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EXISTENCE OF POSITIVE SOLUTIONS FOR A QUASILINEAR KIRCHHOFF PROBLEM WITH SOBOLEV-HARDY EXPONENT

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Abstract. This paper deals with a class of quasilinear Kirchhoff type problem involving Sobolev-Hardy exponents. By using Mountain Pass Theorem, we show the existence of positive solutions.

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1. Introduction

In this work, we consider the existence of a positive solution to the following $p-$Kirchhoff type problem with singular weight

$$\begin{cases} -\left(a\left(\int_{\Omega}|\nabla u|^{p}\,dx\right)^{\tau-1}+b\right)\triangle_{p}u=|x|^{-s}\,|u|^{p^{*}(s)-2}u+\lambda|u|^{q-2}u & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^{N}$ is an open bounded domain with C^{1} boundary and $0\in\Omega$, $1<p<N$, $a>0$, $b\geq0$, $\lambda>0$, $\tau>1$, $q>p$. We denote by $\|.\|$ the usual norm in $W_{0}^{1,p}(\Omega)$ given by $\|u\|^{p}=\int_{\Omega}|\nabla u|^{p}\,dx$. For $0\leq s<p$, the quantity $p^{*}(s)=p(N-s)/(N-p)$ defines the critical Hardy Sobolev exponent. Such problems are often referred to as being nonlocal because of the presence of the integral over the entire domain Ω. It is called degenerate if $b=0$ and $a>0$ and non degenerate if $b>0$ and $a\geq0$.

Problem (1.1) is related to the stationary version of the following Kirchhoff equation [15]

$$\rho\frac{\partial^{2}u}{\partial t^{2}}-\left(\frac{\rho_{0}}{h}+\frac{E}{2L}\int_{0}^{L}\left|\frac{\partial u}{\partial x}\right|^{2}dx\right)\frac{\partial^{2}u}{\partial x^{2}}=0.$$

The relation above is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters ρ, ρ_{0}, h, E and L have the following meanings: L denotes the length of the string, h stands for the area of the cross-section, E is the Young modulus of the material, ρ the mass density and ρ_{0} represents the initial tension. These problems also serve to model other physical phenomena as biological systems where u describes a process which depends on the average of itself (for example, population density). The presence of the nonlocal term makes the theoretical study of these problems so difficult, then they have attracted the attention of many researchers in particular after the work of Lions [16], where a functional analysis approach was proposed to attack them.

In the last few years, great attention has been paid to the study of Kirchhoff problems involving critical nonlinearities. It is worth mentioning that the first work on the Kirchhoff type problem with critical Sobolev exponent is Alves, Correa and Figueiredo in [1]. After that, many researchers dedicated to the study of several kinds of elliptic Kirchhoff equations with critical exponent in a bounded domain or in the whole space $\mathbb{R}^{N}$, some interesting studies can be found in [3, 4, 9, 10, 13, 17, 20, 21, 22, 24] and the references therein. More precisely, Naimen in [21] generalized the results of [6] to the semilinear Kirchhoff problem

$$\begin{cases} -\left(b+a\int_{\Omega}|\nabla u|^{2}\,dx\right)\Delta u=u^{5}+\lambda f(x,u),\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain, $f\in C\left(\overline{\Omega}\times\mathbb{R},\mathbb{R}\right)$ and $\lambda\in\mathbb{R}$. Under several conditions on f and λ, he proved the existence and nonexistence of solutions. In higher dimensions, for $\lambda>0$ sufficiently large, Figueiredo in [9] considers the scenario $N\geq 3$. Moreover, for the p-Laplacian case, Matallah et al. in [4] and Wang et al. in [24] study the existence of solutions for such λ.

On the other hand, the singular p-Kirchhoff equation case is established in [10] for large enough λ. Very recently, Benchira et al. in [5] have generalized the results of [11] to the nonlocal problem (1.1) with $q=p$, $\lambda\in(0,\ b\lambda_1)$, $b>0$, $N\geq p^2$, $a>0$ if $\tau<\dfrac{N}{N-p}$ and $0<a<S^{-\tau}$ if $\tau=\dfrac{N}{N-p}$ (S is the best Sobolev constant for the imbedding $W_0^{1,p}(\Omega)\hookrightarrow L^{p^*(0)}(\Omega)$). Noting that, for $a=0$ and $b=1$, problem (1.1) had been studied in [12] by Ghoussoub and Yuan.

A natural and interesting question is whether results concerning the solutions of problem (1.1) with λ sufficiently large remain valid for all $\lambda>0$.

Motivated by [5] and [21], we give a positive answer and we generalize some results of [12] to the nonlocal problem (1.1). In our setting, a typical difficulty occurs in proving the existence of solutions because of the lack of the compactness of the Sobolev Hardy embedding $W_0^{1,p}(\Omega)\to L_{p^*(s)}\left(\Omega,\ |x|^{-s}\right)$.

We provide a sketch of the proof for the existence of solutions for problem (1). The main tool is variational methods. More precisely, using the Mountain Pass Theorem [2], we obtain a critical point of the corresponding energy. The principal difficulties appear in the fact that the problem (1.1) contains the critical Sobolev Hardy exponent, then the functional energy does not satisfy the Palais Smale condition in all range, to overcome the lack of compactness, we need to determine a good level of the Palais Smale condition and verify that the critical value is contained in the range of this level. This is the key point to obtain the existence of a Mountain Pass solution.

This paper is composed by two sections in addition to the introduction. In Section 2, we give some preliminary results that we will use later. Section 3 is devoted to the main result.

2. Preliminaries

In this work, we use the following notations: $\to$ (resp $\rightharpoonup$) denotes strong (resp. weak) convergence, $o_n(1)$ denotes $o_n(1)\to 0$ as $n\to+\infty$, $B_R(x_0)$ is the ball centred at x_0 and of radius R, $u^- = \max\{-u, 0\}$ and C, C_1, $C_2, \ldots$, denote various positive constants.

The following Sobolev Hardy inequality is essentially due to Caffarelli, Kohn and Nirenberg [7]. Assume that $1 < p < N$, $0 \le s < p$ and $r \le p^*(s)$. Then

$$\left(\int_\Omega \frac{|u|^r}{|x|^s} dx\right)^p \le C\left(\int_\Omega |\nabla u|^p\, dx\right)^r \text{ for all } u \in C_0^\infty(\Omega)$$

for some positive constant C. Note that the embedding of $W_0^{1,p}(\Omega)$ into $L_r\left(\Omega,\ |x|^{-s}\right)$ is compact if $r < p^*(s)$.

We define the best Sobolev Hardy constant for the embedding $W_0^{1,p}(\Omega) \hookrightarrow L_{p^*(s)}\left(\Omega,\ |x|^{-s}\right)$ as

$$S := \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p}{\left(\int_\Omega |x|^{-s}|u|^{p^*(s)} dx\right)^{p/p^*(s)}}. \tag{2.1}$$

Recall that the infimum S is attained in $\mathbb{R}^N$ by the functions of the form

$$u_\varepsilon(x) := \left(\varepsilon(N-s)\left(\frac{N-p}{p-1}\right)^{p-1}\right)^{\frac{N-p}{p(p-s)}} \left(\varepsilon + |x|^{\frac{p-s}{p-1}}\right)^{\frac{p-N}{p-s}},\ \varepsilon > 0.$$

Moreover, u_ε satisfies

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p dx = \int_{\mathbb{R}^N} |x|^{-s}|u_\varepsilon|^{p^*(s)} dx = S^{\frac{p^*(s)}{p^*(s)-p}}.$$

(See Theorem 3.1 in [12]).

Let R be a positive constant and set $\varphi \in C_0^\infty(\Omega)$ such that $0 \le \varphi(x) \le 1$ for $|x| \le R$ and $\varphi(x) \equiv 1$ for $|x| \le \frac{R}{2}$ and $B_R(0) \subset \Omega$. Let $v_\varepsilon(x) = \varphi(x)\, u_\varepsilon(x)$ and taking $z_\varepsilon(x) = v_\varepsilon(x)\left(\int_\Omega |x|^{-s}|v_\varepsilon(x)|^{p^*(s)} dx\right)^{\frac{-1}{p^*(s)}}$ such that

$$\int_\Omega |x|^{-s}|z_\varepsilon|^{p^*(s)} dx = 1.$$

Then we have the well-known estimates as $\varepsilon \to 0$:

$$\begin{cases} \|z_\varepsilon\|^p = S + O\left(\varepsilon^{\frac{N-p}{p-s}}\right) \\ \int_\Omega |z_\varepsilon|^q\, dx = \begin{cases} O\left(\varepsilon^{\frac{(p-1)(N-p)}{p(p-s)}(p^*(0)-q)}\right) & \text{if } q > \frac{p^*(0)(p-1)}{p}, \\ O\left(\varepsilon^{\frac{N-p}{p(p-s)}q}|\ln\varepsilon|\right) & \text{if } q = \frac{p^*(0)(p-1)}{p}, \\ O\left(\varepsilon^{\frac{N-p}{p(p-s)}q}\right) & \text{if } q < \frac{p^*(0)(p-1)}{p}. \end{cases} \end{cases} \tag{2.2}$$

(See [12]).

The energy functional corresponding to the problem (1.1) is given by

$$E(u)=\frac{a}{\tau p}\|u\|^{\tau p}+\frac{b}{p}\|u\|^{p}-\frac{1}{p^{*}(s)}\int_{\Omega}\frac{|u|^{p^{*}(s)}}{|x|^{s}}dx-\frac{\lambda}{q}\int_{\Omega}|u|^{q}dx,\quad \forall u\in W_{0}^{1,p}(\Omega).$$

Notice that E is well defined in $W_{0}^{1,p}(\Omega)$ and belongs to $C^{1}\left(W_{0}^{1,p}(\Omega),\ \mathbb{R}\right)$.

We say that $u\in W_{0}^{1,p}(\Omega)\setminus\{0\}$ is a weak solution of (1.1), if for any $v\in W_{0}^{1,p}(\Omega)$

$$\left(a\|u\|^{p(\tau-1)}+b\right)\int_{\Omega}|\nabla u|^{p-2}\nabla u\nabla v dx-\int_{\Omega}\frac{|u|^{p^{*}(s)-2}}{|x|^{s}}uv dx-\lambda\int_{\Omega}|u|^{q-2}uv dx=0.$$

Hence, a critical point of functional E is a weak solution of the problem (1.1).

Definition 2.1. Let $c\in\mathbb{R}$, a sequence $\{u_n\}\subset W_{0}^{1,p}(\Omega)$ is called a $(PS)_c$ sequence (Palais Smale sequence at level c) if

$$E(u_n)\to c \text{ and } E'(u_n)\to 0 \quad \text{as} \quad n\to+\infty.$$

We say that E satisfies the Palais Smale condition at level c, if any $(PS)_c$ sequence contains a convergent subsequence in $W_{0}^{1,p}(\Omega)$.

By [5] we have the following result.

Lemma 2.2. *Let $a>0$, $b\geq 0$ and $\tau>1$, $\theta>1$. For $y\geq 0$, we consider the function $f_\theta:\mathbb{R}^{+}\to\mathbb{R}^{*}$ given by*

$$f_\theta(y)=S^{-1}y^{\theta}-aS^{\tau-1}y-b.$$

1. *If $b\neq 0$, then the equation $f_\theta(y)=0$ has a unique positive solution $y_0>\left(\frac{a}{\theta}S^{\tau}\right)^{\frac{1}{\theta-1}}$ and $f_\theta(y)\geq 0$ for all $y\geq y_0$.*
2. *If $b=0$, then the equation $f_\theta(y)=0$ has a unique positive solution $y_1=(aS^{\tau})^{\frac{1}{\theta-1}}$ and $f_\theta(y)\geq 0$ for all $y\geq y_1$.*

Remark 2.3. The authors in [4], [10] and [24] used the truncation method to show the existence of solution under the condition "λ sufficiently large". Our objective in this paper is the existence of solution for all $\lambda>0$.

To use the Mountain Pass Theorem, we should verify that the functional E exhibits the Mountain Pass geometry.

Lemma 2.4. *Suppose that $1<p<N$, $a>0$, $b\geq 0$, $0\leq s<p$, $q<p^{*}(0)$ and $1<\tau<\min\left\{\frac{p^{*}(s)}{p},\frac{q}{p}\right\}$. Then there exist $e\in W_{0}^{1,p}(\Omega)\setminus\{0\}$ and positive numbers δ_1 and ρ_1 such that*

1. *$E(u)\geq\delta_1>0$ for $\|u\|=\rho_1$.*

(2) $\|e\| > \rho_1$ *and* $E(e) < 0$.

Proof. (1) Let $u \in W_0^{1,p}(\Omega) \setminus \{0\}$. Then, by Sobolev Hardy and Young inequalities, we have

$$E(u) \geq -\frac{1}{p^*(s)} S^{-p^*(s)/p} \|u\|^{p^*(s)} + \frac{a}{\tau p} \|u\|^{\tau p} + \frac{b}{p} \|u\|^p - \frac{\lambda}{q} C \|u\|^q. \tag{2.3}$$

Let $\rho = \|u\|$. Then, from (2.3), one has

$$E(u) \geq \frac{a}{\tau p} \rho^{\tau p} - \frac{1}{p^*(s)} S^{-p^*(s)/p} \rho^{p^*(s)} - \frac{\lambda}{q} C \rho^q. \tag{2.4}$$

As $a > 0$ and $1 < \tau < \min\left\{\frac{p^*(s)}{p}, \frac{q}{p}\right\}$, there exists a sufficiently small positive numbers ρ_1 and δ_1 such that

$$E(u) \geq \delta_1 > 0 \quad \text{for} \quad \|u\| = \rho_1.$$

(2) Let $u \in W_0^{1,p}(\Omega) \setminus \{0\}$. Then, as $1 < \tau < \min\left\{\frac{p^*(s)}{p}, \frac{q}{p}\right\}$ it holds that $E(tu) \to -\infty$ as $t \to +\infty$, so we can easily find $e \in W_0^{1,p}(\Omega) \setminus \{0\}$ with $\|e\| > \rho_1$ such that $E(e) < 0$ which completes the proof. □

Now, we can define

$$\Gamma = \left\{\xi \in C\left([0,1], W_0^{1,p}(\Omega)\right), \xi(0) = 0,\ \xi(1) = e\right\}$$

and

$$c = \inf_{\xi \in \Gamma} \max_{t \in [0,1]} E(\xi(t)).$$

Let y_0, y_1 be defined in Lemma 2.2 and

$$C^* := \frac{p^*(s) - \tau p}{\tau p p^*(s)} a S^\tau y_*^{\frac{\tau}{\tau-1}} + \frac{p^*(s) - p}{p p^*(s)} b S y_*^{\frac{1}{\tau-1}} \tag{2.5}$$

with

$$y_* = \begin{cases} y_0 & \text{if } b \neq 0, \\ y_1 & \text{if } b = 0. \end{cases}$$

Next, we prove the following lemma, which is important to ensure the local compactness of (PS) sequences for E.

Lemma 2.5. *Assume that* $1 < p < N$, $a > 0$, $b \geq 0$, $0 \leq s < p < q < p^*(0)$, $1 < \tau < \min\left\{\frac{p^*(s)}{p}, \frac{q}{p}\right\}$ *and* $\{u_n\} \subset W_0^{1,p}(\Omega)$ *is a* $(PS)_c$ *sequence for* E *where* $c < C^*$. *Then* $\{u_n\}$ *contains a subsequence converging strongly in* $W_0^{1,p}(\Omega)$.

Proof. For large enough n and $1 < \tau < \min\left\{\frac{p^*(s)}{p}, \frac{q}{p}\right\}$, we get

$$c + o_n(1) - \frac{1}{\min\{q, p^*(s)\}} o_n(1) \|u_n\| = E(u_n) - \frac{1}{\min\{q, p^*(s)\}} \langle E'(u_n), u_n \rangle$$
$$\geq \begin{cases} a\frac{q-\tau p}{\tau pq} \|u_n\|^{\tau p} + b\frac{q-p}{pq} \|u_n\|^p & \text{if } q < p^*(s) \\ a\frac{p^*(s)-\tau p}{\tau pp^*(s)} \|u_n\|^{\tau p} + b\frac{p^*(s)-p}{pp^*(s)} \|u_n\|^p & \text{if } p^*(s) < q. \end{cases}$$

Then $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Consequently, by the concentration compactness principle due to Lions (see [17] and [18]), there exists a subsequence, still denoted by $\{u_n\}$ such that

$$\begin{cases} |\nabla u_n|^p \rightharpoonup d\eta \geq |\nabla u|^p + \sum_{i\in I} \eta_i \widetilde{\delta}_{x_i}, \\ |x|^{-s} |u_n|^{p^*(s)} \rightharpoonup d\gamma = |x|^{-s} |u|^{p^*(s)} + \sum_{i\in I} \gamma_i \widetilde{\delta}_{x_i}, \end{cases} \tag{2.6}$$

where I is an at most countable index set, η_i, γ_i are nonnegative numbers and $\widetilde{\delta}_{x_i}$ is the Dirac mass at x_i. Moreover, by the Sobolev Hardy inequality we infer that

$$\eta_i \geq S\gamma_i^{p/p^*(s)} \text{ for all } i \in I. \tag{2.7}$$

We now claim that $I = \emptyset$. To this end, by contradiction, suppose that $I \neq \emptyset$. Then there exists $j \in I$. For $\varepsilon > 0$, let $\phi_{\varepsilon,j}$ be a smooth cut-off function centered at x_j such that $0 \leq \phi_{\varepsilon,j} \leq 1$, $\phi_{\varepsilon,j}|_{B_\varepsilon(x_j)} = 1$, $\phi_{\varepsilon,j}|_{\Omega\setminus B_{2\varepsilon}(x_j)} = 0$ and $|\nabla\phi_{\varepsilon,j}(x)| \leq 2/\varepsilon$. Clearly $\{\phi_{\varepsilon,j}u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. It follows from $E'(u_n) \to 0$ as $n \to \infty$ that

$$0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\left(a\|u_n\|^{p(\tau-1)} + b\right)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(\phi_{\varepsilon,j}u_n)dx \right.$$
$$\left. - \int_\Omega |x|^{-s}|u_n|^{p^*(s)-2}u_n(\phi_{\varepsilon,j}u_n)\,dx - \lambda\int_\Omega |u_n|^{q-2}u_n(\phi_{\varepsilon,j}u_n)\,dx\right).$$

On the other hand, since $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, we deduce that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega |u_n|^q\phi_{\varepsilon,j}\,dx = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_{\varepsilon,j}dx = 0 \tag{2.8}$$

and

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(a\|u_n\|^{p(\tau-1)} + b\right)\int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_{\varepsilon,j}dx = 0. \tag{2.9}$$

By relations (2.6), (2.8), (2.9) and Hölder's inequality we obtain

$$0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle E'(u_n), \phi_{\varepsilon,j}u_n\rangle \geq (b + a\eta_j^{\tau-1})\eta_j - \gamma_j,$$

that is,

$$\gamma_j \geq b\eta_j + a\eta_j^\tau,$$

hence, by (2.7), we arrive at

$$\gamma_j = 0 \text{ or } S^{-1}\left(\gamma_j\right)^{\frac{p^*(s)-p}{p^*(s)}} - aS^{\tau-1}\left(\gamma_j\right)^{\frac{p}{p^*(s)}(\tau-1)} - b \geq 0. \tag{2.10}$$

Now, we assume that $\gamma_j \neq 0$. Set $y = \left(\gamma_j\right)^{\frac{p(\tau-1)}{p^*(s)}}$ and $\theta = \frac{p^*(s)-p}{p(\tau-1)}$. Then by (2.10) we obtain

$$S^{-1}y^{\theta} - aS^{\tau-1}y - b \geq 0. \tag{2.11}$$

It is clear that $\theta > 1$ since $\tau < \frac{p^*(s)}{p}$. So, from (2.11) and the definition of f_θ in Lemma 2.2, we get

$$f_\theta(y) = S^{-1}y^{\frac{p^*(s)-p}{p(\tau-1)}} - aS^{\tau-1}y - b \geq 0.$$

According to Lemma 2.2, there exist $y_0 > \left(\frac{ap(\tau-1)}{p^*(s)-p}S^\tau\right)^{\frac{p(\tau-1)}{p^*(s)-\tau p}}$ and $y_1 = \left(aS^\tau\right)^{\frac{p(\tau-1)}{p^*(s)-\tau p}}$ such that $f_\theta(y_*) = 0$ and $f_\theta(y) \geq 0$ if $y \geq y_*$ with

$$y_* = \begin{cases} y_0 \text{ if } b \neq 0, \\ y_1 \text{ if } b = 0, \end{cases} \tag{2.12}$$

which implies that $S\left(\gamma_j\right)^{\frac{p}{p^*(s)}} \geq Sy_*^{\frac{1}{\tau-1}}$. So

$$\gamma_j \geq y_*^{\frac{p^*(s)}{p(\tau-1)}}. \tag{2.13}$$

Using relation (2.7), we conclude that

$$\eta_j \geq Sy_*^{\frac{1}{\tau-1}}. \tag{2.14}$$

Note that $1 < \tau < \min\left\{\frac{p^*(s)}{p}, \frac{q}{p}\right\}$, so we distinguish two cases:

Case $1: q \leq p^*(s)$. By (2.13) and (2.14), we have

$$\begin{aligned} c + o_n(1) &= E(u_n) - \frac{1}{q}\left\langle E'(u_n), u_n\right\rangle \\ &= b\frac{q-p}{qp}\|u_n\|^p + \frac{p^*(s)-q}{qp^*(s)}\int_\Omega |x|^{-s}|u_n|^{p^*(s)}dx + a\frac{q-\tau p}{\tau qp}\|u_n\|^{\tau p} \\ &\geq b\frac{q-p}{qp}\left(\|u\|^p + \eta_j\right) + \frac{p^*(s)-q}{qp^*(s)}\left(\int_\Omega |x|^{-s}|u|^{p^*(s)} + \gamma_j\right) \\ &\quad + a\frac{q-\tau p}{\tau qp}\left(\|u\|^p + \eta_j\right)^\tau \\ &\geq b\frac{q-p}{qp}\|u\|^p + \frac{p^*(s)-q}{qp^*(s)}\int_\Omega |x|^{-s}|u|^{p^*(s)} + a\frac{q-\tau p}{\tau qp}\|u\|^{\tau p} \\ &\quad + b\frac{q-p}{qp}\eta_j + \frac{p^*(s)-q}{qp^*(s)}\gamma_j + a\frac{q-\tau p}{\tau qp}\eta_j^\tau, \end{aligned}$$

which implies

$$
\begin{aligned}
c &\geq b\frac{q-p}{qp}Sy_*^{\frac{1}{\tau-1}} + \frac{p^*(s)-q}{qp^*(s)}\left(Sy_*^{\frac{1}{\tau-1}}\right)^{\frac{p^*(s)}{p}} S^{-\frac{p^*(s)}{p}} + a\frac{q-\tau p}{\tau qp}\left(Sy_*^{\frac{1}{\tau-1}}\right)^{\tau} \\
&\geq a\frac{p^*(s)-\tau p}{\tau pp^*(s)}\left(Sy_*^{\frac{1}{\tau-1}}\right)^{\tau} + b\frac{p^*(s)-p}{pp^*(s)}Sy_*^{\frac{1}{\tau-1}} - a\frac{p^*(s)-\tau p}{\tau pp^*(s)}\left(Sy_*^{\frac{1}{\tau-1}}\right)^{\tau} \\
&\quad - b\frac{p^*(s)-p}{pp^*(s)}Sy_*^{\frac{1}{\tau-1}} + b\frac{q-p}{qp}Sy_*^{\frac{1}{\tau-1}} + a\frac{q-\tau p}{\tau qp}\left(Sy_*^{\frac{1}{\tau-1}}\right)^{\tau} \\
&\quad + \frac{p^*(s)-q}{qp^*(s)}\left(Sy_*^{\frac{1}{\tau-1}}\right)^{\frac{p^*(s)}{p}} S^{-\frac{p^*(s)}{p}} \\
&\geq a\frac{p^*(s)-\tau p}{\tau pp^*(s)}\left(Sy_*^{\frac{1}{\tau-1}}\right)^{\tau} + b\frac{p^*(s)-p}{pp^*(s)}Sy_*^{\frac{1}{\tau-1}} - a\frac{p^*(s)-q}{qp^*(s)}\left(Sy_*^{\frac{1}{\tau-1}}\right)^{\tau} \\
&\quad - b\frac{p^*(s)-q}{qp^*(s)}Sy_*^{\frac{1}{\tau-1}} + \frac{p^*(s)-q}{qp^*(s)}\left(Sy_*^{\frac{1}{\tau-1}}\right)^{\frac{p^*(s)}{p}} S^{-\frac{p^*(s)}{p}} \\
&\geq a\frac{p^*(s)-\tau p}{\tau pp^*(s)}S^{\tau}y_*^{\frac{\tau}{\tau-1}} + b\frac{p^*(s)-p}{pp^*(s)}Sy_*^{\frac{1}{\tau-1}} \\
&\quad + \frac{p^*(s)-q}{qp^*(s)}Sy_*^{\frac{1}{\tau-1}}\left(S^{-1}y_*^{\frac{p^*(s)-p}{p(\tau-1)}} - aS^{\tau-1}y_* - b\right) \\
&= C^* + \frac{p^*(s)-q}{qp^*(s)}Sy_*^{\frac{1}{\tau-1}} f_{\theta}(y_*) \\
&= C^*
\end{aligned}
$$

with $\theta = \frac{p^*(s)-p}{p(\tau-1)}$ which leads to a contradiction.

Case $2: p^*(s) < q < p^*(0)$. By (2.13) and (2.14) and Lemma 2.2, we have

$$
\begin{aligned}
c &= \lim_{n\to\infty} I_\lambda(u_n) - \frac{1}{\tau p}\left\langle I'_\lambda(u_n), u_n\right\rangle \\
&= \lim_{n\to\infty} b\left(\frac{1}{p}-\frac{1}{\tau p}\right)\|u_n\|^p + \frac{q-\tau p}{q\tau p}\lambda\int_\Omega |u_n|^q dx + \frac{p^*_\beta-\tau p}{\tau pp^*_\beta}\int_\Omega \frac{|u_n|^{p^*(s)}}{|x|^s} \\
&\geq \lim_{n\to\infty} b\left(\frac{1}{p}-\frac{1}{\tau p}\right)\|u_n\|^p + \frac{p^*_\beta-\tau p}{\tau pp^*_\beta}\int_\Omega \frac{|u_n|^{p^*(s)}}{|x|^s} \\
&\geq b\left(\frac{1}{p}-\frac{1}{\tau p}\right)\left(\|u\|^p+\eta_j\right) + \frac{p^*(s)-\tau p}{\tau pp^*(s)}\left(\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s} + \gamma_{j_0}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\geq b\left(\frac{1}{p}-\frac{1}{\tau p}\right)\|u\|^{p}+\frac{p^{*}(s)-\tau p}{\tau p p^{*}(s)} \int_{\Omega} \frac{|u|^{p^{*}(s)}}{|x|^{s}}+\frac{\tau-1}{\tau p} b \eta_{j}+\frac{p^{*}(s)-\tau p}{\tau p p^{*}(s)} \gamma_{j} \\
&\geq\left(\frac{1}{p}-\frac{1}{\tau p}\right) b \eta_{j}+\frac{p^{*}(s)-\tau p}{\tau p p^{*}(s)} \gamma_{j} \\
&\geq\left(\frac{1}{p}-\frac{1}{\tau p}\right) b S y_{*}^{\frac{1}{\tau-1}}+\frac{p^{*}(s)-\tau p}{\tau p p^{*}(s)}\left(S^{\frac{p^{*}(s)}{p}} y_{*}^{\frac{p^{*}(s)}{p(\tau-1)}}\right) S^{-\frac{p^{*}(s)}{p}} \\
&\geq\left(\frac{1}{p}-\frac{1}{\tau p}\right) b S y_{*}^{\frac{1}{\tau-1}}+\frac{p^{*}(s)-\tau p}{\tau p p^{*}(s)} y_{*}^{\frac{p^{*}(s)}{p(\tau-1)}}+\frac{1}{p^{*}(s)} b S y_{*}^{\frac{1}{\tau-1}}-\frac{1}{p^{*}(s)} b S y_{*}^{\frac{1}{\tau-1}} \\
&\quad -a\left(\frac{1}{\tau p}-\frac{1}{p^{*}(s)}\right) S^{\tau} y_{*}^{\frac{\tau}{\tau-1}}+a\left(\frac{1}{\tau p}-\frac{1}{p^{*}(s)}\right) S^{\tau} y_{*}^{\frac{\tau}{\tau-1}} \\
&\geq a\left(\frac{1}{\tau p}-\frac{1}{p^{*}(s)}\right) S^{\tau} y_{*}^{\frac{\tau}{\tau-1}}+\left(\frac{1}{p}-\frac{1}{p^{*}(s)}\right) b S y_{*}^{\frac{1}{\tau-1}} \\
&\quad +\frac{p^{*}(s)-\tau p}{\tau p p^{*}(s)} y_{*}^{\frac{p^{*}(s)}{p(\tau-1)}}-b \frac{p^{*}(s)-\tau p}{\tau p p^{*}(s)} S y_{*}^{\frac{1}{\tau-1}}-a \frac{p^{*}(s)-\tau p}{\tau p p^{*}(s)} S^{\tau} y_{*}^{\frac{\tau}{\tau-1}} \\
&\geq\left[\frac{p^{*}(s)-\tau p}{\tau p p^{*}(s)} a\left(S y_{*}^{\frac{1}{\tau-1}}\right)^{\tau}+\left(\frac{1}{p}-\frac{1}{p^{*}(s)}\right) b S y_{*}^{\frac{1}{\tau-1}}\right] \\
&\quad +\frac{p^{*}(s)-\tau p}{\tau p p^{*}(s)} S y_{*}^{\frac{1}{\tau-1}}\left(S_{\mu}^{-1} y_{*}^{\frac{p^{*}(s)-p}{p(\tau-1)}}-a S^{\tau-1} y_{*}-b\right) \\
&=\frac{p^{*}(s)-\tau p}{\tau p p^{*}(s)} a\left(S y_{*}^{\frac{1}{\tau-1}}\right)^{\tau}+\left(\frac{1}{p}-\frac{1}{p^{*}(s)}\right) b S y_{*}^{\frac{1}{\tau-1}} \\
&\quad +\frac{p^{*}(s)-\tau p}{\tau p p^{*}(s)} S y_{*}^{\frac{1}{\tau-1}} f_{\tau}\left(y_{*}\right) \\
&=\frac{p^{*}(s)-\tau p}{\tau p p^{*}(s)} a\left(S y_{*}^{\frac{1}{\tau-1}}\right)^{\tau}+\left(\frac{1}{p}-\frac{1}{p^{*}(s)}\right) b S y_{*}^{\frac{1}{\tau-1}}=: C^{*},
\end{aligned}
$$

which entails a contradiction. Hence I is empty and so

$$
\int_{\Omega}|x|^{-s}\left|u_{n}\right|^{p^{*}(s)} dx \rightarrow \int_{\Omega}|x|^{-s}|u|^{p^{*}(s)} dx.
$$

On the other hand, we have

$$
\begin{aligned}
\left\langle E'(u_{n}),\ u_{n}\right\rangle &=\left(a\left\|u_{n}\right\|^{p(\tau-1)}+b\right)\left\|u_{n}\right\|^{p}-\int_{\Omega}|x|^{-s}\left|u_{n}\right|^{p^{*}(s)} dx-\lambda \int_{\Omega}\left|u_{n}\right|^{q} dx \\
&=o_{n}(1) \qquad (2.15)
\end{aligned}
$$

and

$$\left\langle E'(u_n), v\right\rangle = \left(a \left\|u_n\right\|^{p(\tau-1)} + b\right) \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla v dx \tag{2.16}$$
$$- \int_{\Omega} |x|^{-s} |u_n|^{p^*(s)-2} u_n v dx - \lambda \int_{\Omega} |u_n|^{q-2} u_n v dx$$
$$= o_n(1)$$

for any $v \in W_0^{1,p}(\Omega)$. Set $l = \lim \|u_n\|$ as $n \to +\infty$, then from (2.15) and (2.16), we deduce that

$$\left(a l^{p(\tau-1)} + b\right) l^p - \int_{\Omega} |x|^{-s} |u|^{p^*(s)} dx - \lambda \int_{\Omega} |u|^q dx = 0 \tag{2.17}$$

and

$$\left(a l^{p(\tau-1)} + b\right) \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx - \int_{\Omega} |x|^{-s} |u|^{p^*(s)-2} u v dx - \lambda \int_{\Omega} |u|^{q-2} u v dx = 0. \tag{2.18}$$

Taking the test function $v = u$ in (2.18), we get

$$\left(a l^{p(\tau-1)} + b\right) \|u\|^p - \int_{\Omega} |x|^{-s} |u|^{p^*(s)} dx - \lambda \int_{\Omega} |u|^q dx = 0. \tag{2.19}$$

Therefore, the equalities (2.17) and (2.19) imply that $\|u\| = l$. Consequently $\{u_n\}$ converges strongly in $W_0^{1,p}(\Omega)$, which gives the desired result. $\square$

The energy functional E satisfies the Palais Smale condition at level c for any $c < C^*$. So, the existence of critical point follows immediately from the following lemma.

Lemma 2.6. *Let for* $1 < p < N$, $a > 0$, $b \geq 0$, $0 \leq s < p$, $\max\left\{p^*(0) - \frac{p}{p-1}, \frac{p^*(0)(p-1)}{p}\right\} < q < p^*(0)$ *and* $1 < \tau < \min\left\{\frac{p^*(s)}{p}, \frac{q}{p}\right\}$. *Then*

$$\sup_{t \geq 0} E(t z_\varepsilon) < C^*$$

for all $\lambda > 0$.

Proof. We define the functions g and h such that

$$g(t) = E(t z_\varepsilon) = \frac{a}{\tau p} t^{\tau p} \|z_\varepsilon\|^{\tau p} + \frac{b}{p} t^p \|z_\varepsilon\|^p - \frac{t^{p^*(s)}}{p^*(s)} - \frac{\lambda}{q} t^q \int_{\Omega} |z_\varepsilon|^q dx$$

and

$$h(t) = \frac{a}{\tau p} t^{\tau p} \|z_\varepsilon\|^{\tau p} + \frac{b}{p} t^p \|z_\varepsilon\|^p - \frac{S^{-\frac{p^*(s)}{p}}}{p^*(s)} \|z_\varepsilon\|^{p^*(s)} t^{p^*(s)},$$

then

$$g(t) = h(t) - \frac{t^{p^*(s)}}{p^*(s)}\left(1 - S^{-\frac{p^*(s)}{p}} \|z_\varepsilon\|^{p^*(s)}\right) - \lambda \frac{t^q}{q}\int_\Omega |z_\varepsilon|^q dx.$$

Noting $\lim_{t\to+\infty} g(t) = -\infty$ and $g(t) > 0$ when t is close to 0, so $\sup_{t\geq 0} g(t)$ is attained for some $T_\varepsilon > 0$. Furthermore, from $g'(T_\varepsilon) = 0$, it follows that

$$-T_\varepsilon^{p^*(s)-1} + aT_\varepsilon^{\tau p-1}\|z_\varepsilon\|^{\tau p} + bT_\varepsilon^{p-1}\|z_\varepsilon\|^p = \lambda T_\varepsilon^{q-1}\int_\Omega |z_\varepsilon|^q dx \tag{2.20}$$

and

$$-\lambda T_\varepsilon^{q-1}\int_\Omega |z_\varepsilon|^q dx + aT_\varepsilon^{\tau p-1}\|z_\varepsilon\|^{\tau p} + bT_\varepsilon^{p-1}\|z_\varepsilon\|^p = T_\varepsilon^{p^*(s)-1}. \tag{2.21}$$

Multiplying the equation in (2.20) by T_ε^{1-p}, we obtain

$$-T_\varepsilon^{p^*(s)-p} + aT_\varepsilon^{p(\tau-1)}\|z_\varepsilon\|^{\tau p} + b\|z_\varepsilon\|^p > 0.$$

Easy calculations confirm that

$$T_\varepsilon \leq \left(\frac{ap(\tau-1)}{p^*(s)-p}\|z_\varepsilon\|^{\tau p}\right)^{\frac{1}{p^*(s)-\tau p}}.$$

Applying relation (2.2) yields, for sufficiently small ε

$$T_\varepsilon \leq \left(\frac{ap(\tau-1)}{p^*(s)-p}S^\tau\right)^{\frac{1}{p^*(s)-\tau p}} =: t_0. \tag{2.22}$$

On the other hand, we multiply the equation in (2.21) by $T_\varepsilon^{1-\tau p}$ and by recalling (2.22), we obtain

$$\begin{aligned} T_\varepsilon^{p^*(s)-\tau p} &\geq a\|z_\varepsilon\|^{\tau p} - \lambda T_\varepsilon^{q-\tau p}\int_\Omega |z_\varepsilon|^q dx \\ &\geq \|z_\varepsilon\|^{\tau p} - \lambda (t_0)^{q-\tau p}\int_\Omega |z_\varepsilon|^q dx. \end{aligned}$$

From (2.2), we find that for small enough ε

$$T_\varepsilon \geq (aS^\tau)^{\frac{1}{p^*(s)-\tau p}} =: t_1.$$

Now we estimate $g(T_\varepsilon)$. It follows from $h'(t) = 0$ that

$$-S^{-\frac{p^*(s)}{p}}\|z_\varepsilon\|^{p^*(s)} t^{p^*(s)-1} + a\|z_\varepsilon\|^{\tau p} t^{\tau p-1} + b\|z_\varepsilon\|^p t^{p-1} = 0,$$

that is

$$S^{-\frac{p^*(s)}{p}}\|z_\varepsilon\|^{p^*(s)-p} t^{p^*(s)-p} - at^{p(\tau-1)}\|z_\varepsilon\|^{p(\tau-1)} - b = 0. \tag{2.23}$$

Set

$$y = t^{p(\tau-1)}S^{1-\tau}\|z_\varepsilon\|^{p(\tau-1)}$$

and $\theta = \frac{p^*(s)-p}{p(\tau-1)}$. As $\theta > 1$, then by (2.23) we get

$$S^{-1}y^{\theta} - aS^{\tau-1}y - b = f_{\theta}(y) = 0. \tag{2.24}$$

According to Lemma 2.2, $f_{\theta}(y_*) = 0$ where y_* is defined in (2.12). Therefore, $h'(t_*) = 0$ with

$$t_* = S^{\frac{1}{p}} \|z_{\varepsilon}\|^{-1} (y_*)^{\frac{1}{p(\tau-1)}}.$$

Since $f_{\theta}(y)$, it follows that $h'(t)$ is convex and so

$$\max_{t\geq 0} h(t) = h(t_*) = -\frac{1}{p^*(s)} S^{-\frac{p^*(s)}{p}} \|z_{\varepsilon}\|^{p^*(s)} t_*^{p^*(s)} + \frac{a}{\tau p} \|z_{\varepsilon}\|^{\tau p} t_*^{\tau p} + \frac{b}{p} \|z_{\varepsilon}\|^{p} t_*^{p}. \tag{2.25}$$

Since $h'(t_*) = 0$, we have

$$S^{-\frac{p^*(s)}{p}} \|z_{\varepsilon}\|^{p^*(s)} t_*^{p^*(s)} = a \|z_{\varepsilon}\|^{\tau p} t_*^{\tau p} + b \|z_{\varepsilon}\|^{p} t_*^{p}. \tag{2.26}$$

So, we deduce that

$$\begin{aligned}
\max_{t\geq 0} h(t) &= -\frac{1}{p^*(s)} \left(a \|z_{\varepsilon}\|^{\tau p} t_*^{\tau p} + b \|z_{\varepsilon}\|^{p} t_*^{p}\right) + \frac{a}{\tau p} \|z_{\varepsilon}\|^{\tau p} t_*^{\tau p} + \frac{b}{p} \|z_{\varepsilon}\|^{p} t_*^{p} \\
&= a\left(\frac{1}{\tau p} - \frac{1}{p^*(s)}\right) t_*^{\tau p} \|z_{\varepsilon}\|^{\tau p} + b\left(\frac{1}{p} - \frac{1}{p^*(s)}\right) t_*^{p} \|z_{\varepsilon}\|^{p} \\
&= a\left(\frac{1}{\tau p} - \frac{1}{p^*(s)}\right) S^{\tau} y_*^{\frac{\tau}{\tau-1}} + b\left(\frac{1}{p} - \frac{1}{p^*(s)}\right) S y_*^{\frac{1}{\tau-1}} \\
&= C^*.
\end{aligned}$$

Consequently, by (2.2) and the fact that $\max\left\{p^*(0) - \frac{p}{p-1}, \frac{p^*(0)(p-1)}{p}\right\} < q < p^*(0)$, we obtain

$$\begin{aligned}
\sup_{t\geq 0} E(tz_{\varepsilon}) &= g(T_{\varepsilon}) \\
&= h(T_{\varepsilon}) + \frac{1}{p^*(s)} \left(S^{-\frac{p^*(s)}{p}} \|z_{\varepsilon}\|^{p^*(s)} - 1\right) T_{\varepsilon}^{p^*(s)} - \frac{\lambda}{q} T_{\varepsilon}^{q} \int_{\Omega} |z_{\varepsilon}|^q dx \\
&\leq C^* + \frac{1}{p^*(s)} \left(S^{-\frac{p^*(s)}{p}} \|z_{\varepsilon}\|^{p^*(s)} - 1\right) T_{\varepsilon}^{p^*(s)} - \frac{\lambda}{q} T_{\varepsilon}^{q} \int_{\Omega} |z_{\varepsilon}|^q dx \\
&\leq C^* + \frac{1}{p^*(s)} O\left(\varepsilon^{\frac{N-p}{p-s}}\right) (t_1)^{p^*(s)} - \frac{\lambda}{q} (t_0)^q O\left(\varepsilon^{\frac{p-1}{p-s}\left(N - \frac{q(N-p)}{p}\right)}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq C^* + O\left(\varepsilon^{\frac{N-p}{p-s}}\right) - O\left(\varepsilon^{\frac{p-1}{p-s}\left(N - \frac{q(N-p)}{p}\right)}\right) \\
&\leq C^* + O\left(\varepsilon^{\frac{N-p}{p-s}}\right) - O\left(\varepsilon^{\frac{(p-1)(N-p)}{p(p-s)}(p^*(0)-q)}\right) \\
&\leq C^* - O\left(\varepsilon^{\frac{(p-1)(N-p)}{p(p-s)}(p^*(0)-q)}\right).
\end{aligned}
$$

For ε sufficiently small, we obtain $\sup_{t\geq 0} E(tz_\varepsilon) < C^*$ which conclude the desired result. □

3. Main result

The result below provides the existence of a Mountain Pass type solution. We now state our main result.

Theorem 3.1. *Let* $1 < p < N$, $a > 0$, $b \geq 0$, $0 \leq s < p$, $\max\left\{p^*(0) - \frac{p}{p-1}, \frac{p^*(0)(p-1)}{p}\right\} < q < p^*(0)$ *and* $1 < \tau < \min\left\{\frac{p^*(s)}{p}, \frac{q}{p}\right\}$. *Then* (1.1) *has a positive solution for all* $\lambda > 0$.

Proof. Applying Lemma 2.4, E possesses a Mountain Pass geometry. Therefore, from the Mountain Pass Theorem [2], there exists a $(PS)_c$ sequences $\{u_n\} \subset W_0^{1,p}(\Omega)$ of E. According to Lemma 2.5 and Lemma 2.6, $\{u_n\}$ has a subsequence (still denoted by $\{u_n\}$) such that $u_n \to u$ in $W_0^{1,p}(\Omega)$. Hence u is a critical point of E and therefore a solution of (1.1).

Now we show that $u > 0$. To obtain a contradiction, we assume that $u = u^-$. Then

$$
\begin{aligned}
0 &= \left\langle E'(u), u^- \right\rangle \\
&= \left(a\|u\|^{p(\tau-1)} + b\right)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla u^- dx \\
&\quad - \int_\Omega |x|^{-s}|u|^{p^*(s)-2}uu^- dx - \lambda\int_\Omega |u|^{q-2}uu^- dx \\
&\geq \left(a\|u\|^{p(\tau-1)} + b\right)\left\|u^-\right\|^p + \int_\Omega |x|^{-s}|u|^{p^*(s)}dx + \lambda\int_\Omega |u|^q dx, \\
&\geq b\left\|u^-\right\|^p.
\end{aligned}
$$

So $u^- = 0$. By the strong maximum principle [23], we have $u > 0$. Theorem 3.1 can be concluded. □

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