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ON THE SEMILOCAL CONVERGENCE OF MODIFIED NEWTON-TIKHONOV REGULARIZATION METHOD FOR NONLINEAR ILL-POSED PROBLEMS

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Abstract. In this study, we introduce a new modified Newton-Tikhonov method for approximating a solution of nonlinear ill-posed problems. The proposed iteration converges quadratically. Order optimal error bounds are given in case the regularization parameter is chosen a priori and by the adaptive method of Pereverzev and Schock(2005).

1. Introduction

In this study we are concerned with the problem of approximately solving the nonlinear ill-posed operator equation

$$F(x) = f, \tag{1.1}$$

where $F : D(F) \subseteq X \to Y$ is a nonlinear operator between the Hilbert spaces X and Y. Here and below $\langle ., . \rangle$ denote the inner product and $\|.\|$ denote the corresponding norm. We assume throughout that $f^\delta \in Y$ are the available data with

$$\|f - f^\delta\| \leq \delta$$

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and (1.1) has a solution $\hat{x}$ (which need not be unique). Then the problem of recovery of $\hat{x}$ from noisy equation $F(x) = f^{\delta}$ is ill-posed, in the sense that a small perturbation in the data can cause large deviation in the solution.

Further it is assumed that F possesses a locally uniformly bounded Fréchet derivative $F'(.)$ in the domain $D(F)$ of F. A large number of problems in mathematical physics and engineering are solved by finding the solutions of equations in a form like (1.1). If one works with such problems, the measurement data will be distorted by some measurement error. Therefore, one has to consider appropriate regularization techniques for approximately solving (1.1).

Iterative regularization methods are used for approximately solving (1.1). Recall ([13]) that, an iterative method with iterations defined by

$$x_{k+1}^{\delta} = \Phi(x_0^{\delta}, x_1^{\delta}, \cdots, x_k^{\delta}; y^{\delta}),$$

where $x_0^{\delta} := x_0 \in D(F)$ is a known initial approximation of $\hat{x}$, for a known function Φ together with a stopping rule which determines a stopping index $k_{\delta} \in \mathbb{N}$ is called an iterative regularization method if $\|x_{k_{\delta}}^{\delta} - \hat{x}\| \to 0$ as $\delta \to 0$.

The Levenberg-Marquardt method([17], [18], [19], [20], [21], [22], [23], [24]) and iteratively regularized Gauss-Newton method (IRGNA) ([3], [12]) are the well-known iterative regularization methods. In Levenberg-Marquardt method, the iterations are defined by,

$$x_{k+1}^{\delta} = x_k^{\delta} - (A_{k,\delta}^* A_{k,\delta} + \alpha_k I)^{-1} A_{k,\delta}^* (F(x_k^{\delta}) - y^{\delta}), \tag{1.2}$$

where $A_{k,\delta}^* := F'(x_k^{\delta})^*$ is as usual the adjoint of $A_{k,\delta} := F'(x_k^{\delta})$ and (α_k) is a positive sequence of regularization parameter ([12]). In Gauss-Newton method, the iterations are defined by

$$x_{k+1}^{\delta} = x_k^{\delta} - (A_{k,\delta}^* A_{k,\delta} + \alpha_k I)^{-1} [A_{k,\delta}^* (F(x_k^{\delta}) - y^{\delta}) + \alpha_k (x_k^{\delta} - x_0)] \tag{1.3}$$

where $x_0^{\delta} := x_0$ and (α_k) is as in (1.2).

In [3], Bakushinskii obtained local convergence of the method (1.3), under the smoothness assumption

$$\hat{x} - x_0 = (F'(\hat{x})^* F'(\hat{x}))^{\nu} w, \qquad w \in N(F'(\hat{x}))^{\perp} \tag{1.4}$$

with $\nu \geq 1, w \neq 0$ and $F'(.)$ is Lipschitz continuous; $N(F'(\hat{x}))$ denotes the nullspace of $F'(\hat{x})$. For noise free case Bakushinskii ([3]) obtained the rate

$$\|x_k^{\delta} - \hat{x}\| = O(\alpha_k),$$

and Blaschke et.al.([12]) obtained the rate

$$\|x_k^{\delta} - \hat{x}\| = O(\alpha_k^{\nu}), \tag{1.5}$$

for $\frac{1}{2} \leq \nu < 1$.

It is proved in [12], that the rate (1.5) can be obtained for $0 \leq \nu < \frac{1}{2}$ provided $F'(.)$ satifies the following conditions:

$$F'(\bar{x}) = R(\bar{x}, x)F'(x) + Q(\bar{x}, x),$$
$$\|I - R(\bar{x}, x)\| \leq C_R, \quad \bar{x}, x \in B_{2\rho}(x_0),$$
$$\|Q(\bar{x}, x)\| \leq C_Q \|F'(\hat{x})(\bar{x} - x)\|$$

with ρ, C_R and C_Q sufficiently small. In fact in [12], Blaschke et.al. obtained the rate

$$\|x_k^\delta - \hat{x}\| = o(\alpha_k^{\frac{2\nu}{2\nu+1}}), \quad 0 \leq \nu < \frac{1}{2}$$

by choosing the stopping index k_δ according to the discrepancy principle

$$\|F(x_{k_\delta}^\delta) - y^\delta\| \leq \tau\delta < \|F(x_k^\delta) - y^\delta\|, \quad 0 \leq k < k_\delta$$

with $\tau > 1$ chosen sufficiently large. Subsequently, many authors extended, modified, and generalized Bakushinskii's work to obtain error bounds under various contexts(see [4], [5], [6], [7], [8], [9], [10]).

In [13], Mahale and Nair considered a method in which the iterations are defined by

$$x_{k+1}^\delta = x_0 - g_{\alpha_k}(A_0^* A_0) A_0^* [F(x_k^\delta) - y^\delta - A_0(x_k^\delta - x_0)], \quad x_0^\delta := x_0 \tag{1.6}$$

where $A_0 := F'(x_0), (\alpha_k)$ is a sequence of regularization parameters which satisfies,

$$\alpha_k > 0, \quad 1 \leq \frac{\alpha_k}{\alpha_{k+1}} \leq \mu_1, \quad \lim_{k \to 0} \alpha_k = 0 \tag{1.7}$$

for some constant $\mu_1 > 1$ and each g_α, for $\alpha > 0$ is a positive real-valued piecewise continuous function defined on $[0, M]$ with $M \geq \|A_0\|^2$. They choose the stopping index k_δ for this iteration as the positive integer which satisfies

$$\max\{\|F(x_{k_\delta - 1}^\delta) - y^\delta\|, \tilde{\beta_{k_\delta}}\} \leq \tau\delta < \max\{\|F(x_{k-1}^\delta) - y^\delta\|, \tilde{\beta_k}\}, \;\; 1 \leq k < k_\delta$$

where $\tau > 1$ is a sufficiently large constant not depending on δ, and

$$\tilde{\beta_k} := \|F(x_{k-1}^\delta) - y^\delta + A_0(x_k^\delta - x_{k-1}^\delta)\|.$$

In fact, Mahle and Nair obtained an order optimal error estimate, in the sense that an improved order estimate which is applicable for the case of linear ill-posed problems as well is not possible, under the following new source condition on $x_0 - \hat{x}$.

Assumption 1.1. *There exists a continuous, stricly monotonically increasing function* $\varphi : (0, M] \to (0, \infty)$ *satisfying* $\lim_{\lambda \to 0} \varphi(\lambda) = 0$ *and* $\rho_0 > 0$ *such that*

$$x_0 - \hat{x} = [\varphi(A_0^* A_0)]^{1/2} w \tag{1.8}$$

for some $w \in X$ *with* $\|w\| \leq \rho_0$.

In [10], the author considered a particular case of this method, namely, regularized modified Newton's method defined iteratively by

$$x_{k+1}^{\delta} = x_k^{\delta} - (A_0^*A_0 + \alpha I)^{-1}[A_0^*(F(x_k^{\delta}) - y^{\delta}) + \alpha(x_k^{\delta} - x_0)], \quad x_0^{\delta} := x_0 \tag{1.9}$$

for approximately solving (1.1). Using a suitably constructed majorizing sequence (see, [1], p.28), it is proved that the sequence(x_k^{δ}) converges linearly to a solution x_{α}^{δ} of the equation

$$A_0^*F(x_{\alpha}^{\delta}) + \alpha(x_{\alpha}^{\delta} - x_0) = A_0^*y^{\delta} \tag{1.10}$$

and that x_{α}^{δ} is an approximation of $\hat{x}$. The error estimate in this paper was obtained under the following source condition on $x_0 - \hat{x}$.

Assumption 1.2. *There exists a continuous, stricly monotonically increasing function* $\varphi : (0, a_1] \to (0, \infty)$ *with* $a_1 \geq \|F'(\hat{x})\|^2$ *satisfying*

(1) $\lim_{\lambda\to 0} \varphi(\lambda) = 0$;
(2) *for* $\alpha \leq 1, \varphi(\alpha) \geq \alpha$;
(3) $\sup_{\lambda\geq 0} \frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \leq c_{\varphi}\varphi(\alpha), \quad \forall\, \lambda \in (0, a_1]$;
(4) *there exists* $w \in X$ *such that*

$$x_0 - \hat{x} = \varphi(F'(\hat{x})^*F'(\hat{x}))w. \tag{1.11}$$

Later in [11], using a two step Newton method (see, [2]), the author proved that the sequence (x_k^{δ}) in (1.9) converges linarly to the solution x_{α}^{δ} of (1.10). The error estimate in [11] was based on the following source condition

$$x_0 - \hat{x} = \varphi(A_0^*A_0)w,$$

where φ is as in Assumption 1.1 with $a_1 \geq \|A_0\|^2$. In the present paper we improve the semilocal convergence by modifying the method (1.9).

1.1. **The new method.**

In this study we define a new iteration procedure

$$x_{n+1,\alpha}^{\delta} = x_{n,\alpha}^{\delta} - (A_0^*A_n + \alpha I)^{-1}[A_0^*(F(x_{n,\alpha}^{\delta}) - y^{\delta}) + \alpha(x_{n,\alpha}^{\delta} - x_0)], \quad x_{0,\alpha}^{\delta} := x_0 \tag{1.12}$$

where $A_n := F'(x_{n,\alpha}^{\delta})$ and $\alpha > 0$ is the regularization parameter. Using an assumption on the Fréchet derivative of F we prove that the iteration in (1.12) converges quadratically to the solution x_{α}^{δ} of (1.10).

Recall ([14]) that, a sequence (x_n) is said to converge quadratically to x^* if there exists positive reals β, γ such that

$$\|x_{n+1} - x^*\| \leq \beta e^{-\gamma 2^n}$$

for all $n \in \mathbb{N}$. And the convergence of (x_n) to x^* is said to be linear if there exists a positive number $M_0 \in (0, 1)$, such that

$$\|x_{n+1} - x^*\| \leq M_0\|x_n - x^*\|.$$

Quadratically convergent sequence will always eventually converge faster than a linear convergent sequence.

We choose the regularization parameter α from some finite set

$$\{\alpha_0 < \alpha_1 < \cdots < \alpha_N\}$$

using the balancing principle considered by Perverzev and Schock in [15].

The rest of this paper is organized in the following way. In Section 2 we provide the convergence analysis of the proposed method and in Section 3 we provide the error analysis. Finally in Section 4 we provide the details for implementing the method and the algorithm.

2. Convergence Analysis of (1.12)

The following assumption is used extensively for proving the results in this paper.

Assumption 2.1. *There exists a constant $k_0 > 0, r > 0$ such that for every $x, u \in B(x_0, r) \cup B(\hat{x}, r) \subset D(F)$ and $v \in X$, there exists an element $\Phi(x, u, v) \in X$ such that*

$$[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \qquad \|\Phi(x, u, v)\| \leq k_0\|v\|\|x - u\|.$$

In view of Assumption 2.1 there exists an element $\Phi_0(x, x_0, v) \in X$ such that

$$[F'(x) - F'(x_0)]v = F'(x_0)\Phi_0(x, x_0, v), \qquad \|\Phi_0(x, x_0, v)\| \leq l_0\|v\|\|x - x_0\|.$$

Note that

$$l_0 \leq k_0$$

holds in general and $\frac{k_0}{l_0}$ can be arbitrarily large [1], [2]. Let $\delta_0 < \sqrt{\alpha_0}$,

$$\rho < \frac{\sqrt{1 + 2l_0(1 - \frac{\delta_0}{\sqrt{\alpha_0}})} - 1}{l_0},$$

and

$$\gamma_\rho := \frac{l_0}{2}\rho^2 + \rho + \frac{\delta_0}{\sqrt{\alpha_0}}.$$

For $r \leq \frac{2-3k_0}{(2+3l_0)k_0}$, $k_0 \leq \frac{2}{3}$, let $g : (0, 1) \to (0, 1)$ be the function defined by

$$g(t) := \frac{3(1 + l_0 r)k_0}{2(1 - l_0 r)}t, \quad \forall\, t \in (0, 1).$$

Lemma 2.2. *Let $l_0 r < 1$ and $u \in B_r(x_0)$. Then $(A_0^* A_u + \alpha I)$ is invertible:*

(i) $(A_0^* A_u + \alpha I)^{-1} = [I + (A_0^* A_0 + \alpha I)^{-1} A_0^*(A_u - A_0)]^{-1}(A_0^* A_0 + \alpha I)^{-1}$

(ii) $\|(A_0^* A_u + \alpha I)^{-1} A_0^* A_u\| \leq \frac{1 + l_0 r}{1 - l_0 r}$, *where* $A_u := F'(u)$.

Proof. Note that by Assumption 2.1, we have

$$\begin{aligned}\|(A_0^*A_0+\alpha I)^{-1}A_0^*(A_u-A_0)\| &= \sup_{\|v\|\leq 1}\|(A_0^*A_0+\alpha I)^{-1}A_0^*(A_u-A_0)v\| \\ &= \sup_{\|v\|\leq 1}\|(A_0^*A_0+\alpha I)^{-1}A_0^*A_0\Phi_0(u,x_0,v)\| \\ &\leq l_0\|u-x_0\| \leq l_0 r < 1.\end{aligned}$$

So $I + (A_0^*A_0+\alpha I)^{-1}A_0^*(A_u-A_0)$ is invertible. Now (i) follows from the following relation

$$A_0^*A_u+\alpha I = (A_0^*A_0+\alpha I)[I+(A_0^*A_0+\alpha I)^{-1}A_0^*(A_u-A_0)].$$

To prove (ii), observe that by Assumption 2.1 and (i), we have

$$\begin{aligned}\|(A_0^*A_u+\alpha I)^{-1}A_0^*A_u\| &= \sup_{\|v\|\leq 1}\|(A_0^*A_u+\alpha I)^{-1}A_0^*A_u v\| \\ &= \sup_{\|v\|\leq 1}\|(A_0^*A_u+\alpha I)^{-1}A_0^*(A_u-A_0+A_0)v\| \\ &= \sup_{\|v\|\leq 1}\|[I+(A_0^*A_0+\alpha I)^{-1}A_0^*(A_u-A_0)]^{-1} \\ &\quad (A_0^*A_0+\alpha I)^{-1}A_0^*(A_u-A_0+A_0)v\| \\ &\leq \frac{1}{1-k_0r}[\|(A_0^*A_0+\alpha I)^{-1}A_0^*A_0\Phi_0(u,x_0,v)\| \\ &\quad +\|(A_0^*A_0+\alpha I)^{-1}A_0^*A_0v\|] \\ &\leq \frac{1+l_0r}{1-l_0r}.\end{aligned}$$

This completes the proof. □

Theorem 2.3. *Suppose Assumption* 2.1 *holds. Let* $\frac{\gamma_\rho}{1-g(\gamma_\rho)} \leq r \leq \frac{2-3k_0}{(2+3l_0)k_0}$, $\delta \in (0,\delta_0]$. *Then the sequence* $(x_{n,\alpha}^\delta)$ *defined in* (1.12) *is a Cauchy sequence in* $B_r(x_0)$ *and hence converges to* $x_\alpha^\delta \in \overline{B_r(x_0)}$. *Further* x_α^δ *satisfies* (1.10) *and the following estimate holds for all* $n \geq 0$;

$$\|x_{n,\alpha}^\delta - x_\alpha^\delta\| \leq re^{-\gamma 2^n}, \tag{2.1}$$

where $\gamma = -ln(g(\gamma_\rho))$.

Proof. Suppose $x_{n,\alpha}^\delta \in B_r(x_0), \forall n \geq 0$. Then

$$\begin{aligned}
& x^\delta_{n+1,\alpha} - x^\delta_{n,\alpha} \qquad (2.2)\\
&= (A_0^*A_n + \alpha I)^{-1}[A_0^*A_n(x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}) - A_0^*(F(x^\delta_{n,\alpha}) - F(x^\delta_{n-1,\alpha}))]\\
&\quad +(A_0^*A_n + \alpha I)^{-1}A_0^*(A_n - A_{n-1})(A_0^*A_{n-1} + \alpha I)^{-1}\\
&\quad \times[A_0^*(F(x^\delta_{n-1,\alpha}) - y^\delta) + \alpha(x^\delta_{n-1,\alpha} - x_0)]\\
&= (A_0^*A_n + \alpha I)^{-1}A_0^*[A_n(x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}) - (F(x^\delta_{n,\alpha}) - F(x^\delta_{n-1,\alpha}))]\\
&\quad +(A_0^*A_n + \alpha I)^{-1}A_0^*(A_n - A_{n-1})(x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha})\\
&:= \zeta_1 + \zeta_2 \qquad (2.3)
\end{aligned}$$

where

$$\zeta_1 = (A_0^*A_n + \alpha I)^{-1}A_0^*[A_n(x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}) - (F(x^\delta_{n,\alpha}) - F(x^\delta_{n-1,\alpha}))],$$

and

$$\zeta_2 = (A_0^*A_n + \alpha I)^{-1}A_0^*(A_n - A_{n-1})(x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}).$$

So by Fundamental Theorem of Integration,

$$\zeta_1 = (A_0^*A_n+\alpha I)^{-1}A_0^*\Big[\int_0^1 (A_n-F'(x^\delta_{n-1,\alpha}+t(x^\delta_{n,\alpha}-x^\delta_{n-1,\alpha})dt\Big](x^\delta_{n,\alpha}-x^\delta_{n-1,\alpha})$$

and hence by Assumption 2.1 and Lemma 2.2,

$$\begin{aligned}
&\|\zeta_1\|\\
&\le \|(A_0^*A_n + \alpha I)^{-1}A_0^*A_n \int_0^1 \Phi(x^\delta_{n-1,\alpha} + t(x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}), x^\delta_{n,\alpha},\\
&\quad x^\delta_{n-1,\alpha} - x^\delta_{n,\alpha})dt\|\\
&\le \frac{1+l_0r}{1-l_0r}\int_0^1 \Phi(x^\delta_{n-1,\alpha} + t(x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}), x^\delta_{n,\alpha}, x^\delta_{n-1,\alpha} - x^\delta_{n,\alpha})dt\|\\
&\le \frac{(l_0r+1)k_0}{2(1-l_0r)}\|x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}\|^2. \qquad (2.4)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|\zeta_2\| &\le \|(A_0^*A_n + \alpha I)^{-1}A_0^*(A_n - A_{n-1})(x^\delta_{n-1,\alpha} - x^\delta_{n,\alpha})\|\\
&\le \|(A_0^*A_n + \alpha I)^{-1}A_0^*A_n\Phi(x^\delta_{n,\alpha}, x^\delta_{n-1,\alpha}, x^\delta_{n-1,\alpha} - x^\delta_{n,\alpha})\|\\
&\le \frac{(1+l_0r)k_0}{1-l_0r}\|x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}\|^2. \qquad (2.5)
\end{aligned}$$

So by (2.3), (2.4)and (2.5), we have

$$\begin{aligned}
\|x^\delta_{n+1,\alpha} - x^\delta_{n,\alpha}\| &\le \frac{3(1+l_0r)k_0)}{2(1-l_0r)}\|x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}\|^2\\
&\le g(e_n)e_n, \qquad (2.6)
\end{aligned}$$

where

$$e_n := \|x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}\|, \quad n = 1, 2, \cdots .$$

Now using induction we shall prove that $x^\delta_{n,\alpha} \in B_r(x_0)$. Note that

$$\begin{aligned}
e_1 &= \|x^\delta_{1,\alpha} - x_0\| \\
&= \|(A_0^*A_0 + \alpha I)^{-1}A_0^*(F(x_0) - y^\delta)\| \\
&= \|(A_0^*A_0 + \alpha I)^{-1}A_0^*(F(x_0) - F(\hat{x}) - F'(x_0)(x_0 - \hat{x}) \\
&\quad +F'(x_0)(x_0 - \hat{x}) + F(\hat{x}) - y^\delta)\| \\
&\leq \|(A_0^*A_0 + \alpha I)^{-1}A_0^*(\int_0^1 [F'(\hat{x} + t(x_0 - \hat{x})) - F'(x_0)](x_0 - \hat{x})dt \\
&\quad +F'(x_0)(x_0 - \hat{x}) + F(\hat{x}) - y^\delta)\| \\
&\leq \|(A_0^*A_0 + \alpha I)^{-1}A_0^*A_0(\int_0^1 \Phi(x_0, \hat{x} + t(x_0 - \hat{x}), x_0 - \hat{x})\| \\
&\quad +\|(A_0^*A_0 + \alpha I)^{-1}A_0^*F'(x_0)(x_0 - \hat{x})\| \\
&\quad +\|(A_0^*A_0 + \alpha I)^{-1}A_0^*(F(\hat{x}) - y^\delta)\| \\
&\leq \frac{l_0}{2}\rho^2 + \rho + \frac{\delta}{\sqrt{\alpha}} \qquad (2.7) \\
&\leq \gamma_\rho \leq r
\end{aligned}$$

i.e., $x^\delta_{1,\alpha} \in B_r(x_0)$. Now since $\gamma_\rho < 1$, by (2.7), $e_1 < 1$. Therefore by (2.6) and the fact that $g(\mu t) \leq \mu g(t)$, for all $t \in (0, 1)$, we have that $e_n < 1$, $\forall\, n \geq 1$ and

$$g(e_1)^{2^n-1}e_1.$$

Now suppose $x^\delta_{k,\alpha} \in B_r(x_0)$ for some k. Then

$$\begin{aligned}
\|x^\delta_{k+1,\alpha} - x_0\| &\leq \|x^\delta_{k+1,\alpha} - x^\delta_{k,\alpha}\| + \|x^\delta_{k,\alpha} - x^\delta_{k-1,\alpha}\| + \cdots + \|x^\delta_{1,\alpha} - x_0\| \\
&\leq (g(e_1)^{2^k-1} + g(e_1)^{2^{k-1}-1} + \cdots + 1)e_1 \\
&\leq \frac{e_1}{1 - g(e_1)} \\
&\leq \frac{\gamma_\rho}{1 - g(\gamma_\rho)} \leq r.
\end{aligned}$$

Thus by induction $x^\delta_{n,\alpha} \in B_r(x_0)$, $\forall\, n \geq 0$.

Next we shall prove that $(x^\delta_{k+1,\alpha})$ is a Cauchy sequence in $B_r(x_0)$.

$$\begin{aligned}\|x^\delta_{n+m,\alpha}-x^\delta_{n,\alpha}\| &\leq \sum_{i=0}^{m}\|x^\delta_{n+i+1,\alpha}-x^\delta_{n+i,\alpha}\| \qquad (2.8)\\ &\leq \sum_{i=0}^{m} g(e_1)^{2^{n+i}-1}e_1\\ &\leq g(e_1)^{2^n-1}e_1(1+g(e_1)^2+\cdots+g(e_1)^{2^m})\\ &\leq \frac{g(e_1)^{2^n-1}e_1}{1-g(e_1)} \leq \frac{g(\gamma_\rho)^{2^n-1}\gamma_\rho}{1-g(\gamma_\rho)} \leq re^{-\gamma 2^n}. \qquad (2.9)\end{aligned}$$

Thus $(x^\delta_{n,\alpha})$ is a Cauchy sequence in $B_r(x_0)$ and hence converges, say to $x^\delta_\alpha \in \overline{B_r(x_0)}$. Further by letting $n\to\infty$ in (1.12) we obtain

$$F'(x_0)^*(F(x^\delta_\alpha)-y^\delta)+\alpha(x^\delta_\alpha-x_0)=0.$$

The estimate in (2.1) follows by letting m tends to ∞ in (2.9). □

Remark 2.4. Note that if $r\in(r_1,r_2)$ where

$$r_1 := \frac{2+(2l_0-3k_0)\gamma_\rho-\sqrt{(4l_0^2+9k_0^2-36k_0l_0)\gamma_\rho^2-(12k_0+8l_0)\gamma_\rho+4}}{2l_0(2+3k_0\gamma_\rho)}$$

and

$$r_2 := \min\left\{\frac{2+(2l_0-3k_0)\gamma_\rho+\sqrt{(4l_0^2+9k_0^2-36k_0l_0)\gamma_\rho^2-(12k_0+8l_0)\gamma_\rho+4}}{2l_0(2+3k_0\gamma_\rho)},\ \frac{2-3k_0}{(2+3l_0)k_0}\right\},$$

with $\gamma_\rho \leq c_{l_0k_0} := \min\left\{1, \frac{\sqrt{(8l_0-12k_0)^2+16(36k_0l_0-9k_0-4l_0)}-(8l_0+12k_0)}{2(36k_0l_0-9k_0^2-4l_0^2)}\right\}$ then $\frac{\gamma_\rho}{1-g(\gamma_\rho)}\leq r$ and $l_0r<1$.

3. Error analysis

We use the following assumption to obtain an error estimate for $\|x^\delta_\alpha-\hat{x}\|$.

Assumption 3.1. *There exists a continuous, strictly monotonically increasing function $\varphi:(0,a]\to(0,\infty)$ with $a\geq\|F'(x_0)\|^2$ satisfying;*

(i) $\lim_{\lambda\to 0}\varphi(\lambda)=0$.

(ii) $\sup_{\lambda\geq 0}\frac{\alpha\varphi(\lambda)}{\lambda+\alpha}\leq\varphi(\alpha), \quad \forall\,\lambda\in(0,a]$.

(iii) *there exists* $v \in X$ *such that*

$$x_0 - \hat{x} = \varphi(A_0^* A_0)v.$$

Theorem 3.2. *Let* x_α^δ *be as in* (1.10). *Then*

$$\|x_\alpha^\delta - \hat{x}\| \leq \frac{\max\{1, \|v\|\}}{1-q}\Big(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\Big),$$

where $q = l_0 r$.

Proof. Let $M = \int_0^1 F'(\hat{x} + t(x_\alpha^\delta - \hat{x}))dt$. Then

$$F(x_\alpha^\delta) - F(\hat{x}) = M(x_\alpha^\delta - \hat{x})$$

and hence by (1.10), we have

$$(A_0^* M + \alpha I)(x_\alpha^\delta - \hat{x}) = A_0^*(y^\delta - y) + \alpha(x_0 - \hat{x}).$$

Thus

$$\begin{aligned} & x_\alpha^\delta - \hat{x} \\ &= (A_0^* A_0 + \alpha I)^{-1}[A_0^*(y^\delta - y) + \alpha(x_0 - \hat{x}) + A_0^*(A_0 - M)(x_\alpha^\delta - \hat{x})] \\ &= s_1 + s_2 + s_3, \end{aligned} \tag{3.1}$$

where $s_1 := (A_0^* A_0 + \alpha I)^{-1} A_0^*(y^\delta - y), s_2 := (A_0^* A_0 + \alpha I)^{-1}\alpha(x_0 - \hat{x})$ and $s_3 := (A_0^* A_0 + \alpha I)^{-1} A_0^*(A_0 - M)(x_\alpha^\delta - \hat{x})$. Note that

$$\|s_1\| \leq \frac{\delta}{\sqrt{\alpha}}, \tag{3.2}$$

by Assumption 3.1

$$\|s_2\| \leq \varphi(\alpha)\|v\| \tag{3.3}$$

and by Assumption 2.1

$$\|s_3\| \leq l_0 r \|x_\alpha^\delta - \hat{x}\|. \tag{3.4}$$

The result now follows from (3.1), (3.2), (3.3) and (3.4). □

3.1. **Error bounds under source conditions.** Combining the estimates in Theorem 2.3 and Theorem 3.2 we obtain the following.

Theorem 3.3. *Let the assumptions in Theorem* 2.3 *and Theorem* 3.2 *hold and let* $x_{n,\alpha}^\delta$ *be as in* (1.12). *Then*

$$\|x_{n,\alpha}^\delta - \hat{x}\| \leq r e^{-\gamma 2^n} + \frac{\max\{1, \|v\|\}}{1-q}\Big(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\Big).$$

Further if $n_\delta := \min\left\{n : e^{-\gamma 2^n} < \frac{\delta}{\sqrt{\alpha}}\right\}$, *then*

$$\|x^\delta_{n_\delta,\alpha} - \hat{x}\| \le \tilde{C}\Big(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\Big)$$

where $\tilde{C} := r + \frac{\max\{1,\|v\|\}}{1-q}$.

3.2. **A priori choice of the parameter.** Observe that the estimate $\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)$ in Theorem 3.3 is of optimal order for the choice $\alpha := \alpha_\delta$ which satisfies $\frac{\delta}{\sqrt{\alpha_\delta}} = \varphi(\alpha)$. Now, using the function $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}$, $0 < \lambda \le a$, we have $\delta = \sqrt{\alpha}\varphi(\alpha) = \psi(\varphi(\alpha))$ so that $\alpha_\delta = \varphi^{-1}[\psi^{-1}(\delta)]$.

Theorem 3.4. *Let* $\psi(\lambda) = \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \le a$ *and assumptions in Theorem* 3.3 *holds. For* $\delta > 0$, *let* $\alpha_\delta = \varphi^{-1}[\psi^{-1}(\delta)]$ *and let* n_δ *be as in Theorem* 3.3*. Then*

$$\|x^\delta_{n_\delta,\alpha} - \hat{x}\| = O(\psi^{-1}(\delta)).$$

3.3. **Adaptive choice of the parameter.** In the balancing principle considered by Pereverzev and Schock in [15], the regularization parameter $\alpha = \alpha_i$ are selected from some finite set

$$D_N := \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N\}.$$

Let

$$n_i = \min\left\{n : e^{-\gamma 2^n} \le \frac{\delta}{\sqrt{\alpha_i}}\right\}$$

and let $x^\delta_{\alpha_i} := x^\delta_{n_i,\alpha_i}$ where $x^\delta_{n_i,\alpha_i}$ be as in (1.12) with $\alpha = \alpha_i$ and $n = n_i$. Then from Theorem 3.3, we have

$$\|x^\delta_{\alpha_i} - \hat{x}\| \le \tilde{C}\Big(\frac{\delta}{\sqrt{\alpha_i}} + \varphi(\alpha_i)\Big), \quad \forall\, i = 1, 2, \cdots N.$$

Precisely we choose the regularization parameter $\alpha = \alpha_k$ from the set D_N defined by

$$D_N := \{\alpha_i = \mu^i\alpha_0, i = 1, 2, \cdots N\}$$

where $\mu > 1$.

To obtain a conclusion from this parameter choice we considered all possible functions φ satisfying Assumption 2.1 and $\varphi(\alpha_i) \le \frac{\delta}{\sqrt{\alpha_i}}$. Any of such functions is called admissible for $\hat{x}$ and it can be used as a measure for the convergence of $x^\delta_\alpha \to \hat{x}$ (see [16]).

The main result of this section is the following theorem, proof of which is analogous to the proof of Theorem 4.4 in [10].

Theorem 3.5. *Assume that there exists* $i \in \{0, 1, \cdots, N\}$ *such that* $\varphi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}}$. *Let assumptions of Theorem* 3.3 *be satisfied and let*

$$l := \max\left\{i : \varphi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}}\right\} < N,$$

$$k = \max\left\{i : \forall j = 1, 2, \cdots, i; \|x^\delta_{\alpha_i} - x^\delta_{\alpha_j}\| \leq 4\tilde{C}\frac{\delta}{\sqrt{\alpha_j}}\right\}$$

where $\tilde{C}$ *is as in Theorem* 3.3. *Then* $l \leq k$ *and*

$$\|x^\delta_{\alpha_k} - \hat{x}\| \leq 6\tilde{C}\mu\psi^{-1}(\delta).$$

4. Implementation of the method

Finally the balancing algorithm associated with the choice of the parameter specified in Theorem 3.5 involves the following steps:

- Choose $\alpha_0 > 0$ such that $\delta_0 < c_{k_0 l_0}\sqrt{\alpha_0}$ and $\mu > 1$.
- Choose N big enough but not too large and $\alpha_i := \mu^i \alpha_0, i = 0, 1, 2, \cdots, N$.
- Choose $\rho \leq \frac{\sqrt{1 + 2l_0(c_{k_0 l_0} - \frac{\delta_0}{\sqrt{\alpha_0}}) - 1}}{l_0}$ where $c_{k_0 l_0}$ is as in Remark 2.4.
- Choose $r \in (r_1, r_2)$.

4.1. Algorithm.

1. Set $i = 0$.
2. Choose $n_i = \min\left\{n : e^{-\gamma 2^n} \leq \frac{\delta}{\sqrt{\alpha_i}}\right\}$.
3. Solve $x^\delta_{n_i, \alpha_i} = x^\delta_{\alpha_i}$ by using the iteration (1.12) with $n = n_i$ and $\alpha = \alpha_i$.
4. If $\|x^\delta_{\alpha_i} - x^\delta_{\alpha_j}\| > 4\tilde{C}\frac{\delta}{\sqrt{\alpha_j}}$, $j < i$, then take $k = i - 1$ and return $x^\delta_{\alpha_k}$.
5. Else set $i = i + 1$ and return to Step 2.

References

[1] I.K. Argyros, *Convergence and Application of Newton-type Iterations,* Springer (2008).

[2] I.K. Argyros and S. Hilout, *A convergence analysis for directional two-step Newton methods*, Numer. Algor., **55** (2010), 503–528.

[3] A.B. Bakushinskii, *The problem of the convergence of the iteratively regularized Gauss-Newton method*, Comput. Math. Phy., **32** (1992), 1353–1359.

[4] A.B. Bakushinskii, *Iterative methods without saturation for solving degenerate nonlinear operator equations,* Dokl. Akad. Nauk, **344** (1995), 7–8, MR1361018.

[5] T. Hohage, *Logarithmic convergence rate of the iteratively regularized Gauss-Newton method for an inverse potential and an inverse scattering problem,* Inverse Problems, **13** (1997), 1279–1299.

[6] T. Hohage, *Regularization of exponentially ill-posed problems*, Numer. Funct. Anal&Optim., **21** (2000), 439–464.

[7] B. Kaltenbacher, *A posteriori parameter choice strategies for some Newton-type methods for the regularization of nonlinear ill-posed problems*, Numerische Mathematik, **79** (1998), 501–528.

[8] B. Kaltenbacher, *A note on logarithmic convergence rates for nonlinear Tikhonov regularization*, J. Inv. Ill-Posed Problems, **16** (2008), 79–88.

[9] S. Langer and T. Hohage, *Convergence analysis of an inexact iteratively regularized Gauss-Newton method under general source conditions*, J. of Inverse & Ill-Posed Problems, **15** (2007), 19–35.

[10] S. George, *On convergence of regularized modified Newton's method for nonlinear ill-posed problems*, J. Inv. Ill-Posed Problems, **18** (2010), 133–146.

[11] S. George, *Newton type iteration for Tikhonov regularization of nonlinear ill-posed problems*, J. Math., **2013**, Article ID 439316, 9 pages (2013).

[12] B. Blaschke, A. Neubauer and O. Scherzer, *On convergence rates for the iteratively regularized Gauss-Newton method*, IMA Journal on Numerical Analysis, **17** (1997), 421–436.

[13] P. Mahale and M.T. Nair, *A Simplified generalized Gauss-Newton method for nonlinear ill-posed problems*, Mathematics of Computation, **78(265)** (2009), 171–184.

[14] J.M. Ortega and W.C. Rheinboldt, *Iterative solution of nonlinear equations in general variables*, Academic Press, New York and London (1970).

[15] S.V. Perverzev and E. Schock, *On the adaptive selection of the parameter in regularization of ill-posed problems*, SIAM J. Numer. Anal., **43** (2005), 2060–2076.

[16] S. Lu and S.V. Pereverzev, *Sparsity reconstruction by the standard Tikhonov method*, RICAM-Report No. 2008-17.

[17] K. Levenberg, *A Method for the solution of certain problems in least squares*, Quart. Appl .Math., **2** (1944), 164–168.

[18] D. Marquardt, *An Algorithm for least-squares estimation of nonlinear parameters*, SIAM J. Appl. Math., **11** (1963), 431–441.

[19] M. Hanke, *A regularizing Levenberg-Marquardt scheme, with applications to inverse groundwater filtration problems*, Inverse Problems, **13** (1997), 79–95.

[20] M. Kanke, *The regularizing Levenberg-Marquardt scheme is of optimal order*, J. Integral Equations Appl., **22** (2010), 259–283.

[21] M. Hochbruck and M. Honig, *On the convergence of a regularizing Levenberg-Marquardt scheme for nonlinear ill-posed problems*, Numer. Math., **115** (2010), 71–79.

[22] Q. Jin, *On a regularized Levenberg-Marquardt method for solving nonlinear inverse problems*, Numer. Math., **16** (2010), 229–259.

[23] P. Pornsawad and C. Bockmann, *Convergence rate analysis of the first stage Runga-Kutta-type regularizations*, Inverse Problems, **26** (2010), 035–050.

[24] C. Bockmann, A. Kammanee and A. Braunb, *Logarithmic convergence of Levenberg-Marquardt method with application to an inverse potential problem*, J. Inv. Ill-Posed Problems, **19** (2011), 345–367.