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ON THE EXISTENCE OF NON-INCREASING INTEGRABLE SOLUTIONS FOR A NONLINEAR INTEGRO-DIFFERENTIAL EQUATION

Faez N. Ghaffoori[1], Rawaa Ibrahim Esa[2], Wagdy G. El-Sayed[3] and Maan A. Rasheed[4]

[1]Department of Mathematics, College of Basic Education,
Mustansiriyah University, Baghdad, Iraq
e-mail: fng_2022@uomustansiriyah.edu.iq

[2]Department of Computer Sciences, College of Basic Education,
Mustansiriyah University, Baghdad, Iraq
e-mail: rawaaa_e.edbs@uomustansiriyah.edu.iq

[3]Faculty of Science, Alexandria University, Alexandria, Egypt
e-mail: wagdygoma@alexu.edu.eg

[4]Department of Mathematics, College of Science for Women,
University of Baghdad, Baghdad, Iraq
e-mail: maan.a@csw.uobaghdad.edu.iq

Abstract. Many problems from physics, engineering and economics can be efficiently solved when dealing with nonlinear integral equations which involve both integrals and derivatives of unknown function. In this work, the existence of non-increasing integrable solutions for a nonlinear integro-differential equation is investigated using the Darbo fixed point theorem via the Hausdorff measure technique of non-compactness.

1. Introduction

The solutions of differential and integro-differential equations have a major role in the fields of science and engineering. Therefore, many authors

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[0]Corresponding author: M. A. Rasheed(maan.a@csw.uobaghdad.edu.iq).

have considered the theoretical and numerical solutions for some types of these equations [1, 2, 17, 20, 21, 22, 23, 24, 27, 28]. The non-linear integro-differential equations are used to model several phenomena in science and technology includes chemistry, biology, physics, vibration, acoustic signals, signal processing, fluid dynamics and viscoelasticity. Moreover, Integro-differential equations are used mostly to study discontinuous stochastic processes [25]. In this work, we investigate the solvability of the nonlinear integro-differential equation:

$$u(t) = q(t) + \int_0^t p(t,s) f\left(s, u'(s)\right) ds, \quad t \geq 0, \tag{1.1}$$

in the class $L^1(R^+)$of Lebesgue integrable functions on R^+. This equation was investigated before in different Banach spaces or by using different fixed point theorems under some various assumptions, see [4, 8, 9, 10, 11, 12, 13, 14, 15, 16].

Indeed, equation (1.1) is a general form of that reduced from many mathematical modeling such as the resultant equation in treating the spread of the COVID-19 disease [26].

The aim of this work is to find sufficient conditions under which the solution of problem (1.1) are non-increasing. The rest of this article is organized as follows: in the next section, we derive the equivalent integral equation. Also, some needed definitions and theorems are stated. In the third section, the main existence results are presented. In the last section, some conclusions and possible future works are stated.

2. Preliminaries

First of all, we will transform (1.1) to an equivalent integral equation by differentiate both sides of equation (1.1) with respect to t, so, we get

$$u'(t) = q'(t) + p(t,t) f\left(t, u'(t)\right) + \int_0^t \frac{\partial p}{\partial t}(t,s) f\left(s, u'(s)\right) ds.$$

Put

$$x(t) = u'(t), \quad q'(t) = h(t), \quad p(t,t) = g(t), \quad \frac{\partial p(t,s)}{\partial t} = k(t,s).$$

Then, we get

$$x(t) = h(t) + g(t) f(t, x(t)) + \int_0^t k(t,s) f(s, x(s)) ds, \quad t \geq 0. \tag{2.1}$$

Note that the two equations (1.1) and (2.1) are equivalent and they are solvable together, where

$$u(t) = q(0) + \int_0^t x(s)ds.$$

In the rest of this section, we give short notes for some definitions and theorems that will be needed later to investigate the solvability of the nonlinear Volterra integral equation (2.1).

Definition 2.1. ([3, 5], Superposition operator) Assume that a function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions that is, it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. Then, to every function $x(t)$ being measurable on I, we may assign the function

$$F(x)(t) = f(t, x(t)), \quad t \in I.$$

The operator F in such way called the superposition operator generated by the function f and we have the following theorem.

Theorem 2.2. ([5]) *The superposition operator* $\mathbf{F}$ *generated by the function* f *maps continuously the space* $L^1(I)$ *into* $L^1(I)$*, if and only if*

$$|f(t, x)| \le a_1(t) + b|x|, \quad \forall\, t \in I, x \in \mathbb{R},$$

where $a(t) \in L^1(I)$ *and* $b \ge 0$.

Definition 2.3. ([5], Linear integral operators) Consider the following linear integral operator:

$$(Kx)(t) = \int_0^\infty k(t, s)x(s)ds, \quad t \in \mathbb{R}^+,$$

where $k(t, s) : R^+ \times R^+ \to R$ is measurable with respect to its both variables.

Theorem 2.4. ([18, 19]) *Assume that* $k(t, s) = k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is measurable on* $\mathbb{R}^+$*such that the integral operator* $(Kx)(t) = \int_0^\infty k(t, s)x(s)ds,\ t \ge 0$ *maps* $L^1(\mathbb{R}^+)$ *into itself. Then* K *transforms the set of non-increasing functions from* $L^1(\mathbb{R}^+)$ *into itself, if and only if for any* $A > 0$*, the following implication is true.*

$$t_1 < t_2 \quad \Rightarrow \quad \int_0^A k(t_1, s)\, ds \ge \int_0^A k(t_2, s)\, ds.$$

Note that, in the case when K *transforms* $L^1(\mathbb{R}^+)$*into itself, then* K *is continuous and bounded with norm* $\|K\|$ *([18]).*

Definition 2.5. (Measures of non-compactness) Let E be a Banach space with norm $\|\cdot\|$ and zero element θ. Let X be a nonempty and bounded subset of E and B_r be a closed ball in E with center at θ and radius r.

The Hausdorff measure of noncompactness $\chi(X)$ is defined as [6]:

$$\chi(X) = \inf\{r > 0 : \text{ there exists a finite subset } Y \text{ of } E, \ \ X \subset Y + B_r\}.$$

Another measure was defined in the space $L^1(\mathbb{R}^+)$[4]. For $\varepsilon > 0$, let

$$c(X) = \lim_{\epsilon \to 0}\left\{\sup_{\mathrm{x} \in \mathrm{X}}\left\{\sup\left[\int_D |x(t)|dt, \ \ D \subset I, \ \ \text{meas } (D) \leq \epsilon\right]\right\}\right\}$$

and

$$d(X) = \lim_{T \to \infty}\left\{\sup\left[\int_T^{\infty} |x(t)|dt, x \in X\right]\right\}.$$

Where meas D denotes the lebesgue measure of the subset D. Form

$$\gamma(X) = c(X) + d(X).$$

Where the function γ is a regular measure of weak noncompactness in the space $L^1(\mathbb{R}^+)$, and we have the following theorem.

Theorem 2.6. ([4, 5]) *Let X be a nonempty, bounded and compact in measure subset of E. Then*

$$\chi(X) \leq \gamma(X) \leq 2\chi(X).$$

For compactness in measure, we have the following theorem.

Theorem 2.7. ([7]) *Let X be a bounded subset of E, consisting of functions which are almost everywhere non-decreasing (or non-increasing) on the interval $\mathbb{R}^+$. Then X is compact in measure.*

Theorem 2.8. ([5, 6]) *Let Q be a nonempty, bounded, closed and convex subset of E and let $H : Q \to Q$ be a continuous transformation which is a contraction with respect to the measure of non-compactness μ, that is, there exist $q \in [0, 1)$ such that*

$$\mu(A(X)) \leq q\mu(X)$$

for any nonempty subset X of E. Then A has at least one fixed point in the set Q.

3. Main results

Consider the operator H associated with integral equation (2.1).

$$Hx = h(t) + g(t)f(t, x(t)) + \int_0^t k(t,s)f(s,x(s))ds, \quad t \geq 0. \tag{3.1}$$

Then equation (2.1) becomes

$$x = Hx = h + gFx + \lambda KFx, \tag{3.2}$$

where

$$(Fx)(t) = f(t,x), \quad (Kx)(t) = \int_0^t k(t,s)x(s)ds$$

are the superposition operator and the linear integral operator, respectively.

We shall treat the equation (2.1) under the following assumptions:

(i) $h : \mathbb{R}^+ \to \mathbb{R}$ such that $h \in L^1(\mathbb{R}^+)$ and $g : \mathbb{R}^+ \to \mathbb{R}$ is bounded function such that $M = \sup_{t\in\mathbb{R}^+} |g(t)|$, and h and g are almost everywhere positive and non-increasing in $\mathbb{R}^+$,

(ii) $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$satisfies the caratheodory conditions and there are positive function $a \in L^1(\mathbb{R}^+)$ and constant $b \geq 0$ such that

$$|f(t,x)| \leq a(t) + b|x|, \quad \forall\, t, x.$$

Moreover, $f(t,x)$ is assumed to be non-increasing on $\mathbb{R}^+ \times \mathbb{R}$ for all t, x.

(iii) $k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ satisfies caratheodory conditions such that the linear operator K defined as

$$(Kx)(t) = \int_0^t k(t,s)x(s)ds, \quad t > 0$$

maps the space $L^1(\mathbb{R}^+)$ into itself, also, for all $A > 0$ and for all $t \in \mathbb{R}^+$, we have

$$t_1 < t_2 \;\; \Rightarrow \;\; \int_0^A k(t_1, s)\, ds \geq \int_0^A k(t_2, s)\, ds.$$

(iv) $q = b(M + \|K\|) < 1$.

Then, we can prove the following existence theorem.

Theorem 3.1. *Assume that the assumptions $(i) - (iv)$ are satisfied. Then the equation (2.1) has at least one integrable solution on $L^1(\mathbb{R}^+)$ being non-increasing on $\mathbb{R}^+$.*

Proof. First, note that the space $L^1\left(\mathbb{R}^+\right)$ is Banach space. Next, we will prove that the operator H maps continuously the space $L^1\left(\mathbb{R}^+\right)$ into itself, we have

$$\begin{aligned}
&\int_0^\infty |(Hx)(\mathrm{t})|dt \\
&= \int_0^\infty \left| h(t) + g(t)f(t,x(t)) + \int_0^t k(t,s)f(s,x(s))ds \right| dt \\
&\leq \int_0^\infty |h(t) + g(t)f(t,x(t))|dt + \int_0^\infty \left| \int_0^t k(t,s)f(s,x(s))ds \right| dt \\
&\leq \int_0^\infty |h(t)|dt + \int_0^\infty |g(t)|[a(t) + b|x(t)|]dt + \|K\|[a(s) + b|x(s)|]ds \\
&\leq \|h\| + M\|a\| + bM \int_0^\infty |x(t)|dt + \|K\|\|a\| + b\|K\| \int_0^\infty |x(s)|ds \\
&\leq \|h\| + [M + \|K\|]\|a\| + [bM + b\|K\|] \int_0^\infty |x(t)|dt \\
&< \infty.
\end{aligned}$$

Then due to assumptions (i),(ii),(iii) and Theorem 2.2, we see that the operator H maps continuously the space $L^1\left(\mathbb{R}^+\right)$ into itself.

In view of our assumptions and for $x \in B_r$, we have

$$\begin{aligned}
\|Hx\| &= \int_0^\infty \left| h(t) + g(t)f(t,x(t)) + \int_0^t k(t,s)f(s,x(s))ds \right| dt \\
&= \|h\| + \|gF\| + \|KFx\| \\
&\leq \|h\| + \sup_{t \in R^+} |g|\|F\| + \|K\|\|Fx\| \\
&\leq \|h\| + M \int_0^\infty |f(t,x(t))|dt + \|K\| \int_0^\infty |f(t,x(t))|dt \\
&\leq \|h\| + M \int_0^\infty [a(t) + b|x(t)|]dt + \|K\| \int_0^\infty [a(t) + b|x(t)|]dt \\
&\leq \|h\| + M\|a\| + bM \int_0^\infty |x(t)|\Big] dt + \|K\|\|a\| + b\|K\| \int_0^\infty |x(t)|\Big] dt \\
&\leq \|h\| + [M + \|K\|]\|a\| + b[M + \|K\|]\|x\|.
\end{aligned}$$

Then

$$\|Hx\| \leq \|h\| + [M + \|K\|]\|a\| + b[M + \|K\|]r \leq r.$$

So, H transforms B_r into B_r, where

$$r \leq \frac{\|h\| + [M + \|K\|]\|a\|}{1 - b[M + \|K\|]}.$$

Using assumption (iv), we see that $r > 0$. Next, let

$$Q_r = \left\{ x \in B_r : x \text{ is a.e. positive and non-increasing on } \mathbb{R}^+ \right\}.$$

Then Q_r is nonempty, bounded, convex, closed and compact in measure.

For convexity, let $x_1, x_2 \in Q_r$, with $\|x_i\| < r$, $i = 1, 2$, $0 < \lambda \leq 1$ and $x = \lambda x_1 + (1 - \lambda) x_2$. Then

$$\begin{aligned} \|x\| &= \|\lambda x_1 + (1 - \lambda) x_2\| \\ &\leq \lambda \|x_1\| + |1 - \lambda| \|x_2\| \\ &< \lambda r + (1 - \lambda) r \\ &= r, \end{aligned}$$

it implies that $x \in Q_r$ and hence Q_r is convex.

To show that Q_r is closed [4], let us take a sequence $\{x_n\}$ which converges to x (in the norm of $L^1(\mathbb{R}^+)$). Then $\{x_n\}$ converges in measure to x and using Vitali theorem, we deduce that, there exists a subsequence of our sequence which converges to x almost everywhere on R^+. Hence we see that x is non-increasing almost everywhere on $\mathbb{R}^+$ which means that $x \in Q_r$ then Q_r is closed.

We can prove that Q_r is compact in measure by using Theorem 2.7.

Next, by taking $x \in Q_r$, then $x(t)$ is almost everywhere positive and non-increasing on $\mathbb{R}^+$ and consequently, due to the assumption (ii), $Fx(t)$ is also of the same type, in virtue of the assumption (iii) and Theorem 2.4, we deduce that KFx is also positive and non-increasing on $\mathbb{R}^+$. Further, the assumption (i) permits us to deduce that

$$Hx(t) = h(t) + g(t)Fx(t) + KFx(t)$$

is also a.e. positive and non-increasing on $\mathbb{R}^+$. This fact together with assertion $H : B_r \to B_r$, gives that self-mapping of the set Q_r, since the operator K is continuous and F is continuous in view Theorem 2.2, we conclude that H maps continuously Q_r into Q_r.

Now, we will prove that $\beta(HX) \leq q\beta(X)$, for any bounded subset X of Q_r, take an arbitrary number $\varepsilon > 0$ and a set $D \subset \mathbb{R}^+$such that meas $(D) \leq \varepsilon$. Hence, for any $x \in X$, we see that

$$\int_D |(Hx)(\mathrm{t})| dt = \int_D \left| h(t) + g(t) f(t, x(t)) + \int_0^t k(t,s) f(s, x(s)) ds \right| dt$$

$$\begin{aligned}
&\leq \int_D |h(t)|dt + \int_D |g(t)f(t,x(t))|dt \\
&\quad + \int_D \left| \int_0^t k(t,s)f(s,x(s))ds \right| dt \\
&\leq \int_D |h(t)|dt + \int_D |g(t)||f(t,x(t))|dt + \|KFx\|_{L^1(D)} \\
&\leq \int_D |h(t)|dt + M\int_D [a(t)+b|x(t)|]dt + \|K\|_{L^1(D)} \int_D |f(t,x(s))|ds \\
&\leq \int_D |h(t)|dt + M\int_D a(t)dt + bM\int_D |x(t)|dt \\
&\quad + \|K\|_{L^1(D)} \int_D a(s)ds + \mathrm{b}\|K\|_{L^1(D)} \int_D |x(t)|dt \\
&\leq \int_D |h(t)|dt + M\int_D a(t)dt + \left[bM + b\|K\|_{L^1(D)}\right] \int_D |x(t)|dt,
\end{aligned}$$

where $\|K\|_{L^1(D)}$ denotes the norm of the operator $K : L^1(\mathbb{R}^+) \to L^1(\mathbb{R}^+)$. Since

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left\{\sup\left\{\int_D |h(t)|dt : D\subset\mathbb{R}^+,\ \ \mathrm{meas(D)}\leq\varepsilon\right\}\right\} \\
&\quad = \lim_{\varepsilon\to 0}\sup\left[\int_D a(t)dt : D\subset\mathbb{R}^+,\ \ \mathrm{meas(D)}\leq\varepsilon\right\}\right] \\
&\quad = 0.
\end{aligned}$$

Hence, we get

$$c(HX)\leq qc(X), \quad q = b\left[M + \|K\|_{L^1(D)}\right]. \tag{3.3}$$

For $T > 0$, any $x \in X$, we have

$$\begin{aligned}
&\int_T^\infty |(Hx)(\mathrm{t})|dt \\
&\quad = \int_T^\infty \left| h(t) + g(t)f(t,x(t)) + \int_0^t k(t,s)f(s,x(s))ds \right| dt \\
&\quad \leq \int_T^\infty |h(t)+g(t)f(t,x(t))|dt + \int_T^\infty \left| \int_0^t k(t,s)f(s,x(s))ds \right| dt \\
&\quad \leq \int_T^\infty |h(t)|dt + \int_T^\infty |g(t)||f(t,x(t))|dt + \int_T^\infty \left| \int_0^t k(t,s)f(s,x(s))ds \right| dt \\
&\quad \leq \int_T^\infty |h(t)|dt + M\int_T^\infty [a(t)+b|x(t)|]dt + \|K\| \int_T^\infty [a(s)+b|x(s)|]ds \\
&\quad \leq [bM + b\|K\|] \int_T^\infty |x(t)|dt.
\end{aligned}$$

Hence, we get

$$d(HX) \leq qd(X). \tag{3.4}$$

Combine equations (3.2) and (3.3), we deduce that

$$\gamma(HX) \leq q\gamma(X).$$

Using Theorem (2.6), where Q_r is compact in measure and $X \subset Q_r$, we get

$$\chi(HX) \leq q\chi(X).$$

Hence, we can apply Darbo fixed point Theorem 2.8, where all its conditions are satisfied. So Eq. (2.1) has at least one solution in $L^1\left(\mathbb{R}^+\right)$ and the same for Eq. (1.1). □

4. Conclusions

The existence of non-increasing integrable solutions for a nonlinear integro-differential equation is discussed in this article. The equivalent integral equation is derived first. Then, the Darbo fixed point theorem is used to prove the main results. Based on the obtained results, finding exact or numerical solutions might be considered for mathematical models that involve particular types of equations in different fields of study.

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