w_n) = 0$$

and $\Delta\text{-}\lim_{n\to\infty} w_n = w$. Then $tw = w$.

3. Main results

Definition 3.1. Let G be a nonempty, bounded, closed and convex subset of a complete $CAT(0)$ space X. Let $t : G \to G$ be a single-valued asymptotically nonexpansive mapping, and let $T : G \to FB(G)$ be a multivalued asymptotically nonexpansive mapping. Assume $P_T : G \to G$ is a nonexpansive and $Fix(t) \cap Fix(T) \neq \emptyset$. For a given $w_1 \in G$, the sequence $\{w_n\}$ in the iteration scheme is defined by

$$\begin{cases} v_n = (1 - \beta_n)w_n \oplus \beta_n z_n, \\ w_{n+1} = (1 - \alpha_n)z_n \oplus \alpha_n t^n v_n, \end{cases} \tag{3.1}$$

for $n \in \mathbb{N}$, where $z_n \in P_T(t^n w_n)$ and $0 < a \leq \alpha_n, \beta_n \leq b < 1$.

Lemma 3.2. *Let G be a nonempty, bounded, closed, and convex subset of a complete CAT(0) space X, and let $T : G \to FB(G)$ be a multivalued asymptotically nonexpansive mapping, where $Fix(t) \cap Fix(T) \neq \emptyset$, and P_T is a nonexpansive mapping. If $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, then the following statements are equivalent:*

(1) *$w \in Fix(T)$, that is, $w \in Tv$.*
(2) *$P_T(v) = \{v\}$, that is, $v = w$ for all $w \in P_T(v)$.*
(3) *$v \in Fix(P_T)$, that is, $v \in P_T(v)$.*

Moreover, $Fix(T) = Fix(P_T)$.

Proof. We will prove the equivalence of the three statements by showing the logical sequence: (1) ⇒ (2) ⇒ (3) ⇒ (1).

(1) ⇒ (2): Assume $v \in \text{Fix}(T)$, which means $v \in Tv$. By the definition of the distance from a point to a set, $\text{dist}(v, Tv) = \inf\{d(v, w) : w \in Tv\}$. Since $v \in Tv$, the minimum distance is achieved when $w = v$, so $\text{dist}(v, Tv) = d(v, v) = 0$. By the definition of the selection mapping $P_T(v) = \{w \in Tv : d(v, w) = \text{dist}(v, Tv)\}$, any $w \in P_T(v)$ must satisfy $d(v, w) = \text{dist}(v, Tv) = 0$. This implies $d(v, w) = 0$, which holds if and only if $w = v$. Therefore, $P_T(v) = \{v\}$.

(2) ⇒ (3): Assume $P_T(v) = \{v\}$. By definition, the set $P_T(v)$ is a single element, which is v. This means $v \in P_T(v)$. By the definition of a fixed point, $v \in \text{Fix}(P_T)$.

(3) ⇒ (1): Assume $v \in \text{Fix}(P_T)$, which means $v \in P_T(v)$. From the definition of P_T we know that $P_T(v)$ is a subset of Tv. Therefore, if $v \in P_T(v)$, it must also be true that $v \in Tv$. This implies that $v \in \text{Fix}(T)$.

Since we have shown that (1) ⇒ (2) ⇒ (3) ⇒ (1), the three statements are equivalent.

To prove that $\text{Fix}(T) = \text{Fix}(P_T)$, we can use the equivalences we just proved.

(i) From (1) ⇒ (3), if $v \in \text{Fix}(T)$, then $v \in \text{Fix}(P_T)$. This means $\text{Fix}(T) \subseteq \text{Fix}(P_T)$.
(ii) From (3) ⇒ (1), if $v \in \text{Fix}(P_T)$, then $v \in \text{Fix}(T)$. This means $\text{Fix}(P_T) \subseteq \text{Fix}(T)$.

Since both set inclusions hold, we conclude that $\text{Fix}(T) = \text{Fix}(P_T)$. This completes the proof. □

Lemma 3.3. *Let G be a nonempty, bounded, closed and convex subset of a complete $CAT(0)$ space X. Let $t : G \to G$ be a single-valued asymptotically nonexpansive mapping, and let $T : G \to FB(G)$ be a multivalued asymptotically nonexpansive mapping, where $Fix(t) \cap Fix(T) \neq \emptyset$, and P_T is a nonexpansive mapping. If $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, then for $w_1 \in G$, the sequence $\{w_n\}$ generated by the iteration scheme (3.1) satisfies $\lim_{n\to\infty} d(w_n, x^*)$ exists, where $x^* \in Fix(t) \cap Fix(T)$.*

Proof. Let $x^* \in \text{Fix}(t) \cap \text{Fix}(T)$. By Lemma 3.2, we know that x^* is also a fixed point of P_T, that is, $x^* \in \text{Fix}(P_T)$.

Consider the distance from w_{n+1} to x^*,

$$\begin{aligned} d(w_{n+1}, x^*) &= d((1-\alpha_n)z_n \oplus \alpha_n t^n v_n, x^*) \\ &\leq (1-\alpha_n)d(z_n, x^*) + \alpha_n d(t^n v_n, x^*). \end{aligned}$$

Since $z_n \in P_T(t^n w_n)$, we have

$$d(z_n, x^*) = \operatorname{dist}(z_n, \{x^*\}) \leq H(P_T(t^n w_n), P_T(x^*)).$$

Because P_T is nonexpansive, this implies

$$H(P_T(t^n w_n), P_T(x^*)) \leq d(t^n w_n, x^*).$$

As t is asymptotically nonexpansive, $d(t^n w_n, x^*) = d(t^n w_n, t^n x^*) \leq k_n d(w_n, x^*)$. Thus, we have

$$d(z_n, x^*) \leq k_n d(w_n, x^*).$$

Similarly, for the term $d(t^n v_n, x^*)$, we get

$$d(t^n v_n, x^*) = d(t^n v_n, t^n x^*) \leq k_n d(v_n, x^*).$$

And we know that

$$\begin{aligned} d(v_n, x^*) &= d((1-\beta_n)w_n \oplus \beta_n z_n, x^*) \\ &\leq (1-\beta_n)d(w_n, x^*) + \beta_n d(z_n, x^*) \\ &\leq (1-\beta_n)d(w_n, x^*) + \beta_n k_n d(w_n, x^*) \\ &= (1-\beta_n+\beta_n k_n)d(w_n, x^*). \end{aligned}$$

By substituting this back, we have

$$\begin{aligned} d(t^n v_n, x^*) &\leq k_n d(v_n, x^*) \\ &\leq k_n(1-\beta_n+\beta_n k_n)d(w_n, x^*) \\ &= (k_n - \beta_n k_n + \beta_n k_n^2)d(w_n, x^*). \end{aligned}$$

Substituting the bounds for $d(z_n, x^*)$ and $d(t^n v_n, x^*)$ into the first inequality:

$$\begin{aligned} d(w_{n+1}, x^*) &\leq (1-\alpha_n)d(z_n, x^*) + \alpha_n d(t^n v_n, x^*) \\ &\leq (1-\alpha_n)k_n d(w_n, x^*) + \alpha_n(k_n - \beta_n k_n + \beta_n k_n^2)d(w_n, x^*) \\ &= [k_n - \alpha_n k_n + \alpha_n k_n - \alpha_n\beta_n k_n + \alpha_n\beta_n k_n^2]d(w_n, x^*) \\ &= [k_n + \alpha_n\beta_n k_n(k_n-1)]d(w_n, x^*). \end{aligned}$$

Let $a_n = d(w_n, x^*)$. Then

$$a_{n+1} \leq (k_n + \alpha_n\beta_n k_n(k_n-1))a_n.$$

Since $\sum_{n=1}^{\infty}(k_n-1) < \infty$, we know that $k_n \to 1$ as $n \to \infty$. This implies that the sequence $\{k_n\}$ is bounded, so there exists $M > 0$ such that $k_n \leq M$ for all n. Also, $0 < a \leq \alpha_n, \beta_n \leq b < 1$. The term $b_n = k_n + \alpha_n\beta_n k_n(k_n-1)$ can be written as $b_n = 1 + (k_n-1) + \alpha_n\beta_n k_n(k_n-1) = 1 + (k_n-1)(1+\alpha_n\beta_n k_n)$. The sum $\sum_{n=1}^{\infty}(k_n-1)(1+\alpha_n\beta_n k_n)$ converges because $(1+\alpha_n\beta_n k_n)$ is bounded and $\sum_{n=1}^{\infty}(k_n-1)$ converges. By Lemma 2.8, since $a_{n+1}\leq(1+b_n)a_n$ and $\sum b_n$ converges, the limit of $\{a_n\}$ exists. Therefore, $\lim_{n\to\infty} d(w_n,x^*)$ exists. □

Lemma 3.4. *Let G be a nonempty, bounded, closed and convex subset of a complete $CAT(0)$ space X. Let $t: G\to G$ be a single-valued asymptotically nonexpansive mapping, and let $T: G\to FB(G)$ be a multivalued asymptotically nonexpansive mapping, where $Fix(t)\cap Fix(T)\neq\emptyset$, and P_T is a nonexpansive mapping. If $\sum_{n=1}^{\infty}(k_n-1)<\infty$, then the sequence $\{w_n\}$ generated by the iteration scheme* (3.1) *satisfies*

$$\lim_{n\to\infty} d(t^n v_n, w_n)=0 \text{ and } \lim_{n\to\infty} d(w_n,z_n)=0.$$

Proof. Let $w_1\in G$ and $x^*\in \mathrm{Fix}(t)\cap\mathrm{Fix}(T)$. By Lemma 3.2, we have $x^*\in P_T(x^*)=\{x^*\}$ and by Lemma 3.3, the limit $\lim_{n\to\infty} d(w_n,x^*)$ exists. Let's denote this limit by c. We consider the distance $d(t^n v_n,x^*)$ as follows:

$$\begin{aligned}
d(t^n v_n,x^*) &= d(t^n v_n,t^n x^*)\\
&\leq k_n d(v_n,x^*)\\
&= k_n d((1-\beta_n)w_n\oplus\beta_n z_n,x^*)\\
&\leq (1-\beta_n)k_n d(w_n,x^*)+\beta_n k_n d(z_n,x^*).
\end{aligned}$$

Since $z_n\in P_T(t^n w_n)$ and $x^*\in P_T(x^*)$, we know that

$$d(z_n,x^*)=\mathrm{dist}(z_n,\{x^*\})\leq H(P_T(t^n w_n),P_T(x^*)).$$

Since P_T is a nonexpansive mapping, we have

$$H(P_T(t^n w_n),P_T(x^*))\leq d(t^n w_n,x^*).$$

Furthermore, as t is an asymptotically nonexpansive mapping, $d(t^n w_n,x^*)=d(t^n w_n,t^n x^*)\leq k_n d(w_n,x^*)$. Thus,

$$\begin{aligned}
d(t^n v_n,x^*) &\leq (1-\beta_n)k_n d(w_n,x^*)+\beta_n k_n H(P_T(t^n w_n),P_T(x^*))\\
&\leq (1-\beta_n)k_n d(w_n,x^*)+\beta_n k_n d(t^n w_n,x^*)\\
&= (1-\beta_n)k_n d(w_n,x^*)+\beta_n k_n d(t^n w_n,t^n x^*)\\
&\leq (1-\beta_n)k_n d(w_n,x^*)+\beta_n k_n^2 d(w_n,x^*)\\
&\leq (k_n-\beta_n k_n+\beta_n k_n^2)d(w_n,x^*)\\
&= (k_n-1+1+\beta_n k_n^2-\beta_n k_n)d(w_n,x^*)\\
&= (1+(1+\beta_n k_n)(k_n-1))d(w_n,x^*).
\end{aligned}$$

Taking the limit superior of the last inequality, we get

$$\limsup_{n\to\infty} d(t^n v_n, x^*) \le \limsup_{n\to\infty}(1 + (1 + \beta_n k_n)(k_n - 1))d(w_n, x^*).$$

Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, we know that $k_n \to 1$ as $n \to \infty$ and $\beta_n \in (0, 1)$, which implies the term $(1 + \beta_n k_n)(k_n - 1)$ converges to 0. Thus,

$$\limsup_{n\to\infty} d(t^n v_n, x^*) \le \limsup_{n\to\infty} d(w_n, x^*) = c. \tag{3.2}$$

Similarly, we obtain

$$\limsup_{n\to\infty} d(z_n, x^*) \le \limsup_{n\to\infty} k_n d(w_n, x^*) = \limsup_{n\to\infty} d(w_n, x^*) = c. \tag{3.3}$$

Furthermore, from Lemma 3.3 and the definition of the sequence $\{w_n\}$, we have

$$c = \lim_{n\to\infty} d(w_{n+1}, x^*) = \lim_{n\to\infty} d((1 - \alpha_n)z_n \oplus \alpha_n t^n v_n, x^*). \tag{3.4}$$

By (3.2), (3.3), (3.4) and Lemma 2.7, we conclude that

$$\lim_{n\to\infty} d(t^n v_n, w_n) = 0 \text{ and } \lim_{n\to\infty} d(w_n, z_n) = 0.$$

This completes the proof. □

Lemma 3.5. *Let G be a nonempty, bounded, closed and convex subset of a complete $CAT(0)$ space X. Let $t : G \to G$ be a single-valued asymptotically nonexpansive mapping, and let $T : G \to FB(G)$ be a multivalued asymptotically nonexpansive mapping, where $Fix(t) \cap Fix(T) \neq \emptyset$, and P_T is a nonexpansive mapping. If $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, then the sequence $\{w_n\}$ generated by the iteration scheme* (3.1) *satisfies*

$$\lim_{n\to\infty} d(t^n w_n, w_n) = 0.$$

Proof. Let $w_1 \in G$ and $x^* \in Fix(t) \cap Fix(T)$. By Lemma 3.2, we have $x^* \in P_T(x^*) = \{x^*\}$. From Lemma 3.3, the limit $\lim_{n\to\infty} d(w_n, x^*)$ exists. Let's denote this limit as c. We consider the distance $d(t^n v_n, x^*)$ as follows:

$$\begin{aligned}
d(t^n w_n, w_n) &\le d(t^n w_n, t^n v_n) + d(t^n v_n, w_n)\\
&\le k_n d(w_n, v_n) + d(t^n v_n, w_n)\\
&= k_n d(w_n, (1 - \beta_n)w_n \oplus \beta_n z_n) + d(t^n v_n, w_n)\\
&\le k_n[(1 - \beta_n)d(w_n, w_n) + \beta_n d(w_n, z_n)] + d(t^n v_n, w_n)\\
&= k_n \beta_n d(w_n, z_n) + d(t^n v_n, w_n).
\end{aligned}$$

Then, we have

$$\lim_{n\to\infty} d(t^n w_n, w_n) \le \lim_{n\to\infty} k_n \beta_n d(w_n, z_n) + \lim_{n\to\infty} d(t^n v_n, w_n).$$

Using Lemma 3.4, we know that

$$\lim_{n\to\infty} d(w_n, z_n) = 0$$

and

$$\lim_{n\to\infty} d(t^n v_n, w_n) = 0.$$

Since $\lim_{n\to\infty} k_n = 1$ and $\{\beta_n\}$ is a bounded sequence, the limit of the first term on the right-hand side is 0. Therefore,

$$\lim_{n\to\infty} d(t^n w_n, w_n) = 0.$$

This completes the proof. $\square$

Lemma 3.6. *Let G be a nonempty, bounded, closed and convex subset of a complete $CAT(0)$ space X. Let $t : G \to G$ be a single-valued asymptotically nonexpansive mapping, and let $T : G \to FB(G)$ be a multivalued asymptotically nonexpansive mapping, where $Fix(t) \cap Fix(T) \neq \emptyset$, and P_T is a nonexpansive mapping. If $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, then the sequence $\{w_n\}$ generated by the iteration scheme* (3.1) *satisfies*

$$\lim_{n\to\infty} d(tw_n, w_n) = 0.$$

Proof. Let $w_1 \in G$ and $x^* \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$. By Lemma 3.2, we have $x^* \in P_T(x^*) = \{x^*\}$. From Lemma 3.3, the limit $\lim_{n\to\infty} d(w_n, x^*)$ exists. Let's denote this limit by c. We consider the distance $d(t^n v_n, x^*)$ as follows:

$$\begin{aligned}
d(tw_n, w_n) &= d(w_n, tw_n)\\
&\leq d(w_n, t^n w_n) + d(t^n w_n, tw_n)\\
&= d(w_n, t^n w_n) + d(t(t^{n-1} w_n), tw_n)\\
&\leq d(w_n, t^n w_n) + k_1 d(t^{n-1} w_n, w_n)\\
&\leq d(w_n, t^n w_n) + k_1[d(t^{n-1} w_n, t^{n-1} w_{n-1}) + d(t^{n-1} w_{n-1}, w_n)]\\
&\leq d(w_n, t^n w_n) + k_1 k_{n-1} d(w_n, w_{n-1}) + k_1 d(t^{n-1} w_{n-1}, w_n)\\
&= d(w_n, t^n w_n) + k_1 k_{n-1} d((1-\alpha_{n-1}) z_{n-1} \oplus \alpha_{n-1} t^{n-1} v_{n-1}, w_{n-1})\\
&\quad + k_1 d(t^{n-1} w_{n-1}, (1-\alpha_{n-1}) z_{n-1} \oplus \alpha_{n-1} t^{n-1} v_{n-1})\\
&\leq d(w_n, t^n w_n) + k_1 k_{n-1}[(1-\alpha_{n-1}) d(z_{n-1}, w_{n-1})\\
&\quad + \alpha_{n-1} d(t^{n-1} v_{n-1}, w_{n-1})]\\
&\quad + k_1[(1-\alpha_{n-1}) d(t^{n-1} w_{n-1}, z_{n-1}) + \alpha_{n-1} d(t^{n-1} w_{n-1}, t^{n-1} v_{n-1})].
\end{aligned}$$

As $n \to \infty$, we know that

$$\lim_{n\to\infty} d(z_{n-1}, w_{n-1}) = 0$$

and

$$\lim_{n\to\infty} d(t^{n-1}v_{n-1}, w_{n-1}) = 0.$$

From Lemma 3.4, we also have

$$\lim_{n\to\infty} d(t^{n-1}w_{n-1}, z_{n-1}) = \lim_{n\to\infty} d(t^{n-1}v_{n-1}, w_{n-1}) = 0$$

and

$$\lim_{n\to\infty} d(t^{n-1}w_{n-1}, t^{n-1}v_{n-1}) = 0.$$

From the first term, we know that $\lim_{n\to\infty} d(w_n, t^n w_n) = 0$ from Lemma 3.5. Also, all the other terms on the right-hand side approach zero as $n \to \infty$ due to Lemma 3.4 and the boundedness of the sequences involved. Therefore,

$$\lim_{n\to\infty} d(tw_n, w_n) = 0.$$

This completes the proof. □

Theorem 3.7. *Let G be a nonempty, bounded, closed and convex subset of a complete $CAT(0)$ space X. Let $t : G \to G$ be a single-valued asymptotically nonexpansive mapping, and let $T : G \to FB(G)$ be a multivalued asymptotically nonexpansive mapping, where $Fix(t) \cap Fix(T) \neq \emptyset$, and P_T is a nonexpansive mapping. If $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, the sequence $\{w_n\}$ generated by the iteration scheme* (3.1)*, then $\{w_n\}$ is Δ-convergent to a common fixed point of t and T.*

Proof. Let $\{w_n\}$ be the sequence generated by the iterative process (3.1) and let x^* be a common fixed point of t and T. By Lemma 3.3, we know that $\lim_{n\to\infty} d(w_n, x^*)$ exists. To prove the theorem, we must show that $\{w_n\}$ is Δ-convergent to a common fixed point of t and T. Let $A \subset G$ be the asymptotic center of $\{w_n\}$. By Lemma 2.3, A consists of exactly one point. Let's call this point $w \in A$. We need to prove that $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$.

First, we prove that $w \in \mathrm{Fix}(T)$. Since $w \in A$ and A is the asymptotic center of $\{w_n\}$, we have

$$\limsup_{n\to\infty} d(w_n, w) = \min_{y\in G} \limsup_{n\to\infty} d(w_n, y).$$

By Lemma 3.4, we know that $\lim_{n\to\infty} d(w_n, z_n) = 0$. Since $\{w_n\}$ is a bounded sequence, the sequence $\{z_n\}$ is also bounded. Let $\{w_{n_k}\}$ be any subsequence of $\{w_n\}$ that Δ-converges to a point $y \in G$. From the relation $\lim_{n\to\infty} d(w_n, z_n) = 0$, by Lemma 2.4(1), we can deduce that the corresponding subsequence $\{z_{n_k}\}$ also Δ-converges to the same point y. Since $z_{n_k} \in P_T(t^{n_k} w_{n_k})$ for all $k \in \mathbb{N}$, and $y \in G$, it follows from Lemma 2.5 that $y \in T(y)$, and thus $y \in \mathrm{Fix}(T)$. Therefore, every Δ-convergent subsequence of $\{w_n\}$ converges to a fixed point of T. Since the set of asymptotic centers is a single point, this implies $w \in \text{Fix}(T)$.

Next, we prove that $w \in \text{Fix}(t)$. From Lemma 3.6, we know that

$$\lim_{n\to\infty} d(w_n, tw_n) = 0.$$

Since $\{w_n\}$ is Δ-convergent to w, by Lemma 2.4(2), we can deduce that $w = tw$. Therefore, $w \in \text{Fix}(t)$.

Since $w \in \text{Fix}(t)$ and $w \in \text{Fix}(T)$, we conclude that $\{w_n\}$ is Δ-convergent to a common fixed point of t and T. □

Remark 3.8. (Relation to demiclosedness) In the proof of Theorem 3.7, Lemmas 3.4-3.6 yield

$$d\big(t^n v_n, w_n\big) \to 0 \quad \text{and} \quad d(w_n, z_n) \to 0.$$

Since $\{w_n\}$ is bounded in a $CAT(0)$ space, it admits a Δ-cluster point $\bar{w}$. The demiclosedness principle for (asymptotically) nonexpansive single-valued mappings in $CAT(0)$ spaces (see Corollary 2.10) then gives

$$\bar{w} \in Fix(t).$$

On the multivalued side, the coupling through the nonexpansive metric selection P_T ensures $Fix(T) = Fix(P_T)$ and together with $d\big(w_n, P_T(t^n w_n)\big) \to 0$, implies

$$\bar{w} \in Fix(P_T) = Fix(T).$$

Consequently, $\bar{w} \in Fix(t) \cap Fix(T)$, which is precisely the step that upgrades the convergence of the iteration scheme (3.1) to convergence toward a common fixed point of t and T.

4. Examples

Example 4.1. Let (X, d) be a complete $CAT(0)$ space. For concreteness, take $X = \mathbb{R}^2$ with the Euclidean metric, and let

$$G := \{x \in X : \ \|x\| \leq 1\},$$

which is nonempty, bounded, closed and convex.

Define $t : G \to G$ by $t(x) = \frac{1}{2}x$. Then t is $\frac{1}{2}$-Lipschitz, hence asymptotically nonexpansive with modulus $k_n \equiv 1$ (so $\sum_{n=1}^{\infty}(k_n - 1) < \infty$) and $\text{Fix}(t) = \{0\}$. Define a multivalued mapping $T : G \to FB(G)$ by the constant convex set

$$T(x) \equiv G \quad \text{for all} \quad x \in G.$$

For the Hausdorff metric H on closed bounded subsets of X, we have

$$H(T^n x, T^n y) = H(G, G) = 0$$

for all $x, y \in G$ and all n, so T is asymptotically nonexpansive (with modulus $k_n \equiv 1$). Let P_T be the (unique) metric projection onto G,

$$P_T := P_G : X \to G, \qquad P_G(v) := \arg\min_{w \in G} d(v, w).$$

Then P_G is well-defined and nonexpansive in $CAT(0)$ spaces. Moreover, $\mathrm{Fix}(T) = \{x \in G : \ x \in T(x)\} = G$, hence

$$\mathrm{Fix}(t) \cap \mathrm{Fix}(T) = \{0\} \neq \emptyset.$$

Generate $\{w_n\} \subset G$ by the iteration scheme (3.1) with control sequences $0 < a \leq \alpha_n, \ \beta_n \leq b < 1$ and asymptotic moduli $\{k_n\}$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Since all assumptions of Theorem 3.7 are fulfilled (single-valued t and multivalued T asymptotically nonexpansive, nonexpansive mapping P_T, bounded convex G), the sequence $\{w_n\}$ is Δ-convergent to a common fixed point $0 \in G$ of t and T.

Example 4.2. Let (X, d) be a complete $CAT(0)$ space. For a concrete instance, take $X = \mathbb{R}$ with the usual Euclidean metric and let

$$G := [0, 1],$$

which is nonempty, bounded, closed and convex.

Define $t : G \to G$ by $t(u) = \frac{1}{2}u$. Then t is $\frac{1}{2}$-Lipschitz, hence asymptotically nonexpansive with modulus $k_n \equiv 1$ (so $\sum_{n=1}^{\infty}(k_n - 1) < \infty$), and

$$Fix(t) = \{u \in G : \ u = \tfrac{1}{2}u\} = \{0\}.$$

Define a multivalued mapping $T : G \to FB(G)$ by

$$T(u) := \left[\, 0, \tfrac{1}{3}u \,\right], \ u \in G.$$

Each $T(u)$ is a nonempty closed convex subset of G. With the Hausdorff metric H on closed bounded subsets of X, we have

$$H\big(T(u), T(v)\big) = \left|\tfrac{1}{3}u - \tfrac{1}{3}v\right| \leq \tfrac{1}{3}\,|u - v|, \ u, v \in G,$$

so T is (asymptotically) nonexpansive with modulus $k_n \equiv 1$. Let P_T be the (unique) metric projection onto $T(u)$:

$$P_T(u) := \arg\min_{w \in T(u)} d(u, w).$$

Since $u \geq \frac{1}{3}u$ for $u \in [0, 1]$, the nearest point of u to the interval $[0, \frac{1}{3}u]$ is its right endpoint, hence

$$P_T(u) = \tfrac{1}{3}u \quad \text{for all} \quad u \in G.$$

Therefore, P_T is single-valued and nonexpansive (indeed $|P_T(u) - P_T(v)| = \frac{1}{3}|u - v|$). Moreover,

$$Fix(T) = \{u \in G : \ u \in [0, \tfrac{1}{3}u]\} = \{0\}$$

and thus $Fix(t) \cap Fix(T) = \{0\} \neq \emptyset$.

Generate $\{w_n\} \subset G$ by iteration scheme (3.1) with control sequences $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$ and asymptotic moduli $\{k_n\}$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ (here $k_n \equiv 1$). Since all assumptions of Theorem 3.7 are satisfied (singlevalued t and multivalued T asymptotically nonexpansive, nonexpansive mapping P_T, bounded convex G), the sequence $\{w_n\}$ is Δ-convergent to the common fixed point $0 \in G$.

Example 4.3. Let X be the Hilbert space of real 2×2 symmetric matrices endowed with the Frobenius norm $\|A\|_F = \sqrt{\langle A, A\rangle_F}$. Consider the nonempty, bounded, closed and convex set

$$G := \{A \in \mathbb{R}^{2\times 2}_{\text{sym}} : \ A \succeq 0, \ \|A\|_F \leq 1\}.$$

Define $t : G \to G$ by $t(A) = \frac{1}{2}A$. Then t is $\frac{1}{2}$-Lipschitz, hence asymptotically nonexpansive with modulus $k_n \equiv 1$ (so $\sum_{n=1}^{\infty}(k_n - 1) < \infty$), and

$$Fix(t) = \{A \in G : \ A = \tfrac{1}{2}A\} = \{0\}.$$

Define $T : G \to FB(G)$ by the closed convex segment

$$T(A) := \text{conv}\{0, \tfrac{1}{3}A\} = \big\{\theta\,\tfrac{1}{3}A : \ \theta \in [0,1]\big\} = [\,0, \tfrac{1}{3}A\,].$$

Each $T(A)$ is nonempty, closed, convex and consists of positive semidefinite matrices. Let H denote the Hausdorff distance induced by $\|\cdot\|_F$. Since $H([0,a],[0,b]) = \|a - b\|_F$ for a, b in a Hilbert space, we obtain

$$H\big(T(A), T(B)\big) = \left\|\tfrac{1}{3}A - \tfrac{1}{3}B\right\|_F \leq \tfrac{1}{3}\,\|A - B\|_F, \ (A, B \in G),$$

so T is nonexpansive (hence asymptotically nonexpansive with $k_n \equiv 1$). Let P_T be the nearest point projection onto $T(A)$:

$$P_T(A) := \arg\min_{W \in T(A)} \|A - W\|_F.$$

In a Hilbert space, the metric projection onto a closed convex set is single-valued and nonexpansive. Moreover, along the ray $\{\theta A : \theta \geq 0\}$ one has $\|A - \theta\frac{1}{3}A\|_F = (1 - \frac{\theta}{3})\|A\|_F$, minimized at $\theta = 1$, hence

$$P_T(A) = \tfrac{1}{3}A \quad \text{for all } A \in G.$$

Therefore,

$$Fix(T) = \{A \in G : \ A \in [0, \tfrac{1}{3}A]\} = \{0\}$$

and consequently $Fix(t) \cap Fix(T) = \{0\} \neq \emptyset$.

Generate $\{w_n\} \subset G$ by iteration scheme (3.1) with controls $0 < a \leq \alpha_n$, $\beta_n \leq b < 1$ and asymptotic moduli $\{k_n\}$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ (here $k_n \equiv 1$). All assumptions of Theorem 3.7 are satisfied, hence $\{w_n\}$ is Δ-convergent to the common fixed point $0 \in G$.

5. Application

The iteration scheme presented in Theorem 3.7 can be applied to solve signal reconstruction problems, particularly in medical imaging techniques like Compressed Sensing MRI. The objective is to recover an original signal u^* from incomplete and noisy measurements. This problem is framed as a fixed point problem where the desired solution is a common fixed point of two key operators.

5.1. **Problem Formulation.** Let (X, d) be a complete $CAT(0)$ space, representing the space of all possible signals. We define the following:

(1) $G \subset X$: A nonempty, closed, bounded and convex subset representing the space of feasible signals.
(2) $t : G \to G$: A single-valued asymptotically nonexpansive mapping that models a data fidelity step. This operator ensures the reconstructed signal is consistent with the acquired measurements.
(3) $T : G \to FB(G)$: A multivalued asymptotically nonexpansive mapping that models a structural constraint, such as the sparsity of the signal in a transform domain (e.g., wavelet domain).
(4) P_T: A nonexpansive selection mapping associated with T, which allows us to choose a single element from the set $T(v)$.
(5) $Fix(t) \cap Fix(T) \neq \emptyset$: We assume the existence of a common fixed point u^*, which is the true signal we are trying to recover.

The reconstruction problem is to find a signal that satisfies both the data fidelity and structural constraints simultaneously. This is precisely the common fixed point problem that the iteration scheme (3.1) is designed to solve.

5.2. **Iterative Reconstruction Algorithm.** The sequence $\{w_n\}$ is generated by the iteration scheme (3.1),

$$\begin{cases} v_n = (1 - \beta_n)w_n \oplus \beta_n z_n, \\ w_{n+1} = (1 - \alpha_n)z_n \oplus \alpha_n t^n v_n \end{cases}$$

for $n \in \mathbb{N}$, where $z_n \in P_T(t^n w_n)$ and $0 < a \leq \alpha_n$, $\beta_n \leq b < 1$.

In each iteration, the algorithm performs two main steps:

(1) An update is performed to enforce the structural constraints on the signal, modeled by the multivalued mapping T. The selection $z_n \in P_T(t^n w_n)$ ensures that the structural properties are satisfied.

(2) The signal is then updated again using the data fidelity operator t. The asymptotically nonexpansive nature of the mappings and the specific structure of the iteration guarantee the convergence of the sequence.

5.3. **Convergence and Practical Implications.** According to Theorem 3.7, if $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, the sequence $\{w_n\}$ generated by the iterative process will Δ-converge to a common fixed point of t and T. In the context of signal reconstruction, this means:

(1) The algorithm is guaranteed to converge to a solution that is both consistent with the measured data and adheres to the a priori structural constraints.

(2) The use of $CAT(0)$ space geometry ensures the stability of the iterative updates, which is crucial for handling complex, high-dimensional data.

(3) This approach provides a robust theoretical framework for developing effective algorithms for real-world signal recovery problems, including denoising, compressed sensing, and image restoration from incomplete data.

5.4. **Explanation of the Code.** The MATLAB implementation follows a structured iteration scheme (3.1). Below is a breakdown of the main steps:

(1) **Load MRI Image:** The algorithm uses a standard phantom MRI image for testing.

(2) **Degraded Image:** Adds Gaussian noise with mean 0 and variance 0.01.

(3) **Initialization:** Sets up maximum iterations and arrays to store results (PSNR, SSIM). Let $\alpha_n = 0.5$ and $\beta_n = 0.2$.

The quality of the reconstructed image is measured by the Peak Signal-to-Noise Ratio (PSNR) in decibels (dB). The standard formula is defined as:

$$\text{PSNR} = 10\log_{10}\left(\frac{\text{MAX}_I^2}{\text{MSE}}\right),$$

where MAX_I is the maximum possible pixel value of the image, and MSE is the Mean Squared Error, calculated as:

$$\text{MSE} = \frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N}(u_{ij} - u_{ij}^*)^2,$$

where u being the original image, u^* the reconstructed image, and $M \times N$ the image dimensions.

The Structural Similarity Index Measure (SSIM) between two images x and y is defined as:

$$\text{SSIM} = \left[\frac{2\mu_x\mu_y + C_1}{\mu_x^2 + \mu_y^2 + C_1}\right] \times \left[\frac{2\sigma_{xy} + C_2}{\sigma_x^2 + \sigma_y^2 + C_2}\right] \times \left[\frac{\sigma_x^2 + \sigma_y^2 + C_3}{\mu_x^2 + \mu_y^2 + C_3}\right],$$

where

- μ_x and μ_y are the average pixel intensities of the images x and y, respectively.
- σ_x^2 and σ_y^2 are the variances of the pixel intensities in the images x and y, respectively.
- σ_{xy} is the covariance of the pixel intensities between the two images.
- C_1, C_2, C_3 are small constants used to stabilize the division with weak denominators.

(4) **Iterative Reconstruction:** The main iterative loop consists of:
- Applying a nonexpansive projection (wavelet-based denoising) to remove noise.
- Updating the image using an asymptotically nonexpansive transformation.
- Combining the updates using the fixed point iteration scheme.

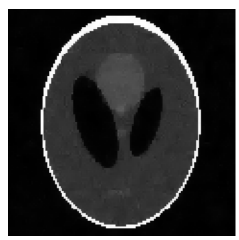
(5) **Convergence Check:** The iteration stops when the relative error falls below a predefined tolerance (10^{-5}).

(6) **Wavelet-Based Denoising:** A Haar wavelet transform is applied to remove small coefficients, ensuring a sparse reconstruction.

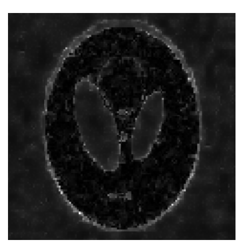
The results are show in Figure 1 and Figure 2.

(A) Original image

(B) Degraded image



(C) PSNR:31.19dB, SSIM: 0.5018, 19 Iterations



(D) Residual

FIGURE 1. Reconstructed image results by iteration scheme (3.1).

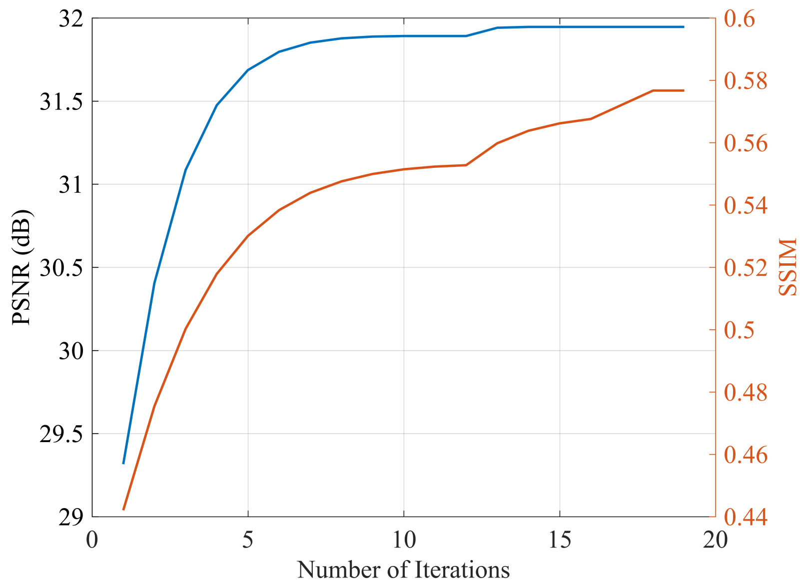


FIGURE 2. Convergence of PSNR and SSIM for reconstruction iteration scheme (3.1).

By applying Theorem 3.7, we guarantee convergence to an optimal reconstructed signal using a structured iterative process. This approach is widely used in compressed sensing, image processing, and denoising algorithms.

6. Conclusion

We studied a coupled iteration for two nonlinear mappings in CAT(0) spaces in which a single-valued asymptotically nonexpansive map $t : G \to G$ is blended with a multivalued asymptotically nonexpansive map $T : G \to FB(G)$ via a nonexpansive mapping P_T. For the explicit two step iteration scheme (3.1) with bounded controls $0 < a \leq \alpha_n, \beta_n \leq b < 1$ and a summable asymptotic modulus $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, we proved that the generated sequence $\{w_n\}$ Δ-converges to a common fixed point of t and T. The analysis hinges on the identification $\mathrm{Fix}(T) = \mathrm{Fix}(P_T)$, quasi-Fejér type estimates yielding asymptotic regularity, and a demiclosedness principle for (asymptotically) nonexpansive mappings adapted to CAT(0) geometry, which together ensure convergence of the mixed scheme.

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