Nonlinear Functional Analysis and Applications
Vol. 30, No. 3 (2025), pp. 961-989
ISSN: 1229-1595(print), 2466-0973(online)
https://doi.org/10.22771/nfaa.2025.30.03.19
http://nfaa.kyungnam.ac.kr/journal-nfaa

NEUTROSOPHIC CONTROLLED METRIC-LIKE SPACES WITH APPLICATION ON NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION

M. Pandiselvi[1], M. Jeyaraman[2], Bahaaeldin Abdalla[3], Kamal Shah[3] and Thabet Abdeljawad[4]

[1]P.G and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai Affiliated to Alagappa University, Karaikudi, Tamil Nadu, India
e-mail: mpandiselvi2612@gmail.com

[2]P.G. and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai Affiliated to Alagappa University, Karaikudi, Tamil Nadu, India
e-mail: jeya.math@gmail.com

[3]Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia
e-mail: babdallah@psu.edu.sa, kshah@psu.edu.sa

[4]Department of Mathematics and Applied Mathematics, School of Science and Technology, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa
e-mail: tabdeljawad@psu.edu.sa

Abstract. In this study, we propose the concept of a neutrosophic controlled metric-like spaces, which extends classical metric notions to better model uncertainty and indeterminacy. We explore its structural properties and present significant examples that illustrate its theoretical relevance. Building on this framework, we establish new fixed point results for contraction mappings tailored to the neutrosophic setting. As an application, we investigate a nonlinear fractional differential equation and prove the existence and uniqueness of its solution, thereby reinforcing the practical value of our theoretical developments.

1. Introduction

Zadeh [29] established the foundation of fuzzy mathematics in 1965. Kramosil and Michalek [12] first introduced the idea of fuzzy metric spaces (FMS),

[0]Received April 9, 2025. Revised June 8, 2025. Accepted June 11, 2025.
[0]2020 Mathematics Subject Classification: 26A33, 34A07, 34A08.
[0]Keywords: Metric like space, neutrosophic b-metric like spaces, neutrosophic controlled metric-like spaces, fixed point, nonlinear fractional differential equation, unique solution.
[0]Corresponding author: Thabet Abdeljawad(tabdeljawad@psu.edu.sa).

which was later modified by George and Veeramani [3]. Amini-Harandi [5] recently proposed the concept of a metric-like space (MLS) and proved several related results. Controlled metric-like spaces ($CMLSs$) were introduced by Mlaiki et al. [16], who also obtained numerous fixed point results for contractive mappings. More recently, controlled fuzzy metric-like spaces ($CFMLSs$) were developed by Sezen [24], extending the idea of controlled-type metric spaces. In this sequence of generalizations, the notion of fuzzy metric-like space ($FMLS$) was defined in [26], and some fixed point results were established in that setting.

Park [20] introduced the concept of intuitionistic fuzzy metric spaces($IFMS$) based on continuous t-norms and t-conorms. Significant contributions to intuitionistic fuzzy topological spaces were made by Sadati and Park [22]. In 1998, Smarandache [27] introduced neutrosophic logic and the concept of the neutrosophic set (NS). The concept of neutrosophic metric spaces (NMS) was later developed by Kirisci and Simsek [10], incorporating the degrees of membership, non-membership, and indeterminacy.

Several researchers [6, 8, 11, 17, 21] have derived fixed point results in intuitionistic fuzzy metric spaces by employing neutrosophic set theory. Authors in [7, 28] also contributed to fixed point theory in neutrosophic metric spaces. Fixed point theory has proven to be a powerful tool for addressing problems in traditional and fractional calculus, particularly in proving the existence of solutions. As a result, it has been widely applied to develop sufficient conditions for the solvability of various mathematical problems [1, 2, 4, 9, 21]. Applied analysis, which finds utility across several disciplines, has also employed different mathematical tools for deriving numerical and analytical solutions [18, 19]. Recently, several works have applied fixed point theory to investigate problems involving fractional calculus [15, 25].

With inspiration from the techniques developed in [16], the objective of this paper is to introduce the concept of a neutrosophic controlled metric-like space ($NCMLS$). Furthermore, we extend existing fixed point results for contraction mappings within this new setting. To demonstrate the utility of the proposed framework, we conclude by applying our results to specific problems in fractional calculus. This study is motivated by the need to further generalize metric structures to better capture uncertainty and indeterminacy, while also providing a robust framework for solving nonlinear fractional differential equations using fixed point theory.

2. Preliminaries

Definition 2.1. ([3]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a CTN if it enjoys the following conditions:

(i) $\zeta * \omega = \omega * \zeta$ for all $\zeta, \omega \in [0,1]$;
(ii) $*$ is continuous;
(iii) $\zeta * 1 = \zeta$ for all $\zeta \in [0,1]$;
(iv) $*$ is associative;
(v) if $\zeta \leq \omega$ and $c \leq d$ with ζ,ω,c, d$\in [0,1]$, then $\zeta * c \leq \omega * d$.

Definition 2.2. ([3]) A binary operation $\odot : [0,1] \times [0,1] \to [0,1]$ is said to be a $CTCN$ if it holds the followings assertions:

(i) $\zeta \odot \omega = \omega \odot \zeta$ for all $\zeta, \omega \in [0,1]$;
(ii) $\odot$ is continuous;
(iii) $\zeta \odot 0 = 0$;
(iv) $\odot$ is associative;
(v) if $\zeta \leq \omega$ and $c \leq d$ with ζ,ω,c, $d \in [0,1]$, then $\zeta \odot c \leq \omega \odot d$.

Definition 2.3. ([21]) Let L be a nonvoid set. The set L along with $J : L \times L \to [1, \infty)$ is said to be a metric-like space (MLS) if the following assertions hold: For all $\zeta, \omega, c \in L$,

(i) $J(\zeta, \omega) = 0$ implies $\zeta = \omega$;
(ii) J is symmetric in ζ, ω;
(iii) $J(\zeta, \omega) \leq J(\zeta, c) + J(c, \omega)$.

Definition 2.4. ([22]) Let L be a nonvoid set, $\Psi : L \times L \to [1, \infty)$ and $J : L \times L \to [1, \infty)$. (L, J) is named a $CMLS$ if it enjoy the following assertions: For all $\zeta, \omega, c \in L$,

(i) $J(\zeta, \omega) = 0 \;\Rightarrow\; \zeta = \omega$;
(ii) J is symmetric in ζ, ω;
(iii) $J(\zeta, \omega) \leq \Psi(\zeta, c)J(\zeta, c) + \Psi(c, \omega)J(c, \omega)$.

Definition 2.5. ([11]) Let L be a nonvoid set and $*$ be a CTN. Let A_b be a FS on $L \times L \times (0, \infty)$. A triplet $(L, A_b, *)$ is said to be a $FMLS$ if the following assertions hold: For all $\zeta, \omega \in L$ and $\delta, \mu > 0$,

(FL1) $A_b(\zeta, \omega, \mu) > 0$;
(FL2) $A_b(\zeta, \omega, \mu) = 1 \;\Rightarrow\; \zeta = \omega$;
(FL3) A_b is symmetric;
(FL4) $A_b(\zeta, c, \delta + \mu) \geq A_b(\zeta, \omega, \delta) * A_b(\omega, c, \mu)$;
(FL5) $A_b(\zeta, \omega, .) : (0, \infty) \to [0, 1]$ is continuous.

Definition 2.6. Let L be a nonempty set. Suppose $*$ is a neutrosophic CTN and $\odot$ be a neutrosophic $CTCN$, $b \geq 1$ and A_b, B_b, C_b are NS on $L \times L \times (0, \infty)$. If $(L, A_b, B_b, C_b, *, \odot)$ the following assertions hold for all $\zeta, \omega, c \in L$ and $\delta, \mu > 0$, then $(L, A_b, B_b, C_b, *, \odot)$ is called a neutrosophic b-metric like pace ($NbMLS$).

(N1) $0 \leq A_b(\zeta,\omega,\mu) \leq 1; 0 \leq B_b(\zeta,\omega,\mu) \leq 1; 0 \leq C_b(\zeta,\omega,\mu) \leq 1$;
(N2) $A_b(\zeta,\omega,\mu) + B_b(\zeta,\omega,\mu) + C_b(\zeta,\omega,\mu) \leq 3$;
(N3) $A_b(\zeta,\omega,\mu) = 1 \ \Rightarrow \ \zeta = \omega$;
(N4) $A_b(\zeta,\omega,\mu) = A_b(\omega,\zeta,\mu)$;
(N5) $A_b(\zeta,c,b(\mu+\delta)) \geq A_b(\zeta,\omega,\mu) * A_b(\omega,c,\delta)$;
(N6) $A_b(\zeta,\omega,.) : [0,\infty) \to [0,1]$ is neutrosophic continuous;
(N7) $\lim_{\mu\to\infty} A_b(\zeta,\omega,\mu) = 1$;
(N8) $B_b(\zeta,\omega,\mu) = 0 \ \Rightarrow \ \zeta = \omega$;
(N9) $B_b(\zeta,\omega,\mu) = B_b(\omega,\zeta,\mu)$;
(N10) $B_b(\zeta,c,b(\mu+\delta)) \leq B_b(\zeta,\omega,\mu) \odot B_b(\omega,c,\delta)$;
(N11) $B_b(\zeta,\omega,.) : [0,\infty) \to [0,1]$ is neutrosophic continuous;
(N12) $\lim_{\mu\to\infty} B_b(\zeta,\omega,\mu) = 0$;
(N13) $C_b(\zeta,\omega,\mu) = 0 \ \Rightarrow \ \omega = \zeta$;
(N14) $C_b(\zeta,\omega,\mu) = C_b(\omega,\zeta,\mu)$;
(N15) $C_b(\zeta,c,b(\mu+\delta)) \leq C_b(\zeta,\omega,\mu) \odot C(\omega,c,\delta)$;
(N16) $C_b(\zeta,\omega,.) : [0,\infty) \to [0,1]$ is neutrosophic continuous;
(N17) $\lim_{\mu\to\infty} C_b(\zeta,\omega,\mu) = 0$;
(N18) if $\mu < 0$ then $A_b(\zeta,\omega,\mu) = 0, B_b(\zeta,\omega,\mu) = 1, C_b(\zeta,\omega,\mu) = 1$.

3. Main results

Definition 3.1. Let L be a nonvoid set. Suppose $*$ is a neutrosophic CTN and $\odot$ be a neutrosophic $CTCN$, $b \geq 1, \Psi : L \times L \to [1,\infty)$ and A_b, B_b, C_b are neutrosophic sets on $L \times L \times (0,\infty)$. If the following assertions hold, then $(L, A_\Psi, B_\Psi, C_\Psi, *, \odot)$ is called neutrosophic controlled metric-like space ($NCMLS$). For all $\zeta,\omega,c \in L$ and $\delta,\mu > 0$,

(NL1) $0 \leq A_\Psi(\zeta,\omega,\mu) \leq 1; 0 \leq B_\Psi(\zeta,\omega,\mu) \leq 1; 0 \leq C_\Psi(\zeta,\omega,\mu) \leq 1$;
(NL2) $A_\Psi(\zeta,\omega,\mu) + B_\Psi(\zeta,\omega,\mu) + C_\Psi(\zeta,\omega,\mu) \leq 3$;
(NL3) $A_\Psi(\zeta,\omega,\mu) = 1 \ \Rightarrow \ \zeta = \omega$;
(NL4) $A_\Psi(\zeta,\omega,\mu) = A_\Psi(\omega,\zeta,\mu)$;
(NL5) $A_\Psi(\zeta,c,(\mu+\delta)) \geq A_\Psi\left(\zeta,\omega,\frac{\mu}{\Psi(\zeta,\omega)}\right) * A_\Psi\left(\omega,c,\frac{\delta}{\Psi(\omega,c)}\right)$;
(NL6) $A_\Psi(\zeta,\omega,.) : [0,\infty) \to [0,1]$ is neutrosophic continuous;
(NL7) $\lim_{\mu\to\infty} A_\Psi(\zeta,\omega,\mu) = 1$;
(NL8) $B_\Psi(\zeta,\omega,\mu) = 0 \ \Rightarrow \ \zeta = \omega$;
(NL9) $B_\Psi(\zeta,\omega,\mu) = B_\Psi(\omega,\zeta,\mu)$;
(NL10) $B_\Psi(\zeta,c,(\mu+\delta)) \leq B_\Psi\left(\zeta,\omega,\frac{\mu}{\Psi(\zeta,\omega)}\right) \odot B_\Psi\left(\omega,c,\frac{\delta}{\Psi(\omega,c)}\right)$;
(NL11) $B_\Psi(\zeta,\omega,.) : [0,\infty) \to [0,1]$ is neutrosophic continuous;
(NL12) $\lim_{\mu\to\infty} B_\Psi(\zeta,\omega,\mu) = 0$;

(NL13) $C_\Psi(\zeta,\omega,\mu)=0 \;\Rightarrow\; \zeta=\omega$;
(NL14) $C_\Psi(\zeta,\omega,\mu)=C_\Psi(\omega,\zeta,\mu)$;
(NL15) $C_\Psi(\zeta,c,(\mu+\delta))\leq C_\Psi\left(\zeta,\omega,\frac{\mu}{\Psi(\zeta,\omega)}\right)\odot C_\Psi\left(\omega,c,\frac{\mu}{\Psi(\omega,c)}\right)$;
(NL16) $C_\Psi(\zeta,\omega,.):[0,\infty)\to[0,1]$ is neutrosophic continuous;
(NL17) $\lim\limits_{\mu\to\infty} C_\Psi(\zeta,\omega,\mu)=0$;
(NL18) if $\mu<0$, then $A_\Psi(\zeta,\omega,\mu)=0$, $B_\Psi(\zeta,\omega,\mu)=1$, $C_\Psi(\zeta,\omega,\mu)=1$.

Example 3.2. Let $L=\{2,4,6\}$ and $\Psi:L\times L\to[1,\infty)$ be a function defined by $\Psi(\zeta,\omega)=\zeta+\omega+1$. Define $A_\Psi,B_\Psi,C_\Psi:L\times L\times(0,\infty)\to[0,1]$ as

$$A_\Psi(\zeta,\omega,\mu)=\frac{\mu}{\mu+\max\{\zeta,\omega\}},\quad B_\Psi(\zeta,\omega,\mu)=\frac{\max\{\zeta,\omega\}}{\mu+\max\{\zeta,\omega\}}$$

and

$$C_\Psi(\zeta,\omega,\mu)=\frac{\max\{\zeta,\omega\}}{\mu}.$$

Then $(L,A_b,B_b,C_b,*,\odot)$ is a $NCMLS$ with $CTN*$ and $CTCN\odot$ given by $\zeta*\omega=\zeta\omega$ and $\zeta\odot\omega=\max\{\zeta,\omega\}$.

In fact, here we have to check NL(5), NL(10), and NL(15), remaining conditions are obvious. Let $\zeta=2$, $\omega=4$, $c=6$. Then,

$$A_\Psi(2,6,\mu+\delta)=\frac{\mu+\delta}{\mu+\delta+\max\{2,6\}}=\frac{\mu+\delta}{\mu+\delta+6},$$
$$A_\Psi\left(2,4,\frac{\mu}{\Psi(2,4)}\right)=\frac{\frac{\mu}{\Psi(2,4)}}{\frac{\mu}{\Psi(2,4)}+\max\{2,4\}}=\frac{\frac{\mu}{7}}{\frac{\mu}{7}+4}=\frac{\mu}{\mu+28},$$
$$A_\Psi\left(4,6,\frac{\delta}{\Psi(4,6)}\right)=\frac{\frac{\delta}{\Psi(4,6)}}{\frac{\mu}{\Psi(4,6)}+\max\{4,6\}}=\frac{\frac{\delta}{11}}{\frac{\delta}{11}+6}=\frac{\delta}{\delta+66}.$$

That is $\frac{\mu+\delta}{\mu+\delta+6}\geq\frac{\mu}{\mu+28}\cdot\frac{\delta}{\delta+66}$. Then it satisfied, for all $\mu,\delta>0$. Hence,

$$A_\Psi(\zeta,c,(\mu+\delta))\geq A_\Psi\left(\zeta,\omega,\frac{\mu}{\Psi(\zeta,\omega)}\right)*A_\psi\left(\omega,c,\frac{\delta}{\Psi(\omega,c)}\right).$$

Now,

$$B_\Psi(2,6,\mu+\delta)=\frac{\max\{2,6\}}{\mu+\delta+\max\{2,6\}}=\frac{6}{\mu+\delta+6}$$

and

$$B_\Psi\left(2,4,\frac{\mu}{\Psi(2,4)}\right)=\frac{\max\{2,4\}}{\frac{\mu}{\Psi(2,4)}+\max\{2,4\}}=\frac{4}{\frac{\mu}{7}+4}=\frac{28}{\mu+28}.$$

In addition,

$$B_\Psi\left(4,6,\frac{\delta}{\Psi(4,6)}\right)=\frac{\max\{4,6\}}{\frac{\delta}{\Psi(4,6)}+\max\{4,6\}}=\frac{6}{\frac{\delta}{11}+6}=\frac{66}{\delta+66}.$$

That is, for all $\mu,\delta>0$,

$$\frac{6}{\mu+\delta+6}\leq\max\left\{\frac{28}{\mu+28},\frac{66}{\delta+66}\right\}.$$

Hence,

$$B(\zeta,c,(\mu+\delta))\leq B_\Psi\left(\zeta,\omega,\frac{\mu}{\Psi(\zeta,\omega)}\right)\odot B_\Psi\left(\omega,c,\frac{\delta}{\Psi(\omega,c)}\right).$$

Now,

$$C_\Psi(2,6,\mu+\delta)=\frac{\max\{2,6\}}{\mu+\delta}=\frac{6}{\mu+\delta}.$$

On the other hand,

$$C_\Psi\left(2,4,\frac{\mu}{\Psi(2,4)}\right)=\frac{\max\{2,4\}}{\mu/7}=\frac{28}{\mu}$$

and

$$C_\Psi\left(4,6,\frac{\delta}{\Psi(4,6)}\right)=\frac{\max\{4,6\}}{\frac{\delta}{11}}=\frac{66}{\delta}.$$

That is, for all $\mu,\delta>0$,

$$\frac{6}{\mu+\delta}\leq\max\left\{\frac{28}{\mu},\frac{66}{\delta}\right\}.$$

Thus, we have

$$C_\Psi(\zeta,c,(\mu+\delta))\leq C_\Psi\left(\zeta,\omega,\frac{\mu}{\Psi(\zeta,\omega)}\right)\odot C_\Psi\left(\omega,c,\frac{\mu}{\Psi(\omega,c)}\right).$$

Similar procedures can be used to check all other cases.
Hence, $(L,A_b,B_b,C_b,*,\odot)$ is $NCMLS$.

Example 3.3. Let $L=\{2,4,6\}$ and $\Psi:L\times L\to[1,\infty)$ be defined by $\Psi(\zeta,\omega)=\zeta+\omega+1$. Define $A_\Psi,B_\Psi,C_\Psi:L\times L\times(0,\infty)\to[0,1]$ as

$$A_\Psi(\zeta,\omega,\mu)=\frac{\mu+\min\{\zeta,\omega\}}{\mu+\max\{\zeta,\omega\}},\quad B_\Psi(\zeta,\omega,\mu)=1-\frac{\mu+\min\{\zeta,\omega\}}{\mu+\max\{\zeta,\omega\}}$$

and

$$C_\Psi(\zeta,\omega,\mu)=\frac{\max\{\zeta,\omega\}-\min\{\zeta,\omega\}}{\mu+\min\{\zeta,\omega\}}.$$

Then $(L,A_b,B_b,C_b,*,\odot)$ is $NCMLS$ with $CTN\zeta*\omega=\min\{\zeta,\omega\}$ and $CTCN\zeta\odot\omega=\max\{\zeta,\omega\}$.

Proposition 3.4. *Let $L = [0,1]$ and $\Psi : L \times L \to [1,\infty)$ be defined by $\Psi(\zeta,\omega) = 2(\zeta+\omega+1)$. Consider $A_\Psi, B_\Psi, C_\Psi : L \times L \times (0,\infty) \to [0,1]$ as*

$$A_\Psi(\zeta,\omega,\mu) = e^{-\frac{\max\{\zeta,\omega\}}{\mu^n}}, \quad B_\Psi(\zeta,\omega,\mu) = 1 - e^{-\frac{\max\{\zeta,\omega\}}{\mu^n}}$$

and

$$C_\Psi(\zeta,\omega,\mu) = e^{\frac{\max\{\zeta,\omega\}}{\mu^n}} - 1$$

for all $\zeta,\omega \in L$ and $\mu > 0$. Then $(L, A_b, B_b, C_b, *, \odot)$ is $NCMLS$ with $CTN \zeta * \omega = \zeta\omega$ and $CTCN \zeta \odot \omega = \max\{\zeta,\omega\}$.

Example 3.5. Let $L = (0,\infty)$, define $A_\Psi, B_\Psi, C_\Psi : L \times L \times (0,\infty) \to [0,1]$ by

$$A_\Psi(\zeta,\omega,\mu) = \frac{\mu}{\mu + \max\{\zeta,\omega\}}; \quad B_\Psi(\zeta,\omega,\mu) = \frac{\max\{\zeta,\omega\}}{\mu + \max\{\zeta,\omega\}}$$

and

$$C_\Psi(\zeta,\omega,\mu) = \frac{\max\{\zeta,\omega\}}{\mu}$$

for all $\zeta,\omega \in L$ and $\mu > 0$. Define CTN $*$ by $\zeta * \omega = \zeta\omega$ and $CTCN$ $\odot$ by $\zeta \odot \omega = \max\{\zeta,\omega\}$, define Ψ by

$$\Psi(\zeta,\omega) = \begin{cases} 1 & \text{if} \quad \zeta = \omega, \\ \frac{1+\max\{\zeta,\omega\}}{\min\{\zeta,\omega\}} & \text{if} \quad \zeta \neq \omega. \end{cases}$$

Then $(L, A_b, B_b, C_b, *, \odot)$ is a $NCMLS$. In fact,

$$\max\{\zeta,c\} \le \Psi(\zeta,\omega)\max\{\zeta,\omega\} + \Psi(\omega,c)\max\{\omega,c\}.$$

This implies,

$$\mu\delta\max\{\zeta,c\} \le \Psi(\zeta,\omega)(\mu\delta+\delta^2)\max\{\zeta,\omega\} + \Psi(\omega,c)(\mu\delta+\delta^2)\max\{\omega,c\}.$$

Then,

$$\mu\delta\max\{\zeta,c\} \le \Psi(\zeta,\omega)(\mu+\delta)\delta\max\{\zeta,\omega\} + \Psi(\omega,c)(\mu+\delta)\mu\max\{\omega,c\}.$$

Therefore, we have

$$\begin{aligned} \mu\delta(\mu+\delta) + \mu\delta\max\{\zeta,c\} \le \mu\delta(\mu+\delta) &+ \Psi(\zeta,\omega)(\mu+\delta)\delta\max\{\zeta,\omega\} \\ &+ \Psi(\omega,c)(\mu+\delta)\mu\max\{\omega,c\}. \end{aligned}$$

This implies

$$\begin{aligned} \mu\delta[(\mu+\delta) + \max\{\zeta,c\}] \le (\mu+\delta)[&\mu\delta + \Psi(\zeta,\omega)\delta\max\{\zeta,\omega\} \\ &+ \Psi(\omega,c)\mu\max\{\omega,c\}] \\ &+ \Psi(\zeta,\omega)\Psi(\omega,c)\max\{\omega,c\}\max\{\omega,c\}]. \end{aligned}$$

Then,

$$\mu\delta[(\mu+\delta)+\max\{\zeta,c\}] \leq (\mu+\delta)[\mu+\Psi(\zeta,\omega)\max\{\zeta,\omega\}][\delta + \Psi(\omega,c)\max\{\omega,c\}].$$

This implies

$$\frac{(\mu+\delta)}{[(\mu+\delta)+\max\{\zeta,c\}]} \geq \frac{\mu\delta}{[\mu+\Psi(\zeta,\omega)\max\{\zeta,\omega\}][\delta+\Psi(\omega,c)\max\{\omega,c\}]}.$$

Hence, we have

$$\frac{(\mu+\delta)}{[(\mu+\delta)+\max\{\zeta,c\}]} \geq \frac{\mu}{[\mu+\Psi(\zeta,\omega)\max\{\zeta,\omega\}]} \cdot \frac{\delta}{[\delta+\Psi(\omega,c)\max\{\omega,c\}]}.$$

Therefore,

$$\frac{(\mu+\delta)}{[(\mu+\delta)+\max\{\zeta,c\}]} \geq \frac{\frac{\mu}{\Psi(\zeta,\omega)}}{\left[\frac{\mu}{\Psi(\zeta,\omega)}+\max\{\zeta,\omega\}\right]} \cdot \frac{\frac{\delta}{\Psi(\omega,c)}}{\left[\frac{\delta}{\Psi(\omega,c)}+\max\{\omega,c\}\right]}.$$

Hence,

$$A_\Psi(\zeta,c,(\mu+\delta)) \geq A_\Psi\left(\zeta,\omega,\frac{\mu}{\Psi(\zeta,\omega)}\right) * A_\Psi\left(\omega,c,\frac{\delta}{\Psi(\omega,c)}\right),$$

this means that (NL5) is satisfied. Since

$$\max\{\zeta,c\} = \max\{\zeta,c\}\max\{1,1\},$$
$$\max\{\zeta,c\} = \max\{\zeta,c\}\max\left\{\frac{\max\{\zeta,\omega\}}{\max\{\zeta,\omega\}},\frac{\max\{\omega,c\}}{\max\{\omega,c\}}\right\},$$
$$\max\{\zeta,c\} \leq [(\mu+\delta)+\max\{\zeta,c\}]\max\left\{\frac{\max\{\zeta,\omega\}}{\max\{\zeta,\omega\}},\frac{\max\{\omega,c\}}{\max\{\omega,c\}}\right\},$$

therefore,

$$\max\{\zeta,c\} \leq [(\mu+\delta)+\max\{\zeta,c\}] \times \max\left\{\frac{\Psi(\zeta,\omega)\max\{\zeta,\omega\}}{\Psi(\zeta,\omega)\max\{\zeta,\omega\}},\frac{\Psi(\omega,c)\max\{\omega,c\}}{\Psi(\omega,c)\max\{\omega,c\}}\right\}.$$

Then,

$$\frac{\max\{\zeta,c\}}{(\mu+\delta)+\max\{\zeta,c\}} \leq \max\left\{\frac{\Psi(\zeta,\omega)\max\{\zeta,\omega\}}{\mu+\Psi(\zeta,\omega)\max\{\zeta,\omega\}},\frac{\Psi(\omega,c)\max\{\omega,c\}}{\delta+\Psi(\omega,c)\max\{\omega,c\}}\right\}.$$

This implies

$$\frac{\max\{\zeta,c\}}{(\mu+\delta)+\max\{\zeta,c\}} \leq \max\left\{\frac{\max\{\zeta,\omega\}}{\frac{\mu}{\Psi(\zeta,\omega)}+\max\{a,\omega\}},\frac{\max\{\omega,c\}}{\frac{\delta}{\Psi(\omega,c)}+\max\{\omega,c\}}\right\}.$$

Hence,

$$B_\Psi(\zeta, c, (\mu+\delta)) \leq B_\Psi\left(\zeta, \omega, \frac{\mu}{\Psi(\zeta,\omega)}\right) \odot B_\Psi\left(\omega, c, \frac{\delta}{\Psi(\omega,c)}\right),$$

this means that (NL10) is satisfied. Since

$$\begin{aligned}
\max\{\zeta, c\} &\leq \max\{\max\{\zeta,\omega\}, \max\{\omega, c\}\},\\
\frac{\max\{\zeta, c\}}{\mu+\delta} &\leq \max\left\{\frac{\max\{\zeta,\omega\}}{\mu+\delta}, \frac{\max\{\omega, c\}}{\mu+\delta}\right\}\\
&\leq \max\left\{\frac{\max\{\zeta,\omega\}}{\mu}, \frac{\max\{\omega, c\}}{\delta}\right\}\\
&\leq \max\left\{\frac{\max\{\zeta,\omega\}}{\frac{\mu}{\Psi(\zeta,\omega)}}, \frac{\max\{\omega, c\}}{\frac{\delta}{\Psi(\omega,c)}}\right\},
\end{aligned}$$

we have

$$C_\Psi(\zeta, c, (\mu+\delta)) \leq C_\Psi\left(\zeta, \omega, \frac{\mu}{\Psi(\zeta,\omega)}\right) \odot C_\Psi\left(\omega, c, \frac{\delta}{\Psi(\omega,c)}\right),$$

this means that (NL15) is satisfied.

Definition 3.6. Let $(L, A_b, B_b, C_b, *, \odot)$ be an $NCMLS$. Let $\{\zeta_s\}$ be a sequence in L

(i) $\{\zeta_\sigma\}$ is said to be G-Cauchy sequence (G-CS) if, for all $\mu > 0$,

$$\lim_{s\to\infty} A_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu), \lim_{\sigma\to\infty} B_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu)$$

and

$$\lim_{\sigma\to\infty} C_\Psi(\zeta, \zeta_{\sigma+\beta}, \mu)$$

exists and is finite.

(ii) $\{\zeta_\sigma\}$ is said to be G-Converging (G-C) to ζ in L if, for all $\mu > 0$,

$$\lim_{\sigma\to\infty} A_\Psi(\zeta_\sigma, \zeta, \mu) = A_\Psi(\zeta, \zeta, \mu),$$

$$\lim_{\sigma\to\infty} B_\Psi(\zeta_\sigma, \zeta, \mu) = B_\Psi(\zeta, \zeta, \mu)$$

and

$$\lim_{\sigma\to\infty} C_\Psi(\zeta_\sigma, \zeta, \mu) = C_\Psi(\zeta, \zeta, \mu).$$

(iii) A $NCMLS$ is said to be complete if, each G-CS converges, that is,

$$\lim_{\sigma\to\infty} A_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu) = \lim_{\sigma\to\infty} A_\Psi(\zeta_\sigma, \zeta, \mu) = A_\Psi(\zeta, \zeta, \mu),$$

$$\lim_{\sigma\to\infty} B_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu) = \lim_{\sigma\to\infty} B_\Psi(\zeta, \zeta, \mu) = B_\Psi(\zeta, \zeta, \mu)$$

and

$$\lim_{\sigma\to\infty} C_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu) = \lim_{\sigma\to\infty} C_\Psi(\zeta_\sigma, \zeta, \mu) = C_\Psi(\zeta, \zeta, \mu).$$

Theorem 3.7. *Suppose $(L, A_b, B_b, C_b, *, \odot)$ is a complete NCMLS in the company of $\Psi : L \times L \to [1, \infty)$. Assume that*

$$\lim_{\mu\to\infty} A_\Psi(\zeta, \omega, \mu) = 1, \lim_{\mu\to\infty} B_\Psi(\zeta, \omega, \mu) = 0 \quad \text{and} \quad \lim_{\mu\to\infty} C_\Psi(\zeta, \omega, \mu) = 0 \tag{3.1}$$

for all $\zeta, \omega \in L$ and $\mu > 0$. Let $I : L \to L$ be such that

$$\begin{aligned} &A_\Psi(I\zeta, I\omega, \vartheta\mu) \geq A_\Psi(\zeta, \omega, \mu); B_\Psi(I\zeta, I\omega, \vartheta\mu) \leq B_\Psi(\zeta, \omega, \mu) \quad \text{and} \\ &C_\Psi(I\zeta, I\omega, \vartheta\mu) \leq C_\Psi(\zeta, \omega, \mu) \end{aligned} \tag{3.2}$$

for all $\zeta, \omega \in L$ and $\mu > 0$, $0 < \vartheta < 1$. In addition, $\omega \in L$,

$$\lim_{\sigma\to\infty} \Psi(\zeta_\sigma, \omega) \text{ and } \lim_{\sigma\to\infty} \Psi(\omega, \zeta_\sigma),$$

$$\lim_{\sigma,m\to\infty} \Psi(\zeta_\sigma, \zeta_m) \text{ and } \lim_{\sigma,m\to\infty} \Psi(\zeta_m, \zeta_\sigma) \tag{3.3}$$

exist and are finite, where $\zeta_\sigma = I^n\zeta = I\zeta_{\sigma-1}$, and $\zeta_0 \in L$ be arbitrary. Then, I has a unique fixed point.

Proof. Take an arbitrary point ζ_0 be in L and set up a sequence $\{\zeta_\sigma\}$ by $\zeta_\sigma = I^n\zeta_0 = I\zeta_{\sigma-1}$, $n \in \mathbb{N}$. Using (3.2) for $\mu > 0$, we examine

$$\begin{aligned} A_\Psi(\zeta_\sigma, \zeta_{\sigma+1}, \vartheta\mu) &= A_\Psi(I\zeta_{\sigma-1}, I\zeta_\sigma, \vartheta\mu) \\ &\geq A_\Psi(\zeta_{\sigma-1}, \zeta_\sigma, \vartheta\mu) \geq A_\Psi\left(\zeta_{\sigma-2}, \zeta_{\sigma-1}, \frac{\mu}{\vartheta}\right) \\ &\geq A_\Psi\left(\zeta_{\sigma-3}, \zeta_{\sigma-2}, \frac{\mu}{\vartheta^2}\right) \geq \cdots \geq A_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{\vartheta^{n-1}}\right) \end{aligned}$$

and

$$\begin{aligned} B_\Psi(\zeta_\sigma, \zeta_{\sigma+1}, \vartheta\mu) &= B_\Psi(I\zeta_{\sigma-1}, I\zeta_\sigma, \vartheta\mu) \\ &\leq B_\Psi(\zeta_{\sigma-1}, \zeta_\sigma, \vartheta\mu) \leq B_\Psi\left(\zeta_{\sigma-2}, \zeta_{\sigma-1}, \frac{\mu}{\vartheta}\right) \\ &\leq B_\Psi\left(\zeta_{\sigma-3}, \zeta_{\sigma-2}, \frac{\mu}{\vartheta^2}\right) \leq \cdots \leq B_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{\vartheta^{n-1}}\right). \end{aligned}$$

In addition,

$$\begin{aligned} C_\Psi(\zeta_\sigma, \zeta_{\sigma+1}, \vartheta\mu) &= C_\Psi(I\zeta_{\sigma-1}, I\zeta_\sigma, \vartheta\mu) \\ &\leq C_\Psi(\zeta_{\sigma-1}, \zeta_\sigma, \vartheta\mu) \leq C_\Psi\left(\zeta_{\sigma-2}, \zeta_{\sigma-1}, \frac{\mu}{\vartheta}\right) \\ &\leq C_\Psi\left(\zeta_{\sigma-3}, \zeta_{\sigma-2}, \frac{\mu}{\vartheta^2}\right) \leq \cdots \leq C_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{\vartheta^{n-1}}\right). \end{aligned}$$

Then, we obtain

$$A_\Psi(\zeta_\sigma, \zeta_{\sigma+1}, \vartheta\mu) \geq A_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{\vartheta^{n-1}}\right); \quad B_\Psi(\zeta_\sigma, \zeta_{\sigma+1}, \vartheta\mu) \leq B_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{\vartheta^{n-1}}\right),$$

$$C_\Psi(\zeta_\sigma, \zeta_{\sigma+1}, \vartheta\mu) \leq C_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{\vartheta^{n-1}}\right). \tag{3.4}$$

For any $\beta \in \mathbb{N}$, using (NL5), (NL10) and (NL15), we deduce

$$\begin{aligned}
&A_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu) \\
&\geq A_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right) * A_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+\beta}, \frac{\mu}{2(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta}))}\right) \\
&\geq A_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right) \\
&\quad * A_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right) \\
&\quad * A_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\right)}\right) \\
&\geq A_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right) \\
&\quad * A_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right) \\
&\quad * A_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+3}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+3})\right)}\right) \\
&\quad * A_\Psi\left(\zeta_{\sigma+3}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+3}, \zeta_{\sigma+\beta})\right)}\right) \\
&\geq A_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right) \\
&\quad * A_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right) \\
&\quad * A_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+3}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+3})\right)}\right) \\
&\quad * A_\Psi\left(\zeta_{\sigma+3}, \zeta_{\sigma+4}, \frac{\mu}{(2)^4\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+3}, \zeta_{\sigma+4})\right)}\right) * \cdots * \\
&\quad * A_\Psi\left(\zeta_{\sigma+\beta-2}, \zeta_{\sigma+\beta-1}, \frac{\mu}{(2)^{\beta-1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-2}, \zeta_{\sigma+\beta-1})\right)}\right) \\
&\quad * A_\Psi\left(\zeta_{\sigma+\beta-1}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^{\beta-1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-1}, \zeta_{\sigma+\beta})\right)}\right)
\end{aligned}$$

and

$$
\begin{aligned}
&B_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu)\\
&\quad \le B_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right) \odot B_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+\beta}, \frac{\mu}{2(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta}))}\right)\\
&\quad \le B_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right)\\
&\qquad \odot B_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right)\\
&\qquad \odot B_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\right)}\right)\\
&\quad \le B_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right)\\
&\qquad \odot B_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right)\\
&\qquad \odot B_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+3}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+3})\right)}\right)\\
&\qquad \odot B_\Psi\left(\zeta_{\sigma+3}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+3}, \zeta_{\sigma+\beta})\right)}\right)\\
&\quad \le B_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right)\\
&\qquad \odot B_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right)\\
&\qquad \odot B_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+3}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+3})\right)}\right)\\
&\qquad \odot B_\Psi\left(\zeta_{\sigma+3}, \zeta_{\sigma+4}, \frac{\mu}{(2)^4\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+3}, \zeta_{\sigma+4})\right)}\right) \odot \cdots \odot\\
&\qquad \odot B_\Psi\left(\zeta_{\sigma+\beta-2}, \zeta_{\sigma+\beta-1}, \frac{\mu}{(2)^{\beta-1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-2}, \zeta_{\sigma+\beta-1})\right)}\right)\\
&\qquad \odot B_\Psi\left(\zeta_{\sigma+\beta-1}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^{\beta-1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-1}, \zeta_{\sigma+\beta})\right)}\right).
\end{aligned}
$$

In addition,

$$
\begin{aligned}
&C_\Psi(\zeta_\sigma, \zeta_{\sigma+1}, \mu)\\
&\quad \le C_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right) \odot C_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+\beta}, \frac{\mu}{2(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta}))}\right)
\end{aligned}
$$

$$\begin{aligned}
&\leq C_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right)\\
&\quad\odot C_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right)\\
&\quad\odot C_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\right)}\right)\\
&\leq C_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right)\\
&\quad\odot C_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right)\\
&\quad\odot C_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+3}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+3})\right)}\right)\\
&\quad\odot C_\Psi\left(\zeta_{\sigma+3}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+3}, \zeta_{\sigma+\beta})\right)}\right)\\
&\leq C_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right)\\
&\quad\odot C_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right)\\
&\quad\odot C_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+3}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+3})\right)}\right)\\
&\quad\odot C_\Psi\left(\zeta_{\sigma+3}, \zeta_{\sigma+4}, \frac{\mu}{(2)^4\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\zeta_{\sigma+4})\right)}\right)\odot\cdots\odot\\
&\quad\odot C_\Psi\left(\zeta_{\sigma+\beta-2}, \zeta_{\sigma+\beta-1}, \frac{\mu}{(2)^{\beta-1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-2}, \zeta_{\sigma+\beta-1})\right)}\right)\\
&\quad\odot C_\Psi\left(\zeta_{\sigma+\beta-1}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^{\beta-1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-1}, \zeta_{\sigma+\beta})\right)}\right).
\end{aligned}$$

Using (3.4) in the above inequalities, we deduce,

$$\begin{aligned}
&A_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu)\\
&\quad\geq A_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{2(\vartheta)^{n-1}(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right)\\
&\qquad * A_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{(2)^2(\vartheta)^n(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2}))}\right)\\
&\qquad * A_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{(2)^3(\vartheta)^{n+1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+3})\right)}\right)
\end{aligned}$$

$$* A_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^4(\vartheta)^{n+2}\left(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+3},\zeta_{\sigma+4})\right)}\right)*\cdots*$$
$$* A_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^{\beta-1}(\vartheta)^{n+\beta-2}\left(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-2},\zeta_{\sigma+\beta-1})\right)}\right)$$
$$* A_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^{\beta-1}(\vartheta)^{n+\beta-1}\left(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-1},\zeta_{\sigma+\beta})\right)}\right)$$

and

$$\begin{aligned}
&B_\Psi(\zeta_\sigma,\zeta_{\sigma+\beta},\mu)\\
&\le B_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{2(\vartheta)^{n-1}(\Psi(\zeta_\sigma,\zeta_{\sigma+1}))}\right)\\
&\odot B_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^2(\vartheta)^{n}(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1},\zeta_{\sigma+2}))}\right)\\
&\odot B_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^3(\vartheta)^{n+1}\left(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2},\zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2},\zeta_{\sigma+3})\right)}\right)\\
&\odot B_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^4(\vartheta)^{n+2}\left(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\cdots\zeta_{\sigma+4})\right)}\right)\odot\cdots\odot\\
&\odot B_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^{\beta-1}(\vartheta)^{n+\beta-2}\left(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-2},\zeta_{\sigma+\beta-1})\right)}\right)\\
&\odot B_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^{\beta-1}(\vartheta)^{n+\beta-1}\left(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-1},\zeta_{\sigma+\beta})\right)}\right).
\end{aligned}$$

In addition,

$$\begin{aligned}
&C_\Psi(\zeta_\sigma,\zeta_{\sigma+\beta},\mu)\\
&\le C_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{2(\vartheta)^{n-1}(\Psi\zeta_\sigma,\zeta_{\sigma+1}))}\right)\\
&\odot C_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^2(\vartheta)^{n}(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1},\zeta_{\sigma+2}))}\right)\\
&\odot C_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^3(\vartheta)^{n+1}\left(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2},\zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2},\zeta_{\sigma+3})\right)}\right)\\
&\odot C_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^4(\vartheta)^{n+2}\left(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\cdots\zeta_{\sigma+4})\right)}\right)\odot\cdots\odot\\
&\odot C_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^{\beta-1}(\vartheta)^{n+\beta-2}\left(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-2},\zeta_{\sigma+\beta-1})\right)}\right)\\
&\odot C_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^{\beta-1}(\vartheta)^{n+\beta-1}\left(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-1},\zeta_{\sigma+\beta})\right)}\right).
\end{aligned}$$

Using (1), for $\sigma \to \infty$, we deduce,

$$\lim_{\sigma\to\infty} A_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu) = 1 * 1 * \cdots * 1 = 1,$$

$$\lim_{\zeta\to\infty} B_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu) = 0 \odot 0 \odot \cdots \odot 0 = 0$$

and

$$\lim_{\sigma\to\infty} C_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu) = 0 \odot 0 \odot \cdots \odot 0 = 0.$$

That is $\{\zeta_\sigma\}$ is a G-CS. Since, $(L, A_b, B_b, C_b, *, \odot)$ is a G-complete $NCMLS$, there exists, $\lim_{\sigma\to\infty} \zeta_\sigma = \zeta$.

Now observe that ζ is a FP of I, using (NL5), (NL10),(NL15) and (1), we deduce

$$\begin{aligned}
A_\Psi(\zeta, I\zeta, \mu) &\geq A_\Psi\left(\zeta, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta, \zeta_{\sigma+1})}) \right) * A_\Psi\left(\zeta_{\sigma+1}, I\zeta, \frac{\mu}{2(\Psi(\zeta_{\sigma+1}, I\zeta))}\right),\\
A_\Psi(\zeta, I\zeta, \mu) &\geq A_\Psi\left(\zeta, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta, \zeta_{\sigma+1})}) \right) * A_\Psi\left(I\zeta_\sigma, I\zeta, \frac{\mu}{2(\Psi(I\zeta_\sigma, I\zeta))}\right),\\
A_\Psi(\zeta, I\zeta, \mu) &\geq A_\Psi\left(\zeta, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta, \zeta_{\sigma+1})}) \right) * A_\Psi\left(\zeta_\sigma, \zeta, \frac{\mu}{2\vartheta(\Psi(\zeta_\sigma, \zeta))}\right)\\
&\to 1 * 1 = 1 \quad \text{as} \quad \sigma \to \infty.\\
B_\Psi(\zeta, I\zeta, \mu) &\leq B_\Psi\left(\zeta, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta, \zeta_{\sigma+1})}) \right) \odot B_\Psi\left(\zeta_{\sigma+1}, I\zeta, \frac{\mu}{2(\Psi(\zeta_{\sigma+1}, I\zeta))}\right),\\
B_\Psi(\zeta, I\zeta, \mu) &\leq B_\Psi\left(\zeta, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta, \zeta_{\sigma+1})}) \right) \odot B_\Psi\left(I\zeta_\sigma, I\zeta, \frac{\mu}{2(\Psi(I\zeta_\sigma, I\zeta))}\right),\\
B_\Psi(\zeta, I\zeta, \mu) &\leq B_\Psi\left(\zeta, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta, \zeta_{\sigma+1})}) \right) \odot B_\Psi\left(\zeta_\sigma, \zeta, \frac{\mu}{2\vartheta(\Psi(\zeta_\sigma, \zeta))}\right)\\
&\to 0 \odot 0 = 0 \quad \text{as} \quad \sigma \to \infty.\\
C_\Psi(\zeta, I\zeta, \mu) &\leq C_\Psi\left(\zeta, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta, \zeta_{\sigma+1})}) \right) \odot C_\Psi\left(\zeta_{\sigma+1}, I\zeta, \frac{\mu}{2(\Psi(\zeta_{\sigma+1}, I\zeta))}\right),\\
C_\Psi(\zeta, I\zeta, \mu) &\leq C_\Psi\left(\zeta, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta, \zeta_{\sigma+1})}) \right) \odot C_\Psi\left(I\zeta_\sigma, I\zeta, \frac{\mu}{2(\Psi(I\zeta_\sigma, I\zeta))}\right),\\
C_\Psi(\zeta, I\zeta, \mu) &\leq C_\Psi\left(\zeta, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta, \zeta_{\sigma+1})}) \right) \odot C_\Psi\left(\zeta_\sigma, \zeta, \frac{\mu}{2\vartheta(\Psi(\zeta_\sigma, \zeta))}\right)\\
&\to 0 \odot 0 = 0 \quad \text{as} \quad \sigma \to \infty.
\end{aligned}$$

Therefore, $I\zeta = \zeta$, that is, ζ is a FP. To check the uniqueness, suppose that $I\alpha = \alpha$ for some $\alpha \in L$. Then,

$$\begin{aligned}1 &\geq A_\Psi(\alpha,\zeta,\mu) = A_\Psi(I\alpha, I\zeta, \mu) \geq A_\Psi\left(\alpha,\zeta,\frac{\mu}{\vartheta}\right) = A_\Psi\left(I\alpha, I\zeta, \frac{\mu}{\vartheta}\right)\\ &\geq A_\Psi\left(\alpha,\zeta,\frac{\mu}{\vartheta^2}\right) \geq \cdots \geq A_\Psi\left(\alpha,\zeta,\frac{\mu}{\vartheta^n}\right) \to 1 \text{ as } n\to\infty,\end{aligned}$$

and

$$\begin{aligned}0 &\leq B_\Psi(\alpha,\zeta,\mu) = B_\Psi(I\alpha, I\zeta, \mu) \leq B_\Psi\left(\alpha,\zeta,\frac{\mu}{\vartheta}\right) = B_\Psi\left(I\alpha, I\zeta, \frac{\mu}{\vartheta}\right)\\ &\leq B_\Psi\left(\alpha,\zeta,\frac{\mu}{\vartheta^2}\right) \leq \cdots \leq B_\Psi\left(\alpha,\zeta,\frac{\mu}{\vartheta^n}\right) \to 0 \text{ as } n\to\infty.\end{aligned}$$

In addition,

$$\begin{aligned}0 &\leq C_\Psi(\alpha,\zeta,\mu) = C_\Psi(I\alpha, I\zeta, \mu) \leq C_\Psi\left(\alpha,\zeta,\frac{\mu}{\vartheta}\right) = C_\Psi\left(I\alpha, I\zeta, \frac{\mu}{\vartheta}\right)\\ &\leq C_\Psi\left(\alpha,\zeta,\frac{\mu}{\vartheta^2}\right) \leq \cdots \leq C_\Psi\left(\alpha,\zeta,\frac{\mu}{\vartheta^n}\right) \to 0 \text{ as } n\to\infty.\end{aligned}$$

By using (NL3), (NL8) and (NL13), we have $\zeta = \alpha$. □

Definition 3.8. Let $(L, A_b, B_b, C_b, *, \odot)$ be a $NCMLS$. A function $I : L \to L$ is neutrosophic controlled (NC) contraction, if for some $0 < \vartheta < 1$,

$$\frac{1}{A_\Psi(I\zeta, I\omega,\mu)} - 1 \leq \vartheta\left[\frac{1}{A_\Psi(\zeta,\omega,\mu)} - 1\right], \tag{3.5}$$

$$B_\Psi(I\zeta, I\omega,\mu) \leq \vartheta B_\Psi(\zeta,\omega,\mu), \tag{3.6}$$

$$C_\Psi(I\zeta, I\omega,\mu) \leq \vartheta C_\Psi(\zeta,\omega,\mu). \tag{3.7}$$

Theorem 3.9. *Let $(L, A_b, B_b, C_b, *, \odot)$ be a G-complete $NCMLS$ with $\Psi : L\times L \to [1,\infty)$ and*

$$\lim_{\mu\to\infty} A_\Psi(\zeta,\omega,\mu) = 1, \lim_{\mu\to\infty} B_\Psi(\zeta,\omega,\mu) = 0 \quad \text{and} \quad \lim_{\mu\to\infty} C_\Psi(\zeta,\omega,\mu) = 0 \tag{3.8}$$

for all $\zeta,\omega \in L$ and $\mu > 0$. Let $I : L \to L$ be a NC contraction. In addition, assume that $\omega \in L$, $\lim_{i\to\infty}\Psi(\zeta_\sigma,\omega)$, $\lim_{\sigma\to\infty}\Psi(\omega,\zeta_\sigma)$, $\lim_{\sigma,m\to\infty}\Psi(\zeta_\sigma,\zeta_m)$ and $\lim_{\sigma,m\to\infty}\Psi(\zeta_m,\zeta_\sigma)$ are finite, where $\zeta_\sigma = I^n\zeta_0 = I\zeta_{\sigma-1}$ for all $\sigma \in \mathbb{N}$, and $\zeta_0 \in L$ be arbitrary. Then, I has a unique fixed point.

Proof. Let ζ_0 be an arbitrary point of L and set up a sequence $\{\zeta_\sigma\}$ by

$$\zeta_\sigma = I^n\zeta_0 = I\zeta_{\sigma-1}, \quad n \in \mathbb{N}.$$

Using (3.5), (3.6) and (3.7) for all $\mu > 0, n > \beta$, we acquire,

$$\begin{aligned}\frac{1}{A_\Psi(\zeta_\sigma, \zeta_{\sigma+1}, \mu)} - 1 &= \frac{1}{A_\Psi(I\zeta_{\sigma-1}, I\zeta_\sigma, \mu)} - 1 \\ &\leq \vartheta \left[\frac{1}{A_\Psi(\zeta_{\sigma-1}, \zeta_\sigma, \mu)} - 1\right] \\ &= \frac{\vartheta}{A_\Psi(\zeta_{\sigma-1}, \zeta_\sigma, \mu)} - \vartheta,\end{aligned}$$

this implies that

$$\begin{aligned}\frac{1}{A_\Psi(\zeta_\sigma, \zeta_{\sigma+1}, \mu)} &\leq \frac{\vartheta}{A_\Psi(\zeta_{\sigma-1}, \zeta_\sigma, \mu)} + (1-\vartheta) \\ &\leq \frac{\vartheta^2}{A_\Psi(\zeta_{\sigma-1}, \zeta_\sigma, \mu)} + \vartheta(1-\vartheta) + (1-\vartheta).\end{aligned}$$

Continuing this way, we acquire

$$\begin{aligned}\frac{1}{A_\Psi(\zeta_\sigma, \zeta_{\sigma+1}, \mu)} &\leq \frac{\vartheta^n}{A_\Psi\zeta_0, \zeta_1, \mu)} + \vartheta^{n-1}(1-\vartheta) + \cdots + \vartheta(1-\vartheta) + (1-\vartheta) \\ &\leq \frac{\vartheta^n}{A_\Psi(\zeta_0, \zeta_1, \mu)} + (1-\vartheta)[(\vartheta^{n-1}) + (\vartheta^{n-2}) + \cdots + 1) \\ &\leq \frac{\vartheta^n}{A_\Psi(\zeta_0, \zeta_1, \mu)} + (1-\vartheta^n),\end{aligned}$$

we obtain

$$\frac{1}{\frac{\vartheta^n}{A_\Psi(\zeta_0,\zeta_1,\mu)} + (1-\vartheta^n)} \leq A_\Psi(\zeta_\sigma, \zeta_{\sigma+1}, \mu) \tag{3.9}$$

and

$$\begin{aligned}B_\Psi(\zeta_\sigma, \zeta_{\sigma+1}, \mu) &= B_\Psi(I\zeta_{\sigma-1}, \zeta_\sigma, \mu) \leq \vartheta B_\Psi(\zeta_{\sigma-1}, \zeta_\sigma, \mu) \\ &= \vartheta B_\Psi(I\zeta_{\sigma-2}, I\zeta_{\sigma-1}, \mu) \\ &\leq \vartheta^2 B_\Psi(\zeta_{\sigma-2}, \zeta_{\sigma-1}, \mu) \leq \cdots \leq \vartheta^n B_\Psi(\zeta_0, \zeta_1, \mu).\end{aligned} \tag{3.10}$$

In addition,

$$\begin{aligned}C_\Psi(\zeta_\sigma, \zeta_{\sigma+1}, \mu) &= C_\Psi(I\zeta_{\sigma-1}, \zeta_\sigma, \mu) \leq \vartheta C_\Psi(\zeta_{\sigma-1}, \zeta_\sigma, \mu) \\ &= \vartheta C_\Psi(I\zeta_{\sigma-2}, I\zeta_{\sigma-1}, \mu) \\ &\leq \vartheta^2 C_\Psi(\zeta_{\sigma-2}, \zeta_{\sigma-1}, \mu) \leq \cdots \leq \vartheta^n C_\Psi(\zeta_0, \zeta_1, \mu).\end{aligned}$$

For any $\beta \in \mathbb{N}$, using (NL5), (NL10) and (NL15), we deduce

$$
\begin{aligned}
&A_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu) \\
&\quad \geq A_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right) * A_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+\beta}, \frac{\mu}{2(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta}))}\right) \\
&\quad \geq A_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right) \\
&\qquad * A_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right) \\
&\qquad * A_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\right)}\right) \\
&\quad \geq A_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right) \\
&\qquad * A_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right) \\
&\qquad * A_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+3}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+3})\right)}\right) \\
&\qquad * A_\Psi\left(\zeta_{\sigma+3}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+3}, \zeta_{\sigma+\beta})\right)}\right) \\
&\quad \geq A_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right) \\
&\qquad * A_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right) \\
&\qquad * A_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+3}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+3})\right)}\right) \\
&\qquad * A_\Psi\left(\zeta_{\sigma+3}, \zeta_{\sigma+4}, \frac{\mu}{(2)^4\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+3}, \zeta_{\sigma+4})\right)}\right) * \cdots * \\
&\qquad * A_\Psi\left(\zeta_{\sigma+\beta-2}, \zeta_{\sigma+\beta-1}, \frac{\mu}{(2)^{\beta-1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-2}, \zeta_{\sigma+\beta-1})\right)}\right) \\
&\qquad * A_\Psi\left(\zeta_{\sigma+\beta-1}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^{\beta-1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-1}, \zeta_{\sigma+\beta})\right)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&B_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu) \\
&\quad \leq B_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right) \odot B_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+\beta}, \frac{\mu}{2(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta}))}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq B_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right)\\
&\quad \odot B_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right)\\
&\quad \odot B_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\right)}\right)\\
&\leq B_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right)\\
&\quad \odot B_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right)\\
&\quad \odot B_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+3}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+3})\right)}\right)\\
&\quad \odot B_\Psi\left(\zeta_{\sigma+3}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+3}, \zeta_{\sigma+\beta})\right)}\right)\\
&\leq B_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right)\\
&\quad \odot B_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right)\\
&\quad \odot B_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+3}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+3})\right)}\right)\\
&\quad \odot B_\Psi\left(\zeta_{\sigma+3}, \zeta_{\sigma+4}, \frac{\mu}{(2)^4\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+3}, \zeta_{\sigma+4})\right)}\right)\odot\cdots\odot\\
&\quad \odot B_\Psi\left(\zeta_{\sigma+\beta-2}, \zeta_{\sigma+\beta-1}, \frac{\mu}{(2)^{\beta-1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-2}, \zeta_{\sigma+\beta-1})\right)}\right)\\
&\quad \odot B_\Psi\left(\zeta_{\sigma+\beta-1}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^{\beta-1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-1}, \zeta_{\sigma+\beta})\right)}\right).
\end{aligned}
$$

In addition,

$$
\begin{aligned}
&C_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu)\\
&\leq C_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right)\odot C_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+\beta}, \frac{\mu}{2(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta}))}\right)\\
&\leq C_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right)\\
&\quad \odot C_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\odot C_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, a_{i+\beta})\Psi(a_{i+2}, a_{i+\beta})\right)}\right)\\
\leq\ & C_\Psi\left(a_i, a_{i+1}, \frac{\mu}{2(\Psi(a_i, a_{i+1}))}\right)\\
&\odot C_\Psi\left(a_{i+1}, a_{i+2}, \frac{\mu}{(2)^2\left(\Psi(a_{i+1}, a_{i+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right)\\
&\odot C_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+3}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+3})\right)}\right)\\
&\odot C_\Psi\left(\zeta_{\sigma+3}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+3}, \zeta_{\sigma+\beta})\right)}\right)\\
\leq\ & C_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right)\\
&\odot C_\Psi\left(\zeta_{\sigma+1}, \zeta_{\sigma+2}, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right)\\
&\odot C_\Psi\left(\zeta_{\sigma+2}, \zeta_{\sigma+3}, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+3})\right)}\right)\\
&\odot C_\Psi\left(\zeta_{\sigma+3}, \zeta_{\sigma+4}, \frac{\mu}{(2)^4\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+3}, \zeta_{\sigma+4})\right)}\right)\odot\cdots\odot\\
&\odot C_\Psi\left(\zeta_{\sigma+\beta-2}, \zeta_{\sigma+\beta-1}, \frac{\mu}{(2)^{\beta-1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-2}, \zeta_{\sigma+\beta-1})\right)}\right)\\
&\odot C_\Psi\left(\zeta_{\sigma+\beta-1}, \zeta_{\sigma+\beta}, \frac{\mu}{(2)^{\beta-1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-1}, \zeta_{\sigma+\beta})\right)}\right),
\end{aligned}
$$

$$
\begin{aligned}
&A_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu)\\
&\geq \frac{1}{\frac{\vartheta^n}{A_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{2(\Psi(\zeta_\sigma,\zeta_{\sigma+1}))}\right)}+(1-\vartheta^n)}\\
&\quad * \frac{1}{\frac{\vartheta^{n+1}}{A_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^2(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1},\zeta_{\sigma+2}))}\right)}+(1-\vartheta^{n+1})}\\
&\quad * \frac{1}{\frac{\vartheta^{n+2}}{A_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^3(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2},\zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+2},\zeta_{\sigma+3}))}\right)}+(1-\vartheta^{n+2})} *\cdots*\\
&\quad * \frac{1}{\frac{\vartheta^{n+\beta-2}}{A_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^{\beta-1}(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta},\zeta_{\sigma+\beta-1}))}\right)}+(1-\vartheta^{n+\beta-2})}
\end{aligned}
$$

$$* \frac{1}{\frac{\vartheta^{n+\beta-1}}{A_\Psi\left(\zeta_0,\zeta_1,\frac{\mu}{(2)^{\beta-1}(\Psi(\zeta_{\sigma+1},\zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-1},\zeta_{\sigma+\beta}))}\right)} + (1-\vartheta^{n+\beta-1})}$$

and

$$\begin{aligned}
&B_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu) \\
&\quad \le \vartheta^n B_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right) \\
&\quad\quad \odot \vartheta^{n+1} B_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right) \\
&\quad\quad \odot \vartheta^{n+2} B_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+3})\right)}\right) \odot \cdots \odot \\
&\quad\quad \odot \vartheta^{n+\beta-2} B_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{(2)^{\beta-1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-2}, \zeta_{\sigma+\beta-1})\right)}\right) \\
&\quad\quad \odot \vartheta^{n+\beta-1} B_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{(2)^{\beta-1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-1}, \zeta_{\sigma+\beta})\right)}\right).
\end{aligned}$$

In addition,

$$\begin{aligned}
&C_\Psi(a_i, a_{i+\beta}, \mu) \\
&\quad \le \vartheta^n C_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{2(\Psi(\zeta_\sigma, \zeta_{\sigma+1}))}\right) \\
&\quad\quad \odot \vartheta^{n+1} C_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{(2)^2\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+2})\right)}\right) \\
&\quad\quad \odot \vartheta^{n+2} C_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{(2)^3\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+2}, \zeta_{\sigma+3})\right)}\right) \odot \cdots \odot \\
&\quad\quad \odot \vartheta^{n+\beta-2} C_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{(2)^{\beta-1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-2}, \zeta_{\sigma+\beta-1})\right)}\right) \\
&\quad\quad \odot \vartheta^{n+\beta-1} C_\Psi\left(\zeta_0, \zeta_1, \frac{\mu}{(2)^{\beta-1}\left(\Psi(\zeta_{\sigma+1}, \zeta_{\sigma+\beta})\cdots\Psi(\zeta_{\sigma+\beta-1}, \zeta_{\sigma+\beta})\right)}\right).
\end{aligned}$$

Therefore,

$$\lim_{i\to\infty} A_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}), \mu) = 1 * 1 * \cdots * 1 = 1,$$
$$\lim_{\sigma\to\infty} B_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu) = 0 \odot 0 \odot \cdots \odot 0 = 0$$

and

$$\lim_{\sigma\to\infty} C_\Psi(\zeta_\sigma, \zeta_{\sigma+\beta}, \mu) = 0 \odot 0 \odot \cdots \odot 0 = 0,$$

that is, $\{\zeta_\sigma\}$ is a G-CS. Since $(L, A_b, B_b, C_b, *, \odot)$ is a G-complete in $NCMLS$, there exists, $\lim \zeta_\sigma = \zeta$ as $\sigma \to \infty$.

Now check that ζ is a FP of I using (NL5), (NL10) and (NL15) we deduce

$$\frac{1}{A_\Psi(I\zeta_\sigma, I\zeta, \mu)} - 1 \leq \vartheta \left[\frac{1}{A_\Psi(\zeta_\sigma, \zeta, \mu)}\right] = \frac{\vartheta}{A_\Psi(\zeta_\sigma, \zeta, \mu)} - \vartheta,$$

it implies that

$$\frac{1}{\frac{\vartheta}{A_\Psi(\zeta_\sigma,\zeta,\mu)} + (1-\vartheta)} \leq A_\Psi(I\zeta_\sigma, I\zeta, \mu),$$

$$B_\Psi(I\zeta_\sigma, I\zeta, \mu) \leq \vartheta B_\Psi(\zeta_\sigma, \zeta, \mu)$$

and

$$C_\Psi(I\zeta_\sigma, I\zeta\mu) \leq \vartheta C_\Psi(\zeta_\zeta, \zeta, \mu).$$

Using the above inequality, we obtain

$$\begin{aligned}
A_\Psi(\zeta, I\zeta, \mu) &\geq A_\Psi\left(\zeta, \zeta_{\sigma+1}, \frac{\mu}{2\Psi(\zeta, \zeta_{\sigma+1})}\right) * A_\Psi\left(\zeta_{\sigma+1}, I\zeta, \frac{\mu}{2\Psi(\zeta_{\sigma+1}, I\zeta)}\right) \\
&\geq A_\Psi\left(\zeta, \zeta_{\sigma+1}, \frac{\mu}{2\Psi(\zeta, \zeta_{\sigma+1})}\right) * A_\Psi\left(I\zeta_\sigma, I\zeta, \frac{\mu}{2\Psi(I\zeta_\sigma, I\zeta)}\right) \\
&\geq A_\Psi\left(\zeta, \zeta_{\sigma+1}, \frac{\mu}{(2\Psi(\zeta, \zeta_{\sigma+1}))}\right) * \frac{1}{\frac{\vartheta}{A_\Psi(I\zeta_\sigma, I\zeta, \frac{\mu}{2\Psi(I\zeta_\sigma, I\zeta)})} + (1-\vartheta)} \\
&\to 1 * 1 = 1 \quad \text{as} \quad \sigma \to \infty
\end{aligned}$$

and

$$\begin{aligned}
B_\Psi(\zeta, I\zeta, \mu) &\leq B_\Psi\left(\zeta, \zeta_{\sigma+1}, \frac{\mu}{2\Psi(\zeta, \zeta_{\sigma+1})}\right) \odot B_\Psi\left(\zeta_{\sigma+1}, I\zeta, \frac{\mu}{2\Psi(\zeta_{\sigma+1}, I\zeta)}\right) \\
&\leq B_\Psi\left(\zeta, \zeta_{\sigma+1}, \frac{\mu}{2\Psi(\zeta, \zeta_{\sigma+1})}\right) \odot B_\Psi\left(I\zeta_\sigma, I\zeta, \frac{\mu}{2\Psi(I\zeta_\sigma, I\zeta)}\right) \\
&\leq B_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2\Psi(\zeta, \zeta_{\sigma+1})}\right) \odot \ \vartheta B_\Psi\left(\zeta_\sigma, \zeta, \frac{\mu}{2\Psi(\zeta_\zeta, \zeta)}\right) \\
&\to 0 \odot 0 = 0 \quad \text{as} \quad i \to \infty.
\end{aligned}$$

In addition,

$$\begin{aligned}
C_\Psi(\zeta, I\zeta, \mu) &\leq C_\Psi\left(\zeta, \zeta_{\sigma+1}, \frac{\mu}{2\Psi(\zeta, \zeta_{\sigma+1})}\right) \odot C_\Psi\left(\zeta_{\sigma+1}, I\zeta, \frac{\mu}{2\Psi(\zeta_{\sigma+1}, I\sigma)}\right) \\
&\leq C_\Psi\left(\zeta, \zeta_{\sigma+1}, \frac{\mu}{2\Psi(\zeta, \zeta_{\sigma+1})}\right) \odot C_\Psi\left(I\zeta_\sigma, I\zeta, \frac{\mu}{2\Psi(I\zeta_\sigma, I\zeta)}\right) \\
&\leq C_\Psi\left(\zeta_\sigma, \zeta_{\sigma+1}, \frac{\mu}{2\Psi(\zeta, \zeta_{\sigma+1})}\right) \odot \vartheta C_\Psi\left(\zeta_\sigma, \zeta, \frac{\mu}{2\Psi(\zeta_\sigma, \zeta)}\right) \\
&\to 0 \odot 0 = 0 \quad \text{as} \quad i \to \infty.
\end{aligned}$$

This indicates that $I\zeta = \zeta$, which is a FP. To demonstrate the uniqueness, take $I\alpha = \alpha$ for some $\alpha \in L$. Then,

$$\begin{aligned}\frac{1}{A_\Psi(\zeta,\alpha,\mu)} - 1 = \frac{1}{A_\Psi(I\zeta, I\alpha,\mu)} - 1 &\leq \vartheta\left[\frac{1}{A_\Psi(\zeta,\alpha,\mu)} - 1\right] \\ &< \frac{1}{A_\Psi(\zeta,\alpha,\mu)} - 1,\end{aligned}$$

which is a contraction, and

$$\begin{aligned}B_\Psi(\zeta,\alpha,\mu) = B_\Psi(I\zeta, I\alpha,\mu) \leq \vartheta B_\Psi(\zeta,\alpha,\mu) < B_\Psi(\zeta,\alpha,\mu);\\ C_\Psi(\zeta,\alpha,\mu) = C_\Psi(I\zeta, I\alpha,\mu) \leq \vartheta C_\Psi(\zeta,\alpha,\mu) < C_\Psi(\zeta,\alpha,\mu)\end{aligned}$$

are contractions. Therefore, we must have

$$A_\Psi(\zeta,\alpha,\mu) = 1;\ \ B_\Psi(\zeta,\alpha,\mu) = 0 \ \text{ and } \ C_\Psi(\zeta,\alpha,\mu) = 0.$$

Hence, $\zeta = \alpha$. $\square$

Example 3.10. Let $L = \left[\frac{1}{8}, 1\right]$. Define Ψ by

$$\Psi(\zeta,\omega) = \begin{cases} 1 & \text{if } \ \zeta = \omega, \\ \frac{1+\max\{\zeta,\omega\}}{\min\{\zeta,\omega\}} & \text{if } \ \zeta \neq \omega \neq 0. \end{cases}$$

Moreover, take,

$$A_\Psi(\zeta,\omega,\mu) = e^{-\frac{\max\{\zeta,\omega\}}{\mu}},\ B_\Psi(\zeta,\omega,\mu) = 1 - e^{-\frac{\max\{\zeta,\omega\}}{\mu}}$$

and

$$C_\Psi(\zeta,\omega,\mu) = e^{\frac{\max\{\zeta,\omega\}}{\mu}-1}$$

with $\zeta * \omega = \zeta\omega$ and $\zeta \odot \omega = \max\{\zeta,\omega\}$. Then $(L, A_b, B_b, C_b, *, \odot)$ is a G-complete $NCMLS$.

Observe that $\lim_{\mu\to\infty} A_\Psi(\zeta,\omega,\mu)$, $\lim_{\mu\to\infty} B_\Psi(\zeta,\omega,\mu)$ and $\lim_{\mu\to\infty} C_\Psi(\zeta,\omega,\mu)$ are all finite. Consider $I : L \to L$, where

$$I(\zeta) = \begin{cases} 0 & \text{if } \ \zeta \in \left[\frac{1}{8}, \frac{1}{4}\right] \\ \frac{a}{8} & \text{if } \ \left]\frac{1}{4}, 1\right] \end{cases}.$$

Then we have four cases:

(i) If $\zeta, \omega \in \left[\frac{1}{8}, \frac{1}{4}\right]$, then $I\zeta = I\omega = 0$.
(ii) If $\zeta \in \left[\frac{1}{8}, \frac{1}{4}\right]$ and $\omega \in \left]\frac{1}{4}, 1\right]$, then $I\zeta = 0$ and $I\omega = \frac{\omega}{8}$.
(iii) If $\omega \in \left[\frac{1}{8}, \frac{1}{4}\right]$ and $a \in \left]\frac{1}{4}, 1\right]$, then $I\omega = 0 = 0$ and $I\zeta = \frac{a}{8}$.
(iv) If $\zeta, \omega \in \left]\frac{1}{4}, 1\right]$, then $I\zeta = \frac{a}{8}$ and $I\omega = \frac{\omega}{8}$.

For all cases (i)-(iv),

$$A_{\Psi}(I\zeta, I\omega, \vartheta\mu) \geq A_{\Psi}(\zeta, \omega, \mu), B_{\Psi}(I\zeta, I\omega, \vartheta\mu) \leq B_{\Psi}(\zeta, \omega, \mu)$$

and

$$C_{\Psi}(I\zeta, I\omega, \vartheta\mu) \leq C_{\Psi}(\zeta, \omega, \mu)$$

are satisfied for $\vartheta \in \,]0,1[$, and also

$$\frac{1}{A_{\Psi}(I\zeta, I\omega, \mu)} - 1 \leq \vartheta \left[\frac{1}{A_{\Psi}(\zeta, \omega, \mu)} - 1\right],$$

$$B_{\Psi}(I\zeta, I\omega, \mu) \leq \vartheta B_{\Psi}(\zeta, \omega, \mu) \text{ and } C_{\Psi}(I\zeta, I\omega, \mu) \leq \vartheta C_{\Psi}(\zeta, \omega, \mu)$$

are satisfied for $\vartheta \in \,]0,1[$. Observe that $\lim_{\sigma \to \infty} \Psi(\zeta_\sigma, \omega)$ and $\lim_{\sigma \to \infty} \Psi(\omega, \zeta_\sigma)$ are finite. Observe as well as that the conditions of Theorem 3.7 and Theorem 3.9 are all met, and 0 is the unique fixed point of I.

4. Application to nonlinear fractional differential equations

In this section, we utilize following theorem to demonstrate the existence and uniqueness of a solution to a nonlinear fractional differential equation $(NFDE)$

$$D^q_{0+}\zeta(\delta) = g(\delta, \zeta(\delta)), \quad 0 < \delta < 1, \tag{4.1}$$

along with the boundary conditions,

$$\zeta(0) + \zeta^{'}(0) = 0, \quad \zeta(1) + \zeta^{'}(1) = 0,$$

where $1 < q \leq 2$ is a number, D^q_{0+} is the Caputo fractional derivative and $\hbar : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function.

Let $L = C([0,1], \mathbb{R})$ refer the space of all continuous functions on $[0,1]$ equipped with the CTN $\zeta * \omega = \zeta\omega$ and $CTCN$ $\zeta \odot \omega = \max\{\zeta, \omega\}$ for all $\zeta, \omega \in [0,1]$. Define an $NCMLS$ on L as follows:

$$A_{\Psi}(\zeta, c, \mu) = \frac{\alpha\mu}{\alpha\mu + \rho \max\{\sup_{\delta\in[0,1]} \zeta(\delta), \sup_{\delta\in[0,1]} c(\delta)\}^6},$$

$$B_{\Psi}(\zeta, c, \mu) = \frac{\rho \max\{\sup_{\delta\in[0,1]} \zeta(\delta), \sup_{\delta\in[0,1])c(\delta)}\}^6}{\alpha\mu + \rho \max\{\sup_{\delta\in[0,1]} \zeta(\delta), \sup_{\delta\in[0,1]} c(\delta)\}^6}$$

and

$$C_{\Psi}(\zeta, c, \mu) = \frac{\rho \max\{\sup_{\delta\in[0,1]} \zeta(\delta), \sup_{\delta\in[0,1]} c(\delta)\}^6}{\alpha\mu}.$$

For all $\mu > 0$ and $\zeta, c \in L$ with

$$\Psi(\zeta, c) = 1 + \max\{\sup_{\delta\in[0,1]} \zeta(\delta), \sup_{\delta\in[0,1]} c(\delta)\}^6.$$

Observe that $\zeta \in L$ solves (4.1) whenever $\zeta \in L$ solves the below integral equation:

$$\begin{aligned}\zeta(\delta) = &\frac{1}{\Gamma(q)}\int_0^1 (1-v)^{q-1}(1-\delta)\hbar(v,\zeta(\delta))cv \\ &+ \frac{1}{\Gamma(q-1)}\int_0^1 (1-v)^{q-2}(1-\delta)\hbar(v,\zeta(\delta))cv \\ &+ \frac{1}{\Gamma(q)}\int_0^\delta (\delta-v)^{q-1}\hbar(v,\zeta(\delta))cv.\end{aligned}$$

Theorem 4.1. *The integral operator $I : L \to L$ is set by*

$$\begin{aligned}\zeta(\delta) = &\frac{1}{\Gamma(q)}\int_0^1 (1-v)^{q-1}(1-\delta)\hbar(v,\zeta(\delta))cv \\ &+ \frac{1}{\Gamma(q-1)}\int_0^1 (1-v)^{q-2}(1-\delta)\hbar(v,\zeta(\delta))cv \\ &+ \frac{1}{\Gamma(q)}\int_0^\delta (\delta-v)^{q-1}\hbar(v,\zeta(\delta))cv,\end{aligned}$$

where $\hbar : [0,1]\times[0,\infty) \to [0,\infty)$ satisfying the following criteria:

$$\begin{aligned}\max&\left\{\sup_{v\in[0,1]} \hbar(v,\zeta(v)), \sup_{v\in[0,1]} \hbar(v,c(v))\right\} \\ &\leq \frac{1}{4}\max\left\{\sup_{v\in[0,1]} \zeta(v), \sup_{v\in[0,1]} c(v)\right\}\end{aligned}$$

for all $\zeta, c \in L$, and

$$\sup_{\delta\in(0,1)} \frac{1}{4096}\left[\frac{1-\delta}{\Gamma(q+1)} + \frac{1-\delta}{\Gamma(q)} + \frac{\delta^q}{\Gamma(q+1)}\right]^6 \leq \omega < 1.$$

Then $NFDE$ has a unique solution.

Proof.

$$
\begin{aligned}
&\max\{I\zeta(v), Ic(v)\}^6\\
&= \begin{pmatrix} \frac{1-\delta}{\Gamma(q)}\int_0^1 (1-v)^{q-1}\max\{\sup_{v\in[0,1]}\hbar(v,\zeta(v)),\sup_{v\in[0,1]}\hbar(v,c(v))\}cv+ \\ \frac{1-\delta}{\Gamma(q-1)}\int_0^1 (1-v)^{q-2}\max\{\sup_{v\in[0,1]}\hbar(v,\zeta(v)),\sup_{v\in[0,1]}\hbar(v,c(v))cv\} \\ +\frac{1-\delta}{\Gamma(q)}\int_0^\delta (\delta-v)^{q-1}\max\{\sup_{v\in[0,1]}\hbar(v,\zeta(v)),\sup_{v\in[0,1]}\hbar(v,c(v))\}cv \end{pmatrix}^6\\
&\leq \begin{pmatrix} \frac{1-\delta}{\Gamma(q)}\int_0^1 (1-v)^{q-1}+\frac{\max\{\sup_{v\in[0,1]}\zeta(v),\sup_{v\in[0,1]}c(v)}{4}cv \\ \frac{1-\delta}{\Gamma(q-1)}\int_0^1 (1-v)^{q-2}\frac{\max\{\sup_{v\in[0,1]}\zeta(v),\sup_{v\in[0,1]}c(v)}{4}cv \\ +\frac{1-\delta}{\Gamma(q)}\int_0^1 (\delta-v)^{q-1}\frac{\max\{\sup_{v\in[0,1]}\zeta(v),\sup_{v\in[0,1]}c(v)}{4}cv \end{pmatrix}^6\\
&\leq \frac{1}{4^6}\max\{\sup_{v\in[0,1]}\zeta(v),\sup_{v\in[0,1]}c(v)\}^6\left(\frac{1-\delta}{\Gamma(q)}\int_0^1 (1-v)^{q-1}cv\right.\\
&\qquad \left.+\frac{1-\delta}{\Gamma(q-1)}\int_0^1 (1-v)^{q-2}cv+\frac{1-\delta}{\Gamma(q)}\int_0^\delta (\delta-v)^{q-1}cv\right)^6\\
&\leq \frac{1}{4^6}\max\{\sup_{v\in[0,1]}\zeta(v),\sup_{v\in[0,1]}c(v)\}^6\sup_{\delta\in[0,1]}\left[\frac{1-\delta}{\Gamma(q+1)}+\frac{1-\delta}{\Gamma(q)}+\frac{\delta^q}{\Gamma(q+1)}\right]^6\\
&= \omega\max\{\sup_{v\in[0,1]}\zeta(v),\sup_{v\in[0,1]}c(v)\}^6,
\end{aligned}
$$

where

$$\omega = \sup_{\delta\in(0,1)}\frac{1}{4096}\left[\frac{1-\delta}{\Gamma(q+1)}+\frac{1-\delta}{\Gamma(q)}+\frac{\delta^q}{\Gamma(q+1)}\right]^6.$$

Therefore, the above inequality implies that

$$\max\{\sup_{v\in[0,1]}I\zeta(v),\sup_{v\in[0,1]}Ic(v)\}^6 \leq \max\{\sup_{v\in[0,1]}\zeta(v),\sup_{v\in[0,1]}c(v)\}^6.$$

Then, we have

$$
\begin{aligned}
\alpha\mu &+ \frac{\rho}{\omega}\max\{\sup_{v\in[0,1]}I\zeta(v),\sup_{v\in[0,1]}Ic(v)\}^6\\
&\leq \alpha\mu+\rho\max\{\sup_{v\in[0,1]}\zeta(v),\sup_{v\in[0,1]}c(v)\}^6.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{\alpha(\omega\mu)}{\alpha(\omega\mu)+\rho\max\{\sup_{s\in[0,1]}I\zeta(v),\sup_{s\in[0,1]}Ic(v)\}^6}\\
&\qquad \geq \frac{\alpha\mu}{\alpha\mu+\rho\max\{\sup_{v\in[0,1]}\zeta(v),\sup_{v\in[0,1]}c(v)\}^6}.
\end{aligned}
$$

Hence, we have

$$A_\Psi(I\zeta, Ic, \omega\mu) \geq A_\Psi(\zeta, c, \mu).$$

And also

$$\frac{\rho \sup_{v\in[0,1]} \max\{\sup_{v\in[0,1]} I\zeta(v), \sup_{v\in[0,1]} Ic(v)\}^6}{\alpha(\omega\mu) + \max\{\sup_{v\in[0,1]} I\zeta(v), \sup_{v\in[0,1]} Ic(v)\}^6}$$
$$\leq \frac{\rho \sup_{v\in[0,1]} \max\{\sup_{v\in[0,1]} \zeta(v), \sup_{v\in[0,1]} c(v)\}^6}{\alpha\mu + \max\{\sup_{v\in[0,1]} I\zeta(v), \sup_{v\in[0,1]} Ic(v)\}^6}.$$

Hence, we have

$$B_\Psi(I\zeta, Ic, \omega\mu) \leq B_\Psi(\zeta, c, \mu).$$

Furthermore,

$$\frac{\rho \sup_{v\in[0,1]} \max\{\sup_{v\in[0,1]} I\zeta(v), \sup_{v\in[0,1]} Ic(v)\}^6}{\alpha(\omega\mu)}$$
$$\leq \frac{\rho \sup_{v\in[0,1]} \max\{\sup_{v\in[0,1]} \zeta(v), \sup_{v\in[0,1]} c(v)\}^6}{.}\alpha\mu.$$

Hence, we have

$$C_\Psi(I\zeta, Ic, \omega\mu) \leq C_\Psi(\zeta, c, \mu)$$

for some $\alpha, \rho > 0$. Observe that the requirements of the Theorem 3.7 and Theorem 3.9 are met. As a result, I has a distinct point. For that reason, the precise $NFDE$ has a unique solution. □

5. Conclusion

In this study, we introduced the concept of a neutrosophic metric-like space, extending traditional metric structures to better handle uncertainty and indeterminacy. We developed the framework of neutrosophic controlled metric-like spaces and established fixed point theorems for contraction mappings on G-complete spaces.

These results generalize classical fixed point theorems and provide useful tools for analyzing systems under vague or incomplete information. To demonstrate the applicability of our findings, we applied them to prove the existence and uniqueness of solutions for a nonlinear fractional differential equation.

We believe the results presented here will be valuable to researchers working in neutrosophic and fuzzy environments, and may inspire further work in generalized fixed point theory, fractional calculus, and applications in uncertain systems.

Acknowledgement: Authors are thankful to Prince Sultan University for APC and support through TAS research Lab.

References

[1] M.A. Alghamdi, N. Hussain and P. Salimi, *Fixed point and coupled fixed point theorems on b-metric-like spaces,* J. Inequ. Appl., **2013** (2013), 1-25.

[2] S.B. Chen, S. Soradi-Zeid, M. Alipour, Y.M. Chu, J.F. Gómez-Aguilar and H. Jahanshahi, *Optimal control of nonlinear time-delay fractional differential equations with Dickson polynomials,* Fractals, **29**(4) (2021), 2150079.

[3] A. George and P. Veeramani, *On some results in fuzzy metric spaces,* Fuzzy Setsd Syst., **64**(3) (1994), 395-399.

[4] S.N. Hajiseyedazizi, M.E. Samei, F. Alzabut and M. Chu, *On multi-step methods for singular fractional q-integro-differential equations,* Open Math., **19**(1) (2021), 1378-1405.

[5] A. Harandi, *Metric-like paces, partial metric spaces and fixed point,* Fixed Point Theory Appl., **2012** (2012), 1-10.

[6] M. Jeyaraman and S. Sowndrarajan, *Some common fixed point theorems for* (Φ, Ψ)*-weak contractions in intuitionistic generalized fuzzy cone metric spaces,* Malaya J. Math., **2019**(1) (2019), 154-159.

[7] M. Jeyaraman and S. Sowndrarajan, *Common fixed point results in neutrosophic metric spaces,* Neutrosophic Sets and Syst., **42**(1) (2021), 208-220.

[8] M. Jeyaraman, M. Suganthi and S.Sowndrarajan, *Fixed point results in non-Archimedean generalized intuitionistic fuzzy metric spaces,* Notes on Intuitionistic Fuzzy Sets, **25**(4) (2019), 48-58.

[9] Z.A. Khan, I. Ahmad and K. Shah, *Applications of fixed point theory to investigate a system of fractional order differential equations,* J. Funct. Spaces, **2021**(1) (2021), 1399764.

[10] M. Kirisci and N. Simsek, *Neutrosophic metric spaces, Mathematical Sciences,* **14**(3) (2020), 241-248.

[11] N. Konwar, *Extension of fixed point results in intuitionistic fuzzy b metric space,* J. Intelligent Fuzzy Syst., **39**(5) (2020), 7831-7841.

[12] I. Kramosil and J. Michálek, *Fuzzy metrics and statistical metric spaces,* Kybernetika, **11**(5) (1975), 336-344.

[13] A. Kumar, *Rosslers system using piecewise derivative,* Results in Physics, **50** (2023), 106555.

[14] A. Kumar, Atul, A. Khursheed, Mati ur Rahmam, Emadifar Homan, Arora Geeta and Ahmed A. Hamoud, *Fractional Stochastic Van der Pol Oscillator with Piecewise Derivatives*, J. Math., **2024** (2024), 2436203.

[15] N. Laksaci, A. Boudaoui, K. Abodayeh, W. Shatanawi and T.A. Shatnawi, *Existence and uniqueness results of coupled fractional-order differential systems involving Riemann-Liouville derivative in the space* $\mathcal{W}_{a^+}^{\Upsilon_1,1}(a,b) \times \mathcal{W}_{a^+}^{\Upsilon_2,1}$ *with Perov's fixed point theorem,* Fractal and Fractional, **5**(4) (2021), 217.

[16] N. Mlaiki, Hassen Aydi, Nizar Souayah, Thabet Abdeljawad, *Controlled metric type spaces and the related contraction principle,* Mathematics, **6**(10) (2018), 6100194

[17] S. Nadaban, *Fuzzy b-metric spaces,* Int. J. Comput. Commu. Control, **11** (2016), 273-281.

[18] Y. Nawaz, M.S. Arif and W. Shatanawi, *A new numerical scheme for time fractional diffusive seair model with non-linear incidence rate: An application to computational biology,* Fractal and Fractional, **6**(2) (2022), 6020078.

[19] Y. Nawaz, M.S. Arif, W. Shatanawi and M.U. Ashraf, *A new unconditionally stable implicit numerical scheme for fractional diffusive epidemic model,* AIMS Mathematics, **7**(8) (2022), 14299-14322.

[20] J.H. Park, *Intuitionistic fuzzy metric spaces,* Chaos Solitons Fractals, **22** (2004), 10391046.

[21] R.S. Rahmat and S.M. Noorani, *Fixed point theorem on intuitionistic fuzzy metric spaces,* Iran. J. Fuzzy Syst., **3**(1) (2006), 23-29.

[22] R. Saadati and J.H. Park, *On the intuitionistic fuzzy topological spaces,* Chaos, Solitons Fractals, **27**(2) (2006), 331-344.

[23] E.Y. Salah, B. Sontakke, A.A. Hamoud, H. Emadifar and A. Kumar, *A fractalfractional order modeling approach to understanding stem cell-chemotherapy combinations for cancer,* Sci. Rep., **15** (2025), 3465.

[24] M.S. Sezen, *Controlled fuzzy metric spaces and some related fixed point results,* Numer. Meth. Partial Diff. Equa., **37**(1) (2021), 583-593.

[25] K. Shah, T.Abdeljawad, B. Abdalla and M.S. Abualrub, *Utilizing fixed point approach to investigate piecewise equations with non-singular type derivative,* AIMS Math., **7**(8) (2022), 14614-14630.

[26] S. Shukla and M. Abbas, *Fixed point results in fuzzy metric-like spaces,* Iran. J. Fuzzy Syst., **11**(5) (2014), 81-92.

[27] F. Smarandache, *Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis and synthetic analysis,* American Research Press, 1998.

[28] S. Sowndrarajan, M. Jeyaraman and F. Smarandache, *Fixed point results for contraction theorems in neutrosophic metric spaces,* Infinite Study, **36** (2020), 308-318.

[29] L.A. Zadeh, *Fuzzy sets,* Inf. Control., **8** (1965), 338-353.