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INVERSE AND OPTIMAL OUTPUT FEEDBACK CONTROL PROBLEMS FOR INFINITE DIMENSIONAL DETERMINISTIC AND STOCHASTIC SYSTEMS WITH UNCERTAIN SEMIGROUP GENERATORS

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Abstract. This paper is concerned with inverse problems and optimal output feedback control problems for semilinear infinite dimensional uncertain systems. We present several interesting typical and nontypical control problems and their solutions. The nontypical problems are related to control of the evolution of measures. We prove existence of optimal feedback control laws for these systems in the presence of uncertainty of the principal operator. We consider both deterministic and stochastic systems. In the last section we present the necessary conditions of optimality for the uncertain stochastic feedback control problem.

1. Introduction

First we consider inverse problems for infinite dimensional semilinear systems where the principal operator is unknown. The objective is to find one such operator from a given class of infinitesimal generators that gives the best fit of the measured data. Next we consider uncertain systems where the principal operator is unknown but the class it belongs to is assumed to be known. In the presence of this uncertainty, we consider feedback control problems for

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a class of semilinear systems and find from a given class, the best linear feedback control law that minimizes the maximum cost. Interestingly, these are all done on general Banach spaces. We also consider stochastic analogs of the aforementioned deterministic systems on Hilbert spaces and present solutions to inverse problems. Finally, we consider control problems for uncertain semilinear stochastic systems and present existence of linear optimal feedback control laws for min-max problems.

Deterministic and stochastic optimal control problems in infinite dimensions have been studied by many authors, see the monographs of Ahmed [1, 2, 4], Curtain and Zwart [14] and Fattorini [16] and the references therein. In Borkar and Govindan [12], the authors initiated a semigroup theory approach for characterization of optimal admissible Markovian controls for semilinear stochastic evolution equations in infinite dimensions with an objective to minimize an associated discounted cost functional. For finite dimensional controlled stochastic differential equations with a nondegenerate diffusion matrix, this task is traditionally accomplished through the HJB equation which is an indirect approach. This requires solving PDEs on R^n and then constructing the feedback control law from the solution. For the infinite dimensional case, the HJB equation is a PDE on an infinite dimensional Hilbert space and certainly it is a formidable computational problem given that existence problem is resolved. The article [12] made an attempt towards obtaining a verification theorem for such controlled systems. Subsequently, Ahmed [5] studied optimal feedback control of infinite dimensional stochastic evolution equations by using the indirect approach. To be precise, some new results were obtained on the question of existence of solutions of HJB equations in infinite dimensional Hilbert spaces. One of those results was applied to prove the existence of an optimal stationary feedback control. These results were later generalized in Ahmed [3] to nonstationary stochastic control problems in Hilbert spaces. King [17] considered problems on existence and regularity of integral representations of feedback operators arising from parabolic control problems. The existence of such representations is important for the design of low order compensators and placement of sensors. In other words, these are problems of existence and smoothness of functional gains for LQR feedback control systems governed by parabolic partial differential equations. It is well known that under suitable stabilizability conditions, a solution to this infinite dimensional LQR problem exists in the form of a bounded linear feedback operator. Recently, Curtain, et al. [13] provided new sufficient conditions under which the feedback operator associated with the Linear Quadratic Regulator design for distributed parameter systems is nuclear or Hilbert-Schmidt. We refer to [13] and the references therein for details. We also refer to Curtain and Zwart [14] for some earlier studies on this topic. The min-max problems arise naturally in the study of control systems with uncertainty [4], [6-11]. Note that uncertainty can occur in many ways. Ahmed and Xiang [8,9] considered uncertain systems wherein uncertainty occurs because of unknown system parameters $\{\sigma\}$. In fact, to a system designer, neither the true value of σ nor its probability law is known, but the range of values it may take, Σ is known. This introduces uncertainty in the system. The designer wishes to find a control policy to minimize the maximum risk or maximize the minimum revenue. In [9, 10], the authors considered min-max problems of optimal control for a general class of nonlinear uncertain evolution equations on Banach space and proved the existence of optimal controls. In [8], Ahmed and Xiang continued the study of optimal control problems for a class of nonlinear evolution equations with uncertain parameters and proved the existence of optimal control and further presented necessary conditions of optimality. The main result therein was applied to quasi-linear partial differential equations with uncertain coefficients. In recent papers, Ahmed [6,7] considered the optimal output feedback boundary control problems for a class of semilinear uncertain parabolic systems, in the sense that the uncertainty appears in the form of a set, and further considered a more general state dependent uncertainty. Using the game-theoretic approach, Ahmed proved the existence of saddle points giving the optimal strategies. Recently, Mordukhovich [18] considered the problem of optimal design of output feedback controller for a class of uncertain systems described by parabolic equations with Dirichlet boundary control. The design variable here is the feedback control law mapping output into control actions on the boundary. Later on, Ahmed [7] considered a more general optimal output feedback boundary control problems for a class of semilinear uncertain parabolic systems. The uncertain initial boundary value problem is converted into an equivalent Cauchy problem described by a differential inclusion in appropriate Banach spaces and proved the existence of saddle points and presented necessary conditions for optimal strategy. Recently, Ahmed and Charalambous considered minimax games in [11] for stochastic uncertain systems in a general set-up with the pay-off being a nonlinear functional of the uncertain measure where the uncertainty is measured in terms of relative entropy between the uncertain and control induced measures. Note that the adversary is the uncertain measure which maximizes the cost while the minimizer is the control induced measure. A new approach to constructive output feedback robust nonlinear controller design based on the min-max LQG control theory and the use of Integral Quadratic Constraints (IQCs) was proposed in Petersen [20]. This approach provides a methodology for constructing robust nonlinear controllers for a class of uncertain nonlinear systems (uncertainty in a different sense) considered over a finite time horizon. For details, see [20].

In this paper, we deal with both uncertain deterministic and stochastic systems. It is interesting to observe that the uncertainty that we deal with here is very different from all those mentioned earlier from the current literature. To be precise, we begin the paper with an inverse (identification) problem in the deterministic case, see Ahmed [2] for a detailed study on the subject. In this paper we consider the infinitesimal generator A (the principal operator) to be unknown, but it is assumed that the class of infinitesimal generators it belongs to is known. The cost functional given by the expression (2.3), representing the identification error for the semilinear system (2.1)-(2.2), is to be minimized with respect to $A \in G$, where G is a known set of unbounded operators. This is the inverse problem which is resolved in Theorem 4. To the best of our knowledge, this problem seems to have never been considered in the literature. We then consider an uncertain control problem wherein again there is uncertainty because the generator A is unknown but belongs to a known family. The problem here is to find a bounded linear state or output feedback operator B that minimizes the cost functional given by (2.6). The existence of an optimal operator B is proved in Theorem 5. Clearly, this is not a standard min-max problem as we look for the existence of operators that minimize the cost. More so, unlike here in Theorem 5, wherein we establish the existence of feedback control operators, Curtain, et al. [12] show the existence of a control variable/input $u(\cdot)$ that minimizes the classical LQR cost functional of a linear controlled system.

In the second part of the paper, we consider the stochastic system (2.7) on a Hilbert space. Again, the problem is to identify the principal operator A in the presence of noise. The goal is to find an operator A that minimizes the cost functional given by the expression (2.9). This is proved in Theorem 8 under some standard hypothesis. Furthermore, a stochastic version of the deterministic problem resolved in Theorem 5 is given in Theorem 9. This result is again very different from the main result of Petersen [20] and others. In [20] the author proves the existence of a control u that minimizes the cost functional. In contrast, our result (Theorem 9) proves directly the existence of an optimal output feedback operator that minimizes the cost functional.

In section 5, the paper also addresses some interesting problems on control of induced measure valued functions. These are nonstandard problems and generally can not be treated in the classical sense. They are dependent on the weak compactness property of the reachable set of measures. Some of the objective functionals are set functions aimed at reducing the Hausdorff dimension of supports of induced measures.

The rest of the paper is organized as follows: In Section 2, some deterministic and stochastic control and inverse problems are formulated and basic background materials presented. Section 3 discusses the main results on inverse problems and optimal feedback control problems for deterministic systems. In Sections 4 similar results are presented for stochastic systems. In Section 5, assuming the principal operator known and fixed, we present some results optimizing certain functionals of induced measures. In section 6, we present necessary conditions of optimality characterizing the optimal set of operators. The proof of this result is rather informal.

2. Problem Formulation

Let X be a Banach space and $M \geq 1$ and $\omega \in R$. Let $\mathcal{G}_0(M,\omega)$ denote the class of infinitesimal generators of C_0-semigroups on X with stability parameters $\{M,\omega\}$ fixed. In other words, every $A \in \mathcal{G}_0(M,\omega)$ generates a C_0-semigroup say $S_A(t), t \geq 0$, on X satisfying

$$\| S_A(t) \|_{\mathcal{L}(X)} \leq M \exp \omega t, \quad t \geq 0.$$

For details on semigroup theory the reader is referred to [1] and the references therein. We consider the following semilinear system including the output equation.

$$dx/dt = Ax + f(x), \quad x(0) = \xi, \quad A \in G \subset \mathcal{G}_0(M,\omega), \tag{2.1}$$

$$y = Lx. \tag{2.2}$$

The first equation represents the state equation and the second one the output equation. The operator $L \in \mathcal{L}(X,Y)$ denotes the sensor (measurement operator) where Y is another Banach space, $f : X \longrightarrow X$ is a continuous nonlinear map and G is a nonempty set. Let $\ell : I \times Y \longrightarrow R$ and define

$$J(A) \equiv \int_0^T \ell(t,y(t))dt \equiv \int_0^T \ell(t,Lx(t))dt. \tag{2.3}$$

Inverse Problem(A): The set G is known, but which A is in force is not known. The problem is to find an $A \in G$ that minimizes the functional (2.3). This is an inverse problem. We are interested in the question of existence of an optimal generator minimizing the functional (2.3).

Uncertain System (Control Problem(B)): Consider the system

$$dx/dt = Ax + By + f(x), \quad x(0) = \xi, \quad A \in G, \tag{2.4}$$

$$y = Lx, \tag{2.5}$$

with $B \in \mathcal{L}(Y,X)$ considered to be a linear output feedback operator. The cost functional is given by

$$J(B) \equiv \sup_{A \in G} \left\{ \int_I \ell(t, y_{A,B}(t))dt \equiv \int_I \ell(t, Lx_{A,B}(t))dt \right\}, \tag{2.6}$$

where $x_{A,B}$ denotes the mild solution of equation (2.4) corresponding $A \in \mathcal{G}_0(M,\omega)$ and $B \in \mathcal{L}(Y,X)$. Let $(\mathcal{L}(Y,X),\tau_{so})$ denote the space of bounded linear operators from Y to X endowed with the strong operator topology and Γ a compact subset of it. The problem is to find a $B \in \Gamma$ that minimizes the functional (2.6). Note that this is a min-max problem on $\Gamma \times G$. Here the set $G \subset \mathcal{G}_0(M,\omega)$ is considered to be the set of uncertainty meaning the principal operator of the semilinear system (2.4) is not precisely known but it is a member of a known family. The objective is to control the system assuming the worst possible situation.

Uncertain Stochastic system (inverse Problem (C)): Let $(\Omega,\mathcal{F},\mathcal{F}_{t\geq 0},P)$ be a complete filtered probability space and let $\mathcal{E}(z)$ denote the expectation of the random variable z. Consider the system

$$dx = Axdt + f(x)dt + CdW, \quad x(0) = \xi, \quad A \in G, \tag{2.7}$$
$$y = Lx, \tag{2.8}$$

where W is an E-valued $\mathcal{F}_t$ Brownian motion and $C \in \mathcal{L}(E,X)$. Here we consider again the following inverse problem. The cost functional (mismatch between measured data and the data generated by the model) is given by

$$J(A) \equiv \mathcal{E}\int_I \ell(t,y_A(t))dt. \tag{2.9}$$

The problem is to find an $A \in G$ that minimizes the functional given by (2.9).

Uncertain Stochastic System (Control Problem)(D): Consider the system

$$dx = Axdt + Bydt + f(x)dt + CdW, \quad x(0) = \xi, \quad A \in G, \tag{2.10}$$
$$y = Lx, \tag{2.11}$$

with $B \in \mathcal{L}(Y,X)$ considered to be a linear output feedback operator. The cost functional is given by

$$J(B) \equiv \sup_{A\in G}\left\{\mathcal{E}\int_I \ell(t,y_A(t))dt \equiv \mathcal{E}\int_I \ell(t,Lx_A(t))dt\right\}. \tag{2.12}$$

Let $(\mathcal{L}(X),\tau_{so})$ denote the space of bounded linear operators in X endowed with the strong operator topology and Γ a compact subset of it. The problem is to find a $B \in \Gamma$ that minimizes the functional (2.12). Note that this is a min-max problem on $(\mathcal{L}(X),\tau_{so})$.

3. Solution of Problems Related to Deterministic Systems

First we consider the inverse problem (A). We need the following lemma.

Lemma 1. *Let $\{X, Y\}$ be a pair of Banach spaces representing the state and output spaces respectively. Consider the system* (2.1) *and* (2.2) *and suppose f is locally Lipschitz having at most linear growth, $L \in \mathcal{L}(X, Y)$. Then, for each $\xi \in X$ and $A \in \mathcal{G}_0(M, \omega)$ the system* (2.1) *has a unique mild solution $x(A, \xi) \in C(I, X) \subset B_\infty(I, X)$ with the output $y \in C(I, Y)$.*

Proof. The proof is standard. We give a brief outline. Since $A \in \mathcal{G}_0(M, \omega)$, there exists a unique C_0-semigroup of operators $S_A(t), t \geq 0$, on X satisfying $\| S_A(t) \|_{\mathcal{L}(X)} \leq M e^{\omega t}, t \geq 0$. Using this semigroup, system (2.1) can be written as an integral equation on the Banach space X given by

$$x(t) = S_A(t)\xi + \int_0^t S_A(t-s) f(x(s)) ds, \quad t \in I \equiv [0, T]. \tag{3.1}$$

Since f has at most linear growth, it is easy to verify that there exists a finite positive number b such that any solution of equation (3.1) (if one exists) is bounded on bounded intervals, that is $\sup\{|x(t)|_X, t \in I\} \leq b$. The fact that $x \in C(I, X)$ follows from the strong continuity of the semigroup $S_A(t), t \geq 0$, and the linear growth and continuity of f. Using the local Lipschitz property of f, one can easily verify that the operator F given by

$$(Fx)(t) \equiv S_A(t)\xi + \int_0^t S_A(t-s) f(x(s)) ds, \quad t \in I, \tag{3.2}$$

has a unique fixed point in the Banach space $C(I, X)$. In fact one first shows that for large enough $n \in N$, the n-th iterate of F given by F^n is a contraction. The assertion then follows from Banach fixed point theorem. □

Definition 2. (D1): The set $\mathcal{G}_0(M, \omega)$ is endowed with the topology of strong convergence of the resolvents denoted τ_{ro} making $(\mathcal{G}_0(M, \omega), \tau_{ro})$ a topological space. A sequence $\{A_n\} \in \mathcal{G}_0(M, \omega)$ is said to be τ_{ro} convergent to $A_0 \in \mathcal{G}_0(M, \omega)$ if, and only if, $R(\lambda, A_n) \xrightarrow{\tau_{so}} R(\lambda, A_0)$ in $(\mathcal{L}(X), \tau_{so})$ for every $\lambda \in (\omega, \infty)$.
(D2): A subset $G \subset \mathcal{G}_0(M, \omega)$ is said to be compact in the resolvent operator topology τ_{ro} if every sequence from G has a subsequence that converges in the τ_{ro} topology to an element of G.

In order to solve the problem (A), we need one more result concerning continuity of the map $A \longrightarrow x_A$. For convenience of notation, we use $\mathcal{S}_0(M, \omega)$ to denote the semigroups corresponding to infinitesimal generators denoted by $\mathcal{G}_0(M, \omega)$.

Lemma 3. *Under the assumptions of Lemma 1, the map $A \longrightarrow x_A$ is continuous with respect to the resolvent operator topology on $\mathcal{G}_0(M,\omega)$ and the supnorm topology on $C(I,X)$.*

Proof. Consider the system (2.1) corresponding to $A = A_n \in \mathcal{G}_0(M,\omega)$ and $A = A_0 \in \mathcal{G}_0(M,\omega)$ respectively and let $x_n \in C(I,X)$ and $x_0 \in C(I,X)$ denote the corresponding mild solutions. We show that as $A_n \xrightarrow{\tau_{ro}} A_0$, $x_n \longrightarrow x_0$ in $C(I,X)$. Since $\{S_n, S_0\} \in \mathcal{S}_0(M,\omega)$ there exists a finite positive number $\tilde{M}$, dependent on $\{M,\omega\}$ and T, such that

$$\sup\{\| S_n(t) \|_{\mathcal{L}(X)}, \| S_0(t) \|_{\mathcal{L}(X)}, t \in I, n \in N\} \leq \tilde{M}.$$

It follows from these bounds and the linear growth assumption for f that there exists a finite positive number b such that

$$\sup\{|x_n(t)|_X, |x_0(t)|_X, t \in I, n \in N\} \leq b. \tag{3.3}$$

Using the integral equation (3.1) for $A = A_n$ and $A = A_0$ and denoting the corresponding solutions by x_n and x_0 respectively and subtracting one from the other we have the identity

$$x_0(t) - x_n(t) = E_n(t) + \int_0^t S_n(t-s)[f(x_0(s)) - f(x_n(s))]ds, \ t \in I, \tag{3.4}$$

where

$$E_n(t) \equiv (S_0(t)\xi - S_n(t)\xi) + \int_0^t (S_0(t-s) - S_n(t-s))f(x_0(s))ds, \ t \in I. \tag{3.5}$$

It follows from Trotter-Kato approximation theory for semigroups [1, Theorem 4.5.4, Remark 4.5.5] that as $A_n \xrightarrow{\tau_{ro}} A_0$, $S_n(t) \xrightarrow{\tau_{so}} S_0(t)$ uniformly on the interval I which is compact. Consequently, for fixed $\xi \in X$, the first term of the expression (3.5) converges strongly in X uniformly on I. Since $x_0 \in C(I,X)$ and f has at most linear growth, it is easy to verify that there exists a $g \in L_1^+(I)$ such that

$$\sup\{|[S_0(t-s) - S_n(t-s)]f(x_0(s))|_X, t \in [s,T]\} \leq g(s), \quad s \in I.$$

Further, it follows from the strong convergence of the semigroup $S_n(t)$ to $S_0(t)$ that

$$|[S_0(t-s) - S_n(t-s)]f(x_0(s))|_X \to 0$$

for every $s \in [0,t]$. Then by Lebesgue dominated convergence theorem

$$\int_0^t (S_0(t-s) - S_n(t-s))f(x_0(s))ds \xrightarrow{s} 0 \text{ in } X$$

uniformly in $t \in I$. In other words $e_n(t) \equiv |E_n(t)|_X \to 0$ uniformly in t on I. Now it follows from the estimate (3.3) and the local Lipschitz property of f

that there exists a finite positive number K_b such that

$$|x_0(t) - x_n(t)|_X \le e_n(t) + \tilde{M} K_b \int_0^t |x_0(s) - x_n(s)|_X \, ds, \quad t \in I. \tag{3.6}$$

Thus it follows from Gronwall inequality applied to (3.6) that $x_n(t) \xrightarrow{s} x_0(t)$ in X uniformly on I. This proves the theorem as stated. □

Now we are prepared to consider the inverse problem **(A)**.

Theorem 4. *Consider the problem* **(A)** *and suppose the set $G \subset \mathcal{G}_0(M, \omega)$ is compact in the resolvent operator topology τ_{ro}. Further, suppose the assumptions of Lemma 1 hold and the function $\ell : I \times Y \longrightarrow R$ is measurable in the first argument and lower semicontinuous in the second on Y and that there exists an $\ell_0 \in L_1(I)$ such that $\ell(t, y) \ge \ell_0(t)$ for all $(t, y) \in I \times Y$. Then problem* **(A)** *has a solution.*

Proof. Since the set $G \subset \mathcal{G}_0(M, \omega)$ is assumed to be compact in the resolvent operator topology τ_{ro}, it suffices to prove that the functional $A \to J(A)$, given by the expression (2.3), is lower semicontinuous with respect to the topology τ_{ro}. Let $\{A_n, A_0\} \in (\mathcal{G}_0(M, \omega), \tau_{ro})$ and $\{x_n, x_0\} \in C(I, X)$ the corresponding solutions of the integral equation (3.1). Let $A_n \xrightarrow{\tau_{ro}} A_0$. Then it follows from Lemma 3 that $x_n(t) \xrightarrow{s} x_0(t)$ in X uniformly in $t \in I$. Since the sensor (output operator) $L \in \mathcal{L}(X, Y)$, it is clear that

$$y_n(t) \equiv L x_n(t) \xrightarrow{s} L x_0(t) \equiv y_0(t) \text{ in } Y$$

for each $t \in I$ (even uniformly). Thus it follows from lower semicontinuity of ℓ in its second argument that

$$\ell(t, y_0(t)) = \ell(t, L x_0(t)) \le \underline{\lim}\, \ell(t, L x_n(t)) = \underline{\lim}\, \ell(t, y_n(t)) \tag{3.7}$$

and hence

$$\begin{aligned} J(A_0) &\equiv \int_I \ell(t, y_0(t)) dt \\ &\le \int_I \underline{\lim}\, \ell(t, y_n(t)) dt \le \underline{\lim} \int_I \ell(t, y_n(t)) dt \equiv \underline{\lim}\, J(A_n). \end{aligned} \tag{3.8}$$

This proves that J is lower semicontinuous in the resolvent operator topology τ_{ro}. Since G is compact in this topology and by assumption $\ell_0 \in L_1(I)$, it is clear that $J(A_0) > -\infty$ and hence J attains its (finite) minimum on G. This completes the proof. □

Now we consider the problem **(B)** with reference to the system (2.4)-(2.5) and the objective functional (2.6). Since $L \in \mathcal{L}(X, Y)$ is fixed, it is obvious that the system (2.4)-(2.5) is equivalent to the system

$$\dot{x} = Ax + BLx + f(x), x(0) = \xi, A \in G, \tag{3.9}$$

where $x_{A,B}$ is the mild solution of equation (3.9) corresponding to the pair A, B. Define the functional

$$\eta(B, A) \equiv \int_I \ell(t, Lx_{A,B}(t))dt \tag{3.10}$$

and note that the functional (2.6) is given by

$$J(B) \equiv \sup\{\eta(B, A), A \in G\}. \tag{3.11}$$

Now we are prepared to state the following result.

Theorem 5. *Consider the system* (3.9) *with* $\xi \in X$, $L \in \mathcal{L}(X, Y)$ *and* f *satisfying the assumptions of Lemma 1. Further, suppose* G *is compact in the resolvent operator topology* τ_{ro} *and* $\Gamma \subset \mathcal{L}(Y, X)$ *is compact in the strong operator topology* τ_{so} *and the integrand* ℓ *is measurable in* $t \in I$ *and continuous in* $y \in Y$ *satisfying*

$$|\ell(t, y)| \leq \alpha(t) + \beta|y|^p, \quad \alpha \in L^+(I), \quad \beta \geq 0, p \in (0, \infty). \tag{3.12}$$

Then the control problem **(B)** *has a solution, in the sense there exists a* $B \in \Gamma$ *at which* J *given by* (3.11) *attains its minimum.*

Proof. Since the operator $L \in \mathcal{L}(X, Y)$ is fixed, it follows from Lemma 3 that, for every fixed $B \in \Gamma$, the map $A \longrightarrow x_{A,B}$ is continuous with respect to the topology τ_{ro} on $\mathcal{G}_0(M, \omega)$ and the uniform topology on $C(I, X)$. Thus it follows from continuity of the map $y \longrightarrow \ell(t, y)$ in Y and (3.12) and the definition (3.10) that the functional $A \longrightarrow \eta(B, A)$ is continuous with respect to the topology τ_{ro}. By hypothesis G is compact in this topology and therefore, for every $B \in \mathcal{L}(Y, X)$, there exists an $A_B \in G$ (not necessarily unique) at which $A \longrightarrow \eta(B, A)$ attains its maximum. Thus the map

$$B \longrightarrow J(B) \equiv \sup\{\eta(A, B), A \in G\} = \eta(B, A_B)$$

is well defined. We show that this functional is continuous with respect to the strong operator topology on $\mathcal{L}(Y, X)$. Let $B_n \xrightarrow{\tau_{so}} B_o$ and let $A_n = A_{B_n} \in G$ denote any element of G at which $A \longrightarrow \eta(B_n, A)$ attains its maximum giving

$$J(B_n) \equiv \eta(A_n, B_n) = \int_I \ell(t, Lx_n(t)) \equiv \int_I \ell(t, y_n(t))dt$$

where $x_n \in C(I, X)$ is the mild solution of the evolution equation

$$dx/dt = A_n x + B_n x + f(x), \quad x(0) = \xi. \tag{3.13}$$

Since Γ is τ_{so} compact there exists a subsequence of the sequence $\{B_n\}$ and a corresponding subsequence of the (maximizing) sequence $\{A_n\} \in G$, all relabeled as the original sequence, such that

$$\begin{aligned} B_n &\xrightarrow{\tau_{so}} B_o \text{ in } \Gamma, \\ A_n &\xrightarrow{\tau_{ro}} A_o \text{ in } G. \end{aligned} \tag{3.14}$$

Corresponding to the pair $\{A_o, B_o\}$, let x_o denote the mild solution of the evolution equation

$$dx/dt = A_o x + B_o x + f(x), \quad x(0) = \xi. \tag{3.15}$$

Let $\{S_n(t), S_o(t)\}$ denote the semigroups corresponding to the infinitesimal generators $\{A_n, A_o\}$ respectively. Using these semigroups we obtain the following integral equations for the pair $\{x_o, x_n\}$:

$$\begin{aligned} x_o(t) = S_o(t)\xi + \int_0^t S_o(t-s)B_o x_o(s)ds \\ + \int_0^t S_o(t-s)f(x_o(s))ds, \end{aligned} \tag{3.16}$$

$$\begin{aligned} x_n(t) = S_n(t)\xi + \int_0^t S_n(t-s)B_n x_n(s)ds \\ + \int_0^t S_n(t-s)f(x_n(s))ds. \end{aligned} \tag{3.17}$$

Subtracting equation (3.17) from equation (3.16) we obtain the following expression

$$\begin{aligned} x_o(t) - x_n(t) = \alpha_n(t) + \beta_n(t) + \int_0^t S_n(t-s)B_n(x_o(s) - x_n(s))ds \\ + \int_0^t S_n(t-s)(f(x_o(s)) - f(x_n(s)))ds, \end{aligned} \tag{3.18}$$

where α_n and β_n are given by

$$\begin{aligned} \alpha_n(t) \equiv (S_o(t) - S_n(t))\xi + \int_0^t (S_o(t-s) - S_n(t-s))B_o x_o(s)ds \\ + \int_0^t (S_o(t-s) - S_n(t-s))f(x_o(s))ds, \ t \in I, \end{aligned} \tag{3.19}$$

and

$$\beta_n(t) \equiv \int_0^t S_n(t-s)(B_o - B_n)x_o(s)ds, \quad t \in I \tag{3.20}$$

respectively. We use the expression (3.18). It follows from linear growth and local Lipschitz property of f and boundedness of the admissible set $\Gamma \subset \mathcal{L}(X)$ that there exists a pair of finite positive numbers $\{K_b, \gamma\}$ such that

$$|x_o(t) - x_n(t)|_X \leq |\alpha_n(t)|_X + |\beta_n(t)|_X + \tilde{M}(\gamma + K_b) \int_0^t |x_o(s) - x_n(s)|_X ds, t \in I. \quad (3.21)$$

Using similar steps as in Lemma 3, involving strong convergence of the semigroups, convergence of B_n to B_o in the strong operator topology, and Lebesgue dominated convergence theorem, it is easy to verify that

$$\lim_{n\to\infty} |\alpha_n(t)|_X = 0 \ \text{ uniformly on } I, \quad (3.22)$$

$$\lim_{n\to\infty} |\beta_n(t)|_X = 0 \text{ uniformly on } I. \quad (3.23)$$

Now by Gronwall inequality, it follows from (3.21) that

$$|x_o(t) - x_n(t)|_X \leq \{|\alpha_n(t)|_X + |\beta_n(t)|_X\} + C \int_0^t \{\alpha_n(s)|_X + |\beta_n(s)|_X\} ds, \quad t \in I. \quad (3.24)$$

where $C \equiv \tilde{M}(\gamma + K_b) \exp\{\tilde{M}(K_b + \gamma)T\}$. From (3.22)-(3.24) we conclude that

$$\lim_{n\to\infty} \| x_o - x_n \|_{C(I,X)} = 0.$$

Thus we have proved that $B \longrightarrow J(B)$ is τ_{so} continuous on Γ. Since Γ is τ_{so} compact, J attains its minimum on Γ. This proves the existence of a solution of the min-max problem as stated. □

4. Solution of Problems Related to Stochastic Systems

Now we consider the problem **(C)** with reference to the system (2.7)-(2.8) and the objective functional (2.9). Again this is an inverse problem in the presence of noise.

Let $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ denote a complete filtered probability space where $\mathcal{F}_t, t \geq 0$, is an increasing family of sub sigma algebras of the sigma algebra $\mathcal{F}$. For simplicity, throughout this section we assume that both X and Y are separable Hilbert spaces and that X has a complete orthonormal basis $\{x_i\}$. Let $\mathcal{X} \equiv L_2(\Omega, X)$ denote the space of second order X valued random variables and $B^a_\infty(I, \mathcal{X})$ denote the class of $\mathcal{F}_t$ adapted second order X valued random processes defined on I. Endowed with the norm topology given by

$$\| x \|_{B^a_\infty(I,\mathcal{X})} \equiv \sup\{|x(t)|_{\mathcal{X}}, t \in I\} \equiv \sup_{t\in I} \left(\int_\Omega |x(t,\omega)|^2_X P(d\omega) \right)^{1/2} \quad (4.1)$$

it is a Banach space.

Let E be another separable Hilbert space and $W \equiv \{W(t), t \geq 0\}$ an $\mathcal{F}_t$ adapted Brownian motion in E with the incremental covariance operator $Q \in \mathcal{L}_s^+(E)$, the class of positive symmetric operators in E. To proceed further, we need the following classical result asserting the existence of a mild solution of equation (2.7).

Lemma 6. *Suppose $A \in \mathcal{G}_0(M, \omega)$, $f : X \longrightarrow X$ is uniformly Lipschitz, $C \in \mathcal{L}(E, X)$ and W a Q Brownian motion in E such that $CQC^* \in \mathcal{L}_1(X)$. Then for any $\mathcal{F}_0$ measurable X valued random variable ξ having finite second moment, equation* (2.7) *has a unique $\mathcal{F}_t$-adapted mild solution $x \in B_\infty^a(I, \mathcal{X})$.*

Proof. The proof is classical [15, Da Prato and Zabczyk, Chapter 7]. □

We are now prepared to consider the inverse problem (**C**). We need the following result.

Theorem 7. *Consider the system* (2.7) *and suppose the assumptions of Lemma 6 hold. Then the map $A \longrightarrow x_A$ is continuous from $\mathcal{G}_0(M, \omega)$ to $B_\infty^a(I, \mathcal{X})$ with respect to their respective topologies.*

Proof. Let $\{A_n, A_o\} \in (\mathcal{G}_0(M, \omega), \tau_{ro})$ and $\{x_n, x_o\} \in B_\infty^a(I, \mathcal{X})$ be the corresponding mild solutions of equation (2.7). Suppose $A_n \xrightarrow{\tau_{ro}} A_o$. Then by Trotter-Kato approximation theory for semigroups we know that, along a subsequence if necessary, $S_n(t) \xrightarrow{\tau_{so}} S_o(t)$ in $\mathcal{L}(X)$ uniformly in $t \in I$. We show that $x_n \xrightarrow{s} x_o$ in $B_\infty^a(I, \mathcal{X})$. Since $\{x_o, x_n\}$ are mild solutions, they satisfy the following integral equations

$$\begin{aligned} x_o(t) = S_o(t)\xi + \int_0^t S_o(t-s) f(x_o(s)) ds \\ + \int_0^t S_o(t-s) C dW(s), \quad t \in I, \end{aligned} \tag{4.2}$$

$$\begin{aligned} x_n(t) = S_n(t)\xi + \int_0^t S_n(t-s) f(x_n(s)) ds \\ + \int_0^t S_n(t-s) C dW(s), \quad t \in I. \end{aligned} \tag{4.3}$$

Subtracting equation (4.3) from equation (4.2) we obtain the following expression

$$(x_o(t) - x_n(t)) = E_n(t) + \int_0^t S_n(t-s)[f(x_o(s)) - f(x_n(s))] ds, t \in I, \tag{4.4}$$

where E_n is given by

$$E_n(t) \equiv \left\{ (S_o(t) - S_n(t))\xi + \int_0^t (S_o(t-s) - S_n(t-s)) f(x_o(s)) ds + \int_0^t (S_o(t-s) - S_n(t-s)) C dW \right\}. \tag{4.5}$$

From the Lipschitz property of f with Lipschitz constant K, it is easy to verify that

$$\mathcal{E}|x_o(t) - x_n(t)|_X^2 \leq 2\left\{ \mathcal{E}|E_n(t)|_X^2 + (\tilde{M}K)^2 T \int_0^t \mathcal{E}|x_o(s) - x_n(s)|_X^2 ds \right\} \tag{4.6}$$

for all $t \in I$. Then using Gronwall Lemma we arrive at the following inequality

$$\mathcal{E}|x_o(t) - x_n(t)|_X^2 \leq 2\left\{ \mathcal{E}|E_n(t)|_X^2 + (\tilde{M}K)^2 T \exp\{(\tilde{M}K)^2 T\} \int_0^t \mathcal{E}|E_n(s)|_X^2 ds \right\} \tag{4.7}$$

for all $t \in I$. We prove that

$$\lim_{n\to\infty} \mathcal{E}|E_n(t)|_X^2 = 0 \text{ uniformly in } t \in I. \tag{4.8}$$

For convenience, we write $E_n(t) \equiv E_{1,n}(t) + E_{2,n}(t) + E_{3,n}(t)$ and prove that each of these components has the property (4.8). Considering the first term, $E_{1,n}(t) \equiv (S_o(t) - S_n(t))\xi$, we have

$$\mathcal{E}|E_{1,n}(t)|_X^2 = Tr\Big((S_o(t) - S_n(t)) P_\xi (S_o^*(t) - S_n^*(t)) \Big), \tag{4.9}$$

where P_ξ is the covariance of the X valued random variable ξ. Since by assumption ξ is (strongly) second order, it is clear that $P_\xi \in \mathcal{L}_1^+(X)$ and so a compact operator. Thus it follows from the convergence of $S_n(t)$ to $S_o(t)$ in the strong operator topology (τ_{so}) uniformly on I that

$$\lim_{n\to\infty} \{\mathcal{E}|E_{1,n}(t)|_X^2\} = \lim_{n\to\infty} Tr\Big((S_o(t) - S_n(t)) P_\xi (S_o^*(t) - S_n^*(t)) \Big) = 0 \tag{4.10}$$

uniformly in $t \in I$. Next considering the third term

$$E_{3,n}(t) \equiv \int_0^t (S_o(t-s) - S_n(t-s)) C dW, \quad t \in I,$$

and recalling that C is time invariant it is easy to verify that

$$\mathcal{E}|E_{3,n}(t)|_X^2 = \int_0^t Tr[(S_o(s) - S_n(s))(CQC^*)(S_o(s)^* - S_n(s)^*)] ds. \tag{4.11}$$

By hypothesis $CQC^* \in \mathcal{L}_1^+(X)$ and therefore a compact operator and since $S_n(t) \xrightarrow{\tau_{so}} S_o(t)$ uniformly on I, it is clear that the integrand converges to zero uniformly in t on I, and hence, by virtue of dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \sup\{\mathcal{E}|E_{3,n}(t)|_X^2, \ \ t \in I\} = 0. \tag{4.12}$$

Next we consider the second term

$$E_{2,n}(t) \equiv \int_0^t (S_o(t-s) - S_n(t-s))f(x_o(s))ds, \quad t \in I.$$

Define $z(t) \equiv f(x_o(t)), t \in I$. Since f is assumed to be uniformly Lipschitz on X, it has at most linear growth, and by Lemma 6, $x_o \in B_\infty^a(I, \mathcal{X})$, and therefore $z \in B_\infty^a(I, \mathcal{X})$. In other words z is a strongly second order X valued random process adapted to $\mathcal{F}_t$. Define the covariance of the process z by

$$(P_z(t)h, h) \equiv \mathcal{E}\{(z(t), h)^2\}.$$

Since $z \in B_\infty^a(I, \mathcal{X})$, it is evident that $P_z(t) \in \mathcal{L}_1^+(X) \subset \mathcal{L}_1(X)$ for each $t \in I$ and that $P_z \in L_\infty(I, \mathcal{L}_1(X)) \subset L_1(I, \mathcal{L}_1(X))$. Using this operator valued function, it is easy to verify that the second moment of the random process $E_{2,n}$ is given by

$$\begin{aligned} &\mathcal{E}\{|E_{2,n}(t)|_X^2\} \\ &= \int_0^t Tr\{(S_o(t-s) - S_n(t-s))P_z(s)(S_o^*(t-s) - S_n^*(t-s))\}ds, \end{aligned} \tag{4.13}$$

for each $t \in I$. Hence it follows from compactness of the operator valued function $P_z(\cdot)$ and strong convergence of the semigroup $S_n(t)$ to $S_o(t)$ uniformly on I that the integrand of the expression (4.13) converges to zero for each $s \in [0, t]$ and $t \in I$. Then by use of Lebesgue dominated convergence theorem, it is easy to verify that

$$\lim_{n\to\infty} \sup\{\mathcal{E}\{|E_{2,n}(t)|_X^2\}, t \in I\} = 0. \tag{4.14}$$

Using (4.10),(4.12) and (4.14) we obtain the proof of (4.8). Thus it follows from (4.7) that

$$\lim_{n\to\infty} \sup\{\mathcal{E}|x_o(t) - x_n(t)|_X^2, t \in I\} = 0. \tag{4.15}$$

This proves the continuity as stated in the theorem. □

Now we can prove the existence of solution of the inverse problem C.

Theorem 8. *Consider the system* (2.7) *with the objective functional* (2.9) *and suppose the assumptions of Theorem 7 hold. Let ℓ be Borel measurable* *on $I \times Y$ and lower semicontinuous in its second argument and suppose there exist $\alpha \in L_1^+(I)$ and $\beta \geq 0$ such that*

$$|\ell(t,y)| \leq \alpha(t) + \beta |y|_Y^2 \tag{4.16}$$

*and $L \in \mathcal{L}(X,Y)$. Then the inverse problem (**C**) admits a solution.*

Proof. By Theorem 7, $A \longrightarrow x_A$ is continuous with respect to the topologies as indicated in the statement of the theorem. Since L is a bounded linear operator from X to Y, it is clear that $A \longrightarrow y_A \equiv Lx_A$ is continuous from $\mathcal{G}_0(M,\omega)$ to $B_\infty^a(I,\mathcal{Y})$ in the respective topologies where $\mathcal{Y} \equiv L_2(\Omega, Y)$. Thus, if $A_n \xrightarrow{\tau_{ro}} A_o$ in $\mathcal{G}_0(M,\omega)$ and $\{x_n, x_o\}$ are the corresponding mild solutions of equation (2.7), then, along a subsequence if necessary, $x_n \xrightarrow{s} x_o$ in $B_\infty^a(I,\mathcal{X})$ and hence $y_n \to y_o$ in $B_\infty^a(I,\mathcal{Y})$. Clearly, $y_n(t) \to y_o(t)$ in $L_2(\Omega,Y)$ for each $t \in I$. Thus it follows from well known Cauchy theorems that $y_n(t) \to y_o(t)$ in probability (in P measure) and hence there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k}(t) \xrightarrow{s} y_o(t)$ (in Y) P-a.s. Relabeling the subsequence $\{y_{n_k}\}_k$ as $\{y_k\}$ it follows from lower semicontinuity of ℓ in its second argument that

$$\ell(t, y_o(t)) \leq \underline{\lim}\, \ell(t, y_k(t)) \;\; P - a.s$$

for all $t \in I$. Since $\{A_k\} \in \mathcal{G}(M,\omega)$ it follows from (4.16) that $\ell(\cdot, y_k(\cdot))$ is uniformly integrable. Thus by Fatou's Lemma, it follows from the above inequality that

$$J(A_o) \equiv \mathcal{E}\int_I \ell(t,y_o(t))dt \leq \underline{\lim}\;\; \mathcal{E}\int_I \ell(t,y_k(t))dt = \underline{\lim}\, J(A_n).$$

This proves that J defined by the expression (2.9) is lower semicontinuous in the τ_{ro} topology of $\mathcal{G}_0(M,\omega)$. Since $G \subset \mathcal{G}_0(M,\omega)$ is compact in this topology, J attains its minimum on it. Thus the problem (**C**) admits a solution. □

An example. We present here a simple example of the cost integrand ℓ such as

$$\ell(t, Lx_A(t)) \equiv < \Xi(Lx_A(t) - y^d(t)), (Lx_A(t) - y^d(t) >_Y$$

where y_d is any observed (measured) data possibly an element of $B_\infty^a(I,\mathcal{Y})$ and $\Xi \in \mathcal{L}_s^+(Y)$ (the class of bounded positive selfadjoint operators).

Uncertain Stochastic System (Control Problem)(D): Now we consider the control problem of the uncertain system (2.10)-(2.11) with the pay-off functional given by (2.12).

Theorem 9. *Consider the system* (2.10) *with $\xi \in \mathcal{X}$ being $\mathcal{F}_0$-measurable, $L \in \mathcal{L}(X,Y)$ and suppose f satisfies the assumption of Lemma 6. Further, suppose G is compact in the resolvent operator topology τ_{ro} and $\Gamma \subset \mathcal{L}(Y,X)$ is* *compact in the strong operator topology τ_{so} and the integrand ℓ is measurable in $t \in I$ and continuous in $y \in Y$ satisfying*

$$|\ell(t,y)| \leq \alpha(t) + \beta |y|^2, \ \ \alpha \in L^+(I), \ \ \beta \geq 0. \tag{4.17}$$

Then the control problem **(D)** *has a solution, in the sense that there exists a $B \in \Gamma$ at which J given by* (2.12) *attains its minimum.*

Proof. The proof is quite similar to that of the deterministic case. First note that we are now concerned with the uncertain feedback control system

$$dx = Axdt + BLxdt + f(x)dt + CdW, \quad t \in I \tag{4.18}$$

with $A \in G$ being uncertain. Since $L \in \mathcal{L}(X,Y)$ is fixed and $\Gamma \subset \mathcal{L}(Y,X)$ is bounded, taking $\tilde{f}(x) \equiv f(x) + BLx$, it is easy to verify that with f replaced by $\tilde{f}$, the results of Lemma 6, Theorem 7 and Theorem 8 remain valid. Let $x_{A,B} \in B^a_\infty(I,\mathcal{X})$ denote the mild solution of equation (4.18) corresponding to any choice of $A \in G \subset \mathcal{G}_0(M,\omega)$ and $B \in \Gamma$. Define the functional

$$\eta(B,A) \equiv \mathcal{E}\left(\int_I \ell(t, Lx_{A,B}(t))dt\right). \tag{4.19}$$

For each $B \in \Gamma$ fixed, it follows from Theorem 7 that $A \longrightarrow x_{A,B}$ is continuous with respect to the topologies mentioned there. By assumption, ℓ is continuous in its second argument. Thus, following similar arguments as in the proof of Theorem 8, it is easy to verify that $A \longrightarrow \eta(A,B)$ is continuous in the resolvent operator topology τ_{ro}. Since G is compact in this topology, for each $B \in \Gamma$ there exists an element $A_B \in G$ such that

$$J(B) = \eta(B, A_B) \equiv \sup\{\eta(B,A), \ \ A \in G\}. \tag{4.20}$$

We must show that $B \longrightarrow J(B)$ is continuous with respect to the strong operator topology of $(\mathcal{L}(Y,X), \tau_{so})$. Let $\{B_n\} \in \Gamma$ and suppose $B_n \xrightarrow{\tau_{so}} B_o$. Then it follows from the above result (see (4.20)) that there exists a sequence $A_n \in G$ such that $J(B_n) = \eta(B_n, A_n)$. Let $x_n \equiv x_{A_n,B_n}$ and $y_n = Lx_n \equiv Lx_{A_n,B_n}$. Since G is τ_{ro} compact, there exists a subsequence of the sequence $\{A_n\}$, relabeled as the original sequence, and an element $A_o \in G$ such that $A_n \xrightarrow{\tau_{ro}} A_o$. Let $\{S_n(t), S_o(t)\}$, $t \geq 0$, denote the semigroups corresponding to the sequence of generators $\{A_n, A_o\}$. Note that the mild solutions $\{x_n, x_o\}$ of equation (4.18) corresponding to the pairs $\{A_n, B_n\}$ and $\{A_o, B_o\}$ respectively are given by the solutions of the following integral equations

$$\begin{aligned} x_n(t) = S_n(t)\xi &+ \int_0^t S_n(t-s)B_nLx_n(s)ds \\ &+ \int_0^t S_n(t-s)f(x_n(s))ds + \int_0^t S_n(t-s)CdW, \end{aligned} \tag{4.21}$$

and

$$x_o(t) = S_o(t)\xi + \int_0^t S_o(t-s)B_oLx_o(s)ds + \int_0^t S_o(t-s)f(x_o(s))ds + \int_0^t S_o(t-s)CdW \quad (4.22)$$

respectively. Using the above equations and carrying out some straightforward algebra, it is easy to verify that

$$(x_o(t) - x_n(t)) \equiv \alpha_n(t) + \beta_n(t) + \gamma_n(t) + \int_0^t S_n(t-s)B_nL(x_o(s) - x_n(s))ds + \int_0^t S_n(t-s)(f(x_o(s)) - f(x_n(s)))ds, \quad (4.23)$$

where

$$\alpha_n(t) \equiv (S_o(t) - S_n(t))\xi + \int_0^t (S_o(t-s) - S_n(t-s))B_oLx_o(s)ds + \int_0^t (S_o(t-s) - S_n(t-s))f(x_o(s))ds \quad (4.24)$$

$$\beta_n(t) \equiv \int_0^t S_n(t-s)(B_o - B_n)Lx_o(s)ds \quad (4.25)$$

$$\gamma_n(t) \equiv \int_0^t (S_o(t-s) - S_n(t-s))CdW. \quad (4.26)$$

By use of standard triangle inequality, it is easy to deduce from (4.23) that there exists a positive constant $\kappa > 0$ such that

$$\mathcal{E}|x_o(t) - x_n(t)|_X^2 \leq \kappa \Big\{ \mathcal{E}|\alpha_n(t)|_X^2 + \mathcal{E}|\beta_n(t)|_X^2 + \mathcal{E}|\gamma_n(t)|_X^2 + \big(\tilde{M}\gamma \parallel L \parallel + \tilde{M}K\big)^2 T \int_0^t \mathcal{E}|x_o(s) - x_n(s)|^2 ds \Big\} \quad (4.27)$$

for all $t \in I$, where $\gamma \equiv \sup\{\parallel B \parallel_{\mathcal{L}(Y,X)}, B \in \Gamma\}$ and $\tilde{M} \equiv M \exp(|\omega|T)$. Now we use compactness of the covariance operators P_ξ, CQC^* and strong continuity of the semigroups $\{S_n, S_o\}$ uniformly on compact intervals and follow similar approach as in Theorem 7 to prove that

$$\mathcal{E}|\alpha_n(t)|_X^2 + \mathcal{E}|\beta_n(t)|_X^2 + \mathcal{E}|\gamma_n(t)|_X^2 \to 0$$

uniformly in $t \in I$. Then it follows from the expression (4.27) and Gronwall inequality, that $x_n \xrightarrow{s} x_o$ in $B_\infty^a(I, \mathcal{X})$. Thus, again it follows from Cauchy theorems that along a subsequence, relabeled as the original sequence,

$\ell(t, Lx_n(t)) \to \ell(t, Lx_o(t))$ for all $t \in I$ and P-a.s. Then it follows from (4.17) and Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty} \mathcal{E} \int_I \ell(t, Lx_n(t))dt = \mathcal{E} \int_I \ell(t, Lx_o(t))dt. \tag{4.28}$$

This proves that $\lim_{n\to\infty} J(B_n) = J(B_o)$. Thus we have proved the continuity of the map $B \to J(B)$ as defined by (4.20) with respect to the topology τ_{so} on $\mathcal{L}(Y, X)$. Since Γ is τ_{so} compact, J attains its minimum on Γ and hence the uncertain stochastic feedback control problem (**D**) has a solution. □

5. Control of Induced Measure Valued Functions

Here we consider the systems (2.10)-(2.11) with the principal operator assumed known and fixed. It is only the control operator B that is to be chosen from the admissible set Γ so as to extremize certain functionals of the induced measures. First note that, under the assumptions of Theorem 9, for each $B \in \Gamma$ the system

$$dx = Axdt + BLxdt + f(x)dt + CdW, x(0) = \xi, \quad t \in I \tag{5.1}$$

has a unique mild solution $x^B \in B^a_\infty(I, \mathcal{X})$. Thus for each $t \in I$, $x^B(t) \in L_2(\Omega, X)$ and $\mathcal{F}_t$ measurable. Let $\mathcal{B}(X)$ denote the Borel algebra of subsets of the Hilbert space X and $\mathcal{M}_1(X)$ the space of probability measures on $\mathcal{B}(X)$. Clearly, the measure μ_t^B given by

$$\mu_t^B(V) \equiv P(x_t^B)^{-1}(V) \equiv Prob.\{x^B(t) \in V\}, \quad V \in \mathcal{B}(X)$$

is well defined. Let μ_0 denote the measure $\mu_0 \equiv P\xi^{-1}$ giving the distribution of the initial state ξ. We introduce the reachable set

$$\mathcal{R}(t) \equiv \{\mu \in \mathcal{M}_1(X) : \mu = \mu_t^B, \ \ B \in \Gamma\} \tag{5.2}$$

and show that it is weakly sequentially compact.

Theorem 10. *Consider the system* (5.1)*. Suppose A is the infinitesimal generator of a C_0-semigroup $S(t), t \geq 0$, in $\mathcal{L}(X)$, $L \in \mathcal{L}(X, Y)$ and $B \in \Gamma$ where Γ is compact in the strong operator topology of $\mathcal{L}(Y, X)$, f is uniformly Lipschitz on X, $C \in \mathcal{L}(E, X)$ and W is a Q Brownian motion in E such that $CQC^* \in \mathcal{L}_1^+(X)$, and ξ is $\mathcal{F}_0$-measurable X valued random variable with finite second moment. Then for each $t \in I$, $\mathcal{R}(t)$ is a weakly compact subset of $\mathcal{M}_1(X)$.*

Proof. We show that every sequence $\{\mu_t^n\} \in \mathcal{R}(t)$ has a subsequence that converges weakly to an element $\mu_t^o \in \mathcal{R}(t)$. Since $\mu_t^n \in \mathcal{R}(t)$, there exists a $B_n \in \Gamma$ such that $\mu_t^n = P(x^{B_n}(t))^{-1}$. By compactness of Γ in the strong operator topology, there exists a subsequence of this sequence, relabeled as the original sequence, and $B_o \in \Gamma$ such that $B_n \xrightarrow{\tau_{so}} B_o$. Let $\{x^n, x^o\}$ denote the mild solutions of equation (5.1) corresponding to $\{B_n, B_o\}$ respectively. Then as shown in the proof of Theorem 9 (with A considered fixed) $x^n \to x^o$ in $B_\infty^a(I, \mathcal{X})$ and hence $x^n(t) \to x^o(t)$ in $L_2(\Omega, X)$ for all $t \in I$. It is well known that mean convergence implies convergence in measure and convergence in measure implies the existence of a subsequence that converges P-a.s. So there exists a subsequence x^{n_k} of the sequence x^n and an element $x^o \in B_\infty^a(I, \mathcal{X})$ such that $x^{n_k}(t) \xrightarrow{s} x^o(t)$ in X P-a.s. Define

$$\mu_t^{n_k} = P(x^{n_k}(t))^{-1}, \text{ and } \mu_t^o = P(x^o(t))^{-1}.$$

Clearly, $\{\mu_t^{n_k}, \mu_t^o\} \in \mathcal{R}(t)$ for each $t \in I$. Let $BC(X)$ denote the Banach space of real valued bounded continuous functions on X (endowed with the standard sup norm topology). Then for any $\varphi \in BC(X)$ it is clear that $\varphi(x^{n_k}(t)) \to \varphi(x^o(t))$ with probability one which is equivalent to

$$\int_X \varphi(\zeta)\mu_t^{n_k}(d\zeta) \to \int_X \varphi(\zeta)\mu_t^o(d\zeta).$$

This shows that $\mu_t^{n_k} \xrightarrow{w} \mu_t^o$ for each $t \in I$. Thus we have proved weak sequential compactness of the reachable set $\mathcal{R}(t)$ for each $t \in I$. This completes the proof. □

We use the above result to solve the following mass transfer problem. Let D be a closed subset of X supporting the initial measure μ_0, that is $\mu_0(D) = 1$. Let K, a closed subset of X, denote the target set possibly satisfying $D \cap K = \emptyset$. We seek a control operator $B \in \Gamma$ that maximizes the mass of the measure $\mu_T^B(\cdot)$ on K. Here the objective functional is given by

$$J(B) \equiv \mu_T^B(K) \tag{5.3}$$

where μ^B is the measure induced by the (mild) solution of equation (5.1) corresponding to the control operator $B \in \Gamma$.

Corollary 11. *Consider the system* (5.1) *with the objective functional* (5.3) *and suppose the assumptions of Theorem 10 hold. Then there exists a control operator* $B_0 \in \Gamma$ *such that* $J(B_o) \geq J(B)$ *for all* $B \in \Gamma$.

Proof. It suffices to prove that J given by (5.3) is upper semicontinuous in the strong operator topology of $\mathcal{L}(Y, X)$. Let $\{B_n, B_o\}$ be any sequence from Γ and $\{\mu^n, \mu^o\}$ a sequence of measure valued functions induced by the corresponding solutions of equation (5.1). Suppose $B_n \xrightarrow{\tau_{so}} B_o$. Then it follows from Theorem 10, that along a subsequence, if necessary, $\mu_t^n \xrightarrow{w} \mu_t^o$. Since K is a closed set, it follows from a well known result [19, Parthasarathy, Theorem 6.1, p40] that

$$\overline{\lim}\, \mu_t^n(K) \leq \mu_t^o(K). \tag{5.4}$$

This is the same as $\overline{\lim} J(B_n) \leq J(B_o)$. Thus J is upper semicontinuous on Γ with respect to strong operator topology. Compactness of Γ in this topology implies existence of an optimal operator B_o. This proves the existence of a $B_o \in \Gamma$ such that $J(B_o) \geq J(B)$ for all $B \in \Gamma$. □

This result can be further extended as follows. The objective functional is given by

$$J(B) \equiv \int_I \mu_t^B(K)\lambda(dt). \tag{5.5}$$

The problem is to find an operator $B \in \Gamma$ at which J given by (5.5) attains its maximum. Note that if λ is the Lebesgue measure, maximizing this functional is equivalent to finding a control law that maximizes the residence time of the solution process in the set K.

Corollary 12. *Consider the system* (5.1) *with the objective functional* (5.5) *and suppose the assumptions of Theorem 10 hold and that* λ *is a countably additive positive measure having bounded total variation on* I. *Then there exists a control operator* $B_o \in \Gamma$ *such that* $J(B_o) \geq J(B)$ *for all* $B \in \Gamma$.

Proof. By Corollary 11, the expression (5.4) holds. Clearly, the functions on both the sides of the inequality are bounded measurable and λ integrable over I. By integrating this over the interval I with respect to the measure λ, it is easy to verify that $J(B_o) \geq \overline{\lim} J(B_n)$. Thus the statement of the theorem follows from τ_{so} compactness of the admissible set Γ. □

Note that λ need not be a non atomic measure. In fact we can even choose a purely atomic measure such as $\lambda(dt) \equiv \sum_{i=1}^{\infty} \alpha_i \delta_{t_i}(dt)$ where δ_s denotes the Dirac measure supported at $s \in I$. Since λ is assumed to be a positive measure having bounded variation, it is necessary that $\sum_{i=1}^{\infty} \alpha_i < \infty$.

There are other interesting applications. One such problem is the evasion problem where one wants to avoid approaching a danger zone. Let $D \subset X$ denote the forbidden zone and $D^{\varepsilon} \equiv \{x \in X : d(x, D) < \varepsilon\}$ the open ε neighborhood of the set D. The objective is to stay away from D^{ε} if possible. Thus we may try to find a feedback operator $B \in \Gamma$ that minimizes the functional

$$J(B) \equiv \int_I \mu_t^B(D^{\varepsilon})\lambda(dt). \tag{5.6}$$

Corollary 13. *Consider the system* (5.1) *with the objective functional* (5.6) *and suppose the assumptions of Theorem 10 hold and that* λ *is a countably additive positive measure having bounded total variation on* I. *Then there exists an optimal control operator in* Γ *at which* J *given by* (5.6) *attains its minimum.*

Proof. (Outline) We use the notations and arguments of Corollary 11. Since D^ε is an open set, it follows from [19, Parthasarathy, Theorem 6.1, p40] that $\underline{\lim}\, \mu_t^n(D^\varepsilon) \geq \mu_t^o(D^\varepsilon)$ for all $t \in I$. Hence one can easily verify that the functional J is lower semicontinuous, that is, $J(B_o) \leq \underline{\lim}\, J(B_n)$. Thus by τ_{so} compactness of the admissible set Γ, J attains its infimum on Γ proving existence of an optimal feedback operator. □

Another interesting problem is concerned with minimizing the Hausdorff dimension of the support of the measure induced by the process x^B. Let $\mathcal{K}(X)$ denote the hyper space of compact subsets of the Hilbert space X. This is furnished with the metric topology ρ_H where ρ_H denotes the standard Hausdorff metric on $\mathcal{K}(X)$. It is well known that $(\mathcal{K}(X), \rho_H)$ is a Polish space if X is Polish. In our case X is a separable Hilbert space so a Polish space.

Here, we are concerned with the objective functional given by

$$J(B) \equiv \inf\{\eta(K, B), K \in \mathcal{K}(X)\} \tag{5.7}$$

where

$$\eta(K, B) \equiv \nu(K) + (\beta/T) \int_I \mu_t^B(X \setminus K)\ \lambda(dt)$$

with $\nu : \mathcal{K}(X) \longrightarrow [0, \infty]$ an extended nonnegative real valued set function defined on $\mathcal{K}(X)$.

Theorem 14. *Consider the system* (5.1) *and suppose the assumptions of Theorem 10 hold. Further, suppose ν satisfies the following properties:* (P1)*: $\nu(K_1) \leq \nu(K_2)$ whenever $K_1, K_2 \in \mathcal{K}(X)$ satisfying $K_1 \subset K_2$.* (P2)*: ν is coercive with respect to the Hausdorff dimension d_H in the sense that* $\lim_{d_H(K)\to\infty} \nu(K) = \infty$. *Then, there exists a $B \in \Gamma$ at which J given by* (5.7) *attains its minimum.*

Proof. (Outline) We show that J given by (5.7) is lower semicontinuous with respect to the relative τ_{so} topology on $\Gamma \subset \mathcal{L}(Y, X)$. Since the second term of η is bounded above by $(\beta/T)\lambda(T)$ and ν is coercive there exists a $K^o \in \mathcal{K}(X)$ such that for all $B \in \mathcal{L}(Y, X)$, the minimizing set $\{K : K \in \mathcal{K}(X)\} \subset K^o$. Thus it suffices to restrict η on the compact topological space $(\mathcal{K}(K^o), \rho_H) \times (\Gamma, \tau_{so})$. In other words (5.7) is equivalent to

$$J(B) \equiv \inf\{\eta(K, B), K \in \mathcal{K}(K^o)\}. \tag{5.8}$$

We verify that J given by (5.8) is lower semicontinuous in τ_{so}. Let $\{B_n\}$ be any sequence from Γ such that $B_n \xrightarrow{\tau_{so}} B_o$. Let $\{K_n\} \in \mathcal{K}(K^o)$ the corresponding sequence of minimizers of $K \longrightarrow \eta(K, B_n)$. That is $\eta(K_n, B_n) \leq \eta(K, B_n)$ for all $K \in \mathcal{K}(K^o)$. Since $(\mathcal{K}(K^o), \rho_H)$ is a compact topological space, there exists a subsequence of the sequence $\{K_n, B_n\} \subset \mathcal{K}(K^o) \times \Gamma$, relabeled as the original sequence, and an element $(K_o, B_o) \in \mathcal{K}(K^o) \times \Gamma$ such that

$$(K_n, B_n) \longrightarrow (K_o, B_o)$$

with respect to the topology $\rho_H \times \tau_{so}$. Since, for each $t \in I$, the reachable set $\mathcal{R}(t)$ is weakly compact, it is uniformly tight and hence it follows from compactness of the sets $\{K_n, K_o\}$ that

$$\underline{\lim}\, \mu_t^n(X \setminus K_n) \geq \mu_t^o(X \setminus K_o).$$

This follows from the same argument as (5.4). Now using this it is easy to verify that

$$(\beta/T) \int_I \mu_t^o(X \setminus K_o)\ \lambda(dt) \leq (\beta/T)\, \underline{\lim} \int_I \mu_t^n(X \setminus K_n)\ \lambda(dt). \tag{5.9}$$

By lower semicontinuity of ν we have

$$\nu(K_o) \leq \underline{\lim}\, \nu(K_n). \tag{5.10}$$

Adding up (5.9) and (5.10) and recalling that sum of liminfs is equal to or less than liminf of the sum we arrive at the conclusion that $J(B_o) \leq \underline{\lim}\, J(B_n)$ proving lower semicontinuity of J given by (5.8) and hence (5.7). Thus it follows from τ_{so} compactness of the set Γ that J attains its infimum on Γ. This completes the outline of our proof. □

The results presented above can be easily extended to cover time varying operators by choosing for the admissible class the set $\mathcal{F}_{ad} \equiv B_s(I, \Gamma)$. This is the class of strongly measurable operator valued functions defined on I and taking values from a set $\Gamma \subset \mathcal{L}(Y, X)$ which is compact in the strong operator topology. The set $\mathcal{F}_{ad}$, furnished with the Tychnoff product topology, is compact.

Remark 15. In all the results presented above we have considered admissible feedback operators which are independent of time. As mentioned above, by use of Tychonoff product topology the results can be easily extended to operator valued functions. Time invariant operators, however, are easy to construct and implement and so it is preferred in engineering applications.

6. Necessary Conditions of Optimality

So far we have proved existence of optimal operators extremizing certain objective functionals. These optimal operators can be characterized through necessary conditions of optimality. Using the necessary conditions one can construct the optimal operators. This is a subject of another paper. For completeness we present here one result for the **problem (D)**.

Theorem 16. *Consider the system* (2.10)-(2.11) *with the objective (cost) functional* (2.12) *and suppose the assumptions of Theorem 9 hold and that the set* G *is a convex subset of* $\mathcal{G}_0(M,\omega)$ *and* Γ *is a convex subset of* $\mathcal{L}_s(X)$. *Further suppose* f *is Fréchet differentiable with the derivative* $Df(x) \in \mathcal{L}(X)$ *uniformly bounded. Suppose the integrand* ℓ *is once continuously Fréchet differentiable with the Fréchet derivative along the optimal trajectory* $\ell_y(\cdot, Lx_o(\cdot)) \in L_1^a(I, \mathcal{Y})$. *Then, in order that the pair* $\{A_o, B_o\} \in G \times \Gamma$ *be optimal, it is necessary that there exists a pair of solutions* $\{x_o, \psi\} \in B_\infty^a(I, \mathcal{X})$ *of the evolution equations*

$$\begin{aligned} dx_o &= (A_o x_o + B_o L x_o)dt + f(x_o(t))dt + CdW, \quad x(0) \equiv \xi, \\ d\psi &= -\{A_o^*\psi + (B_o L)^*\psi + (Df(x_o))^*\}dt - L^*\ell_y(t, Lx_o)dt, \psi(T) = 0, \end{aligned} \tag{6.1}$$

and that the following inequalities hold:

$$\begin{aligned} &\mathcal{E}\int_I < (A - A_o)^*\psi, x_o >_X dt \le 0, \ \forall\ A \in G, \\ &\mathcal{E}\int_I < (B - B_o)^*\psi, Lx_o >_Y dt \ge 0, \ \forall\ B \in \Gamma. \end{aligned} \tag{6.2}$$

Proof. (Outline of an informal proof): Clearly the pair $\{A_o, B_o\}$ is optimal if and only if the functional η, given by the expression (4.19), satisfies the following inequalities:

$$\eta(B_o, A) \le \eta(B_o, A_o) \le \eta(B, A_o) \ \forall\ \{B, A\} \in \Gamma \times G. \tag{6.3}$$

In other words the pair $\{A_o, B_o\}$ is a saddle point for η. Now using the left hand inequality of (6.3) and the convexity of the set G, it is easy to verify that

$$\mathcal{E}\int_I < L^*\ell_y(t, Lx_o(t)), z(t) >_X dt \le 0,$$

where $z \in B_\infty^a(I, \mathcal{X})$ is the strong solution of the variational evolution equation

$$dz = [A_o z + B_o L z + Df(x_o)z]dt + (A - A_o)x_o dt, z(0) = 0$$

with $(A - A_o)x_o \in L_1^a(I, \mathcal{X})$. Then it follows from continuity of the functional

$$(A - A_o)x_o \longrightarrow z \longrightarrow \mathcal{E}\int_I < L^*\ell_y(t, Lx_o(t)), z > dt$$

on $L_1^a(I, \mathcal{X})$ that there exists a $\psi \in B_\infty^a(I, \mathcal{X}) \subset L_\infty(I, \mathcal{X})$ such that

$$\mathcal{E}\int_I < (A - A_o)^*\psi, x_o >_X dt \le 0, \ \forall\ A \in G.$$

The function ψ can be chosen as the strong solution of equation (6.1b). Next, using the inequality on the right hand side of the expression (6.3) and the convexity of the set Γ and similar arguments, one can verify the second inequality of (6.2). This way we obtain all the necessary conditions of optimality as stated in the theorem. □

Remark 17. In general, the evolution equations (6.1) do not have strong solutions unless $\{f, C, \ell\}$ satisfy additional properties. Hence our proof of Theorem 16 is rather informal. Here, Yosida regularization may play a significant role. If we are satisfied with ε-optimal solution, we can certainly develop necessary conditions of ε-optimality with formal proof. And this can be done by using Yosida regularization of the set Γ and the operators $\{f, C\}$ including the Fréchet derivative $L^*\ell_y$. We leave it as an open problem.

References

[1] N. U. Ahmed, *Semigroup Theory with Applications to Systems and Control*, Pitman Research Notes in Mathematics Series, Vol. 246, Longman Scientific and Technical Copublished with John Wiley and Sons, Inc., New York, 1991.

[2] N. U. Ahmed, *Optimization and Identification of Systems Governed by Evolution equations on Banach Spaces*, Pitman Research Notes in Mathematics Series, Vol. 184, Longman Scientific and Technical Copublished with John Wiley and Sons, Inc., New York, 1988.

[3] N. U. Ahmed, *Generalized solutions of HJB equations applied to stochastic control on Hilbert space*, Nonlinear Analysis 54 (2003), 495-523.

[4] N. U. Ahmed, *Dynamic: Systems and Control with Applications*, World Scientific Publ., New Jersey, London, Singapore, Beijing, Shanghai, Hong Kong, Taipei, Chennai, 2006.

[5] N. U. Ahmed, *Optimal control of ∞-dimensional stochastic systems via generalized solutions of HJB equations*, Discussiones Mathematicae Differential Inclusions, Control and Optimization 21 (2001), 97–126.

[6] N. U. Ahmed, *Optimal choice of nonlinear output feedback control law for a class of uncertain parabolic systems*, Dynam. Systems Appl. 17, no.3-4 (2008), 571–582.

[7] N. U. Ahmed, *Optimal output feedback boundary control for systems governed by semilinear parabolic inclusions: uncertain systems.* Adv. Nonlinear Var. Inequal. 11(1) (2008), 61–79.

[8] N. U. Ahmed and X. Xiang, *Nonlinear uncertain systems and necessary conditions of optimality.* SIAM J. Control Optim. 35(5) (1997), 1755–1772.

[9] N. U. Ahmed and X. Xiang, *Optimal control of infinite-dimensional uncertain systems.* J. Optim. Theory Appl. 80(2) (1994), 261–272.

[10] N. U. Ahmed, *Optimal Relaxed Controls for Systems Governed by Impulsive Differential Inclusions*, Nonlinear Functional Analysis & Applications, 10(3) (2005), 427–460

[11] N. U. Ahmed and C. D. Charalambous, *Minimax games for stochastic systems subject to relative entropy uncertainty: applications to SDE's on Hilbert Spaces*, J. Mathematics of Control, Signals and Systems, 19 (2007), 65–91.

[12] V. S. Borkar and T. E. Govindan, *Optimal control of semilinear stochastic evolution equations.* Nonlinear Anal. 23(1) (1994), 15–35.

[13] R. Curtain, K. Mikkola and A. Sasane, *The Hilbert–Schmidt property of feedback operators*, J. Math. Anal. Appl., 329 (2007), 1145–1160.
[14] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer, 1995.
[15] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge Univ. Press, Cambridge, 1992.
[16] H. O. Fattorini, *Infinite-dimensional optimization and control theory. Encyclopedia of Mathematics and its Applications*, 62. Cambridge University Press, Cambridge, 1999.
[17] B. B. King, *Representation of feedback operators for parabolic control problems*, Proc. AMS, 128(5) (2000), 1339–1346.
[18] B. Mordukhovich, *Suboptimal feedback control design of constrained parabolic systems in uncertain conditions*, Research Report no. 15, 2006.
[19] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, New York and London, 1967.
[20] I. R. Petersen, *Robust output feedback guaranteed cost control of nonlinear stochastic uncertain systems via an IQC approach.* IEEE Trans. Automat. Control 54, no. 6 (2009), 1299–1304.