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SENSITIVITY ANALYSIS FOR SET-VALUED MIXED VARIATIONAL-HEMIVARIATIONAL INEQUALITY PROBLEMS

Jong Kyu Kim[1] **and Salahuddin**[2]

[1]Department of Mathematics Education, Kyungnam University,
Changwon, Gyeongnam 51767, Korea
e-mail: jongkyuk@kyungnam.ac.ac.kr

[2]Department of Mathematics, College of Science, Jazan University, Jazan-45142,
P.O. Box 114, Kingdom of Saudi Arabia
e-mail: dr_salah12@yahoo.com; smohammad@jazanu.edu.sa

Abstract. In this study, we examine a set-valued mixed variational-hemivariational inequality problem and its inverse. We utilize the concepts of inverse strong monotonicity and Hausdorff Lipschitz continuity to prove that the inverse problem has a solution and that set-valued mixed variational-hemivariational inequality problems are well-posed.

1. Introduction

Hemivariational inequalities were introduced by Panagiotopoulos as a variational formulation of significant classes of unilateral and inequality problems in the mechanical sciences (see [17]). A generalisation of the variational inequality for situations where the function is non-convex and non-smooth is known as the hemivariational inequality. This is based on the concept of Clarkes generalised gradient, as explained in [3, 10, 11, 16]. Recently, many different kinds of variational and hemivariational inequalities have been developed, and the study of variational-hemivariational inequalities has emerged as a new,

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[0]Corresponding author: J. K. Kim(jongkyuk@kyungnam.ac.kr).

innovative, and intriguing field of applied and industrial mathematics. For more information, refer to [2, 20].

Sensitivity analysis is a study that determines the impact of changes in parameters, forcing functions, or sub-models on the state variables of a mathematical model or system. The objective is to understand how alterations in a systems inputs affect its outputs or performance. Sensitivity analysis can help identify the various sources of uncertainty in a systems inputs that influence its outputs. To conduct a sensitivity analysis, one of the systems parameters is modified by a specific percentage while the others remain constant. The model is then run to assess the percentage change in the pre-specified performance indicator. Further information can be found in the cited references: [6, 7, 12, 18].

The realm of inverse and identification problems has emerged as a highly dynamic and rapidly evolving branch of applied and industrial mathematics, due to its vast range of practical applications. Numerous studies, such as [1, 4, 13, 14, 19], have focused on inverse coefficient problems in variational inequalities. However, the problem of inverse parameter identification in variational-hemivariational inequalities has received little attention in the current literature. This research gap provided inspiration for the current study.

We discusses the problem of set-valued mixed variational-hemivariational inequality. The problem is defined as follows. Let $\mathbb{X}$ be a reflexive Banach space, and $\mathscr{K}$ be a nonempty, closed and convex set of constraints in $\mathbb{X}$. Let $\mathscr{P}$ be a normed space of parameters. Consider a pseudomonotone operator $\mathscr{A}: \mathscr{P}\times\mathbb{X} \to 2^{\mathbb{X}^\star}$, a convex functional $\varphi\colon \mathscr{P}\times\mathscr{K}\times\mathscr{K} \to \mathbb{R}$, a locally Lipschitz (in general non-convex) functional $\jmath\colon \mathscr{P}\times\mathbb{X} \to \mathbb{R}$ and a map $f\colon \mathscr{P} \to \mathbb{X}^\star$ with certain properties. For a given $\gamma \in \mathscr{P}$, the problem is to find an element $u = u(\gamma) \in \mathscr{K}$ and $x = x(\gamma) \in \mathscr{A}(\gamma, u)$ such that:

$$\langle x - f(\gamma), v - u\rangle_{\mathbb{X}} + \varphi(\gamma, u, v) - \varphi(\gamma, u, u) + \jmath^0(\gamma, u; v - u) \geq 0,\ \forall v \in \mathscr{K}, \quad (1.1)$$

where $\jmath^0(\gamma, u; v)$ stands for the generalized (Clarke) directional derivative of $\jmath(\gamma, \cdot)$ at a point $u \in \mathbb{X}$ in the direction $v \in \mathbb{X}$.

In this paper, we present a new result that shows the mapping of $\gamma \in \mathscr{P} \rightarrowtail u(\gamma) \in \mathscr{K}$ is Lipschitz continuous. This continuous dependency has the consequence of obtaining the existence of a solution to an inverse problem, where we minimize an appropriate objective functional $\mathcal{T}$ that is defined on the space of admissible parameters $\mathscr{P}_{ad} \subset \mathscr{P}$.

The paper is structured as follows. In the next section, we will provide some necessary preliminary material. Section 3 will be dedicated to proving the existence of a unique solution to the problem. Finally, in Section 4, we will first provide a continuous result of dependency for equation (1.1), and then establish the existence of a solution to the parametric inverse problem.

2. Preliminaries

This section provides fundamental definitions, concepts, facts and properties that will be used later. We will first establish notation and review some definitions. Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a Banach space, $\mathbb{X}^\star$ denotes the dual space of $\mathbb{X}$ and $\langle\cdot,\cdot\rangle_{\mathbb{X}}$ denotes the dual pairing between $\mathbb{X}^\star$ and $\mathbb{X}$. Let $2^{\mathbb{X}} \neq \emptyset$ be set of all subsets of $\mathbb{X}$. The symbols $\mathscr{D}(\mathcal{T})$ and $\mathcal{R}(\mathcal{T})$ stand for the domain and the range of an operator $\mathcal{T} : \mathbb{X} \to 2^{\mathbb{X}^\star}$, respectively, that is

$$\mathscr{D}(\mathcal{T}) = \{u \in \mathbb{X} \mid \mathcal{T}u \neq \emptyset\}$$

and

$$\mathcal{R}(\mathcal{T}) = \bigcup\{\mathcal{T}u \mid u \in \mathbb{X}\}.$$

We denote by $\to$ the strong convergence and by $\rightharpoonup$ the weak convergence.

Definition 2.1. ([3]) A function $f : \mathbb{X} \to \mathbb{R}$ is said to be

(i) (weakly) upper semicontinuous (u.s.c.) at u_0, if for any sequence $\{u_n\}_{n\geq 1} \subset \mathbb{X}$ with $(u_n \rightharpoonup u_0)$ $u_n \to u_0$, we have

$$\limsup f(u_n) \leq f(u_0).$$

(ii) (weakly) lower semicontinuous (l.s.c.) at u_0, if for any sequence $\{u_n\}_{n\geq 1}$ $\subset \mathbb{X}$ with $(u_n \rightharpoonup u_0)$ $u_n \to u_0$, we have

$$f(u_0) \leq \liminf f(u_n).$$

(iii) f is said to be (weakly) u.s.c. (l.s.c.) on $\mathbb{X}$, if for all $u \in \mathbb{X}$, f is (weakly) u.s.c. (l.s.c.) at u.

Definition 2.2. ([8]) Let $\varphi : \mathbb{X} \to \mathbb{R}\bigcup\{+\infty\}$ be a proper convex lower semicontinuous function. The mapping $\partial_c\varphi : \mathbb{X} \to 2^{\mathbb{X}^\star}$ defined by

$$\partial_c\varphi(u) = \{u^\star \in \mathbb{X} \mid \langle u^\star, v-u\rangle_{\mathbb{X}} \leq \varphi(v) - \varphi(u),\ \forall\, v \in \mathbb{X}\},\ \text{for } u \in \mathbb{X}$$

is called the subdifferential of φ. An element $u^\star \in \partial_c\varphi(u)$ is called a subgradient of φ in u.

Definition 2.3. ([3]) Given a locally Lipschitz function $\varphi : \mathbb{X} \to \mathbb{R}$, we denote by $\varphi^0(u; v)$ the Clarke generalized directional derivative of φ at the point $u \in \mathbb{X}$ in the direction $v \in \mathbb{X}$ defined by

$$\varphi^0(u; v) = \limsup_{\lambda\to 0^+,\ \varsigma\to u} \frac{\varphi(\varsigma + \lambda v) - \varphi(\varsigma)}{\lambda}.$$

The Clarke subdifferential or the generalized gradient of φ at $u \in \mathbb{X}$, denoted by $\partial\varphi(u)$, is a subset of $\mathbb{X}^\star$ given by

$$\partial\varphi(u) = \{u^\star \in \mathbb{X}^\star \mid \varphi(u; v) \geq \langle u^\star, v\rangle_{\mathbb{X}},\ \forall\, v \in \mathbb{X}\}.$$

Definition 2.4. ([5, 8]) A single-valued mapping $g : \mathbb{X} \to \mathbb{X}$ is said to be Lipschitz continuous, if there exists a constant $\rho > 0$ such that

$$\|g(v) - g(u)\|_{\mathbb{X}} \leq \rho\|v - u\|_{\mathbb{X}},\ \forall\, u, v \in \mathbb{X}.$$

Definition 2.5. ([5, 15, 20]) Let $\mathcal{T} : \mathbb{X} \to 2^{\mathbb{X}^\star}$ be a set-valued operator. Then $\mathcal{T}$ is said to be

(i) monotone, if

$$\langle \mathcal{T}(v) - \mathcal{T}(u), v - u\rangle_{\mathbb{X}} \geq 0,\ \forall\, u, v \in \mathbb{X};$$

(ii) strongly monotone, if there exists $c > 0$ such that for all $u, v \in \mathbb{X}$,

$$\langle \mathcal{T}(v) - \mathcal{T}(u), v - u\rangle_{\mathbb{X}} \geq c\|v - u\|^2_{\mathbb{X}};$$

(iii) inverse strongly monotone, if there exist $c > 0$ such that for all $u, v \in \mathbb{X}$,

$$\langle \mathcal{T}(v) - \mathcal{T}(u), v - u\rangle_{\mathbb{X}} \geq c\|\mathcal{T}(v) - \mathcal{T}(u)\|^2_{\mathbb{X}};$$

(v) pseudomonotone, if

(a) for all $u \in \mathbb{X}$, the set $\mathcal{T}(u)$ is nonempty, bounded, closed and convex;

(b) the mapping $\mathcal{T}$ is u.s.c. from each finite dimensional subspace of $\mathbb{X}$ to $\mathbb{X}^\star$ endowed with the weak topology;

(c) if $\{u_n\} \subset \mathbb{X}$ with $u_n \rightharpoonup u\ \in \mathbb{X}$, and $u_n^\star \in \mathcal{T}(u_n)$ such that

$$\limsup\langle u_n^\star, u_n - u\rangle_{\mathbb{X}} \leq 0,$$

then for every $v \in \mathbb{X}$, there exists $u^\star(v) \in \mathcal{T}(u)$ such that

$$\langle u^\star(v), u - v\rangle \leq \liminf\langle u_n^\star, u_n - v\rangle_{\mathbb{X}}.$$

(vi) generalized pseudomonotone, if for any sequence $\{u_n\} \subset \mathbb{X}$ with $u_n \rightharpoonup u\ \in \mathbb{X}$, and $u_n^\star \in \mathcal{T}(u_n)$ with $u_n^\star \rightharpoonup u^\star$ such that

$$\limsup\langle u_n^\star, u_n - u\rangle_{\mathbb{X}} \leq 0,$$

we have $u^\star \in \mathcal{T}(u)$ and

$$\langle u_n^\star, u_n\rangle \to \langle u^\star, u\rangle.$$

Lemma 2.6. ([11, Proposition 1.3]) *Let $\mathbb{X}$ be a reflexive Banach space and $\mathcal{T} : \mathbb{X} \to 2^{\mathbb{X}^\star}$ be a set-valued operator.*

(i) *If $\mathcal{T}$ is a pseudomonotone operator, then $\mathcal{T}$ is generalized pseudomonotone.*

(ii) *If $\mathcal{T}$ is a generalized pseudomonotone operator that is bounded, and for each $u \in \mathbb{X}$, $\mathcal{T}(u)$ is a nonempty, closed, and convex subset of $\mathbb{X}^\star$, then $\mathcal{T}$ is pseudomonotone.*

Definition 2.7. ([5]) A single-valued operator $\mathcal{T} : \mathbb{X} \to \mathbb{X}^\star$ is said to be pseudomonotone, if it is bounded and satisfies the inequality

$$\langle \mathcal{T}u, u - v\rangle \le \liminf \langle \mathcal{T}u_n, u_n - v\rangle_{\mathbb{X}}, \ \forall\, v \in \mathbb{X},$$

where $u_n \rightharpoonup u$ in $\mathbb{X}$ with

$$\limsup \langle \mathcal{T}u_n, u_n - u\rangle_{\mathbb{X}} \le 0.$$

Definition 2.8. A set-valued operator $\mathcal{T} : \mathbb{X} \to 2^{\mathbb{X}}$ is called $\hat{\mathcal{H}}$-Lipschitz continuous, if there exists a constant $\xi > 0$ such that for all $u, v \in \mathbb{X}$,

$$\hat{\mathcal{H}}(\mathcal{T}(u), \mathcal{T}(v)) \le \xi \|u - v\|_{\mathbb{X}},$$

where $\hat{\mathcal{H}}$ is the Hausdorff pseudo-metric, that is, for any two nonempty subsets A and B of $\mathbb{X}$

$$\hat{\mathcal{H}}(A, B) = \max\left\{\sup_{u\in A} d(u, B), \sup_{v\in B} d(v, A)\right\}.$$

It should be noted that if the domain of $\hat{\mathcal{H}}$ is restricted to the family of closed, bounded subsets of $\mathbb{X}$, then $\hat{\mathcal{H}}$ becomes the Hausdorff metric.

Lemma 2.9. ([11, Proposition 1.3]) *Let $\mathcal{T} : \mathbb{X} \to \mathbb{X}^\star$ be a single-valued operator defined on a reflexive Banach space $\mathbb{X}$. The operator $\mathcal{T}$ is pseudomonotone if and only if $\mathcal{T}$ is bounded and satisfies the following condition: if*

$$u_n \rightharpoonup u \ \in \mathbb{X}$$

and

$$\limsup \langle \mathcal{T}u_n, u_n - u\rangle_{\mathbb{X}} \le 0,$$

then

$$\mathcal{T}u_n \rightharpoonup \mathcal{T}u \ \in \mathbb{X}$$

and

$$\lim \langle \mathcal{T}u_n, u_n - u\rangle_{\mathbb{X}} = 0.$$

Theorem 2.10. ([9, Theorem 2.2 and Corollary 2.3]) *Let $\mathbb{X}$ be a reflexive Banach space. Consider a bounded pseudomonotone operator $\mathcal{T}_1 : \mathscr{D}(\mathcal{T}_1) \subset \mathbb{X} \to 2^{\mathbb{X}^\star}$ and a maximal monotone operator $\mathcal{T}_2 : \mathscr{D}(\mathcal{T}_2) \subset \mathbb{X} \to 2^{\mathbb{X}^\star}$. Let $f \in \mathbb{X}^\star$. Assume that there exists $u_0 \in \mathbb{X}$ such that*

$$\inf_{\xi\in\mathcal{T}_1(v), \eta\in\mathcal{T}_2(v)} \frac{\langle \xi + \eta, v - u_0\rangle_{\mathbb{X}}}{\|v\|_{\mathbb{X}}} \to \infty \quad \text{as} \ \ \|v\|_{\mathbb{X}} \to \infty, y \in \mathscr{D}(\mathcal{T}_2). \tag{2.1}$$

Then the operator $\mathcal{T}_1 + \mathcal{T}_2$ *is surjective, that is,*

$$f \in \mathcal{R}(\mathcal{T}_1 + \mathcal{T}_2).$$

3. Main results

In this section, we demonstrate the existence and uniqueness of a solution to the set-valued mixed variational-hemivariational inequality problem of the following form:

We assume that $(\mathscr{P}, \|\cdot\|_{\mathscr{P}})$ is a normed space of parameters. Then, for given $\gamma \in \mathscr{P}$, find $u = u(\gamma) \in \mathscr{K}$ and $x = x(\gamma) \in \mathscr{A}(\gamma, u)$ such that

$$\langle x - f(\gamma), v - u\rangle_{\mathbb{X}} + \varphi(\gamma, u, v) - \varphi(\gamma, u, u) + \jmath^0(\gamma, u; v - u) \geq 0, \quad \forall\, v \in \mathscr{K} \tag{3.1}$$

satisfying the following hypotheses:

(1) $\mathscr{K}$ is a nonempty, closed, convex subset of $\mathbb{X}$.

(2) $\mathscr{A} : \mathscr{P} \times \mathbb{X} \to 2^{\mathbb{X}^\star}$ is such that

  (a) $\mathscr{A}(\gamma, \cdot)$ is pseudomonotone for all $\gamma \in \mathscr{P}$.

  (b) there exists $\alpha_{\mathscr{A}} > 0$ such that for all $\gamma \in \mathscr{P}, u_1, u_2 \in \mathbb{X}, x_1 \in \mathscr{A}(\gamma, u_1), x_2 \in \mathscr{A}(\gamma, u_2)$

$$\langle x_1 - x_2, u_1 - u_2\rangle_{\mathbb{X}} \geq \alpha_{\mathscr{A}} \|x_1 - x_2\|^2_{\mathbb{X}^*}. \tag{3.2}$$

  (c) there exists a constant $\rho_{\mathscr{A}} > 0$ such that

$$\hat{\mathcal{H}}(\mathscr{A}(\gamma, u_1), \mathscr{A}(\gamma, u_2)) \leq \rho_{\mathscr{A}} \|u_1 - u_2\|_{\mathbb{X}}, \quad \forall\, u_1, u_2 \in \mathbb{X}. \tag{3.3}$$

(3) $\varphi : \mathscr{P} \times \mathscr{K} \times \mathscr{K} \to \mathbb{R}$ is such that

  (a) $\varphi(\gamma, u, \cdot) : \mathscr{K} \to \mathbb{R}$ is convex and l.s.c. on $\mathscr{K}$, for all $\gamma \in \mathscr{P}$, $u \in \mathscr{K}$.

  (b) there exist $\alpha_\varphi > 0$ and $\beta_\varphi \geq 0$ such that

$$\begin{aligned} &\varphi(\gamma_1, u_1, v_2) - \varphi(\gamma_1, u_1, v_1) + \varphi(\gamma_2, u_2, v_1) - \varphi(\gamma_2, u_2, v_2) \\ &\quad \leq \alpha_\varphi \|u_1 - u_2\|_{\mathbb{X}} \|v_1 - v_2\|_{\mathbb{X}} + \beta_\varphi \|\gamma_1 - \gamma_2\|_{\mathscr{P}} \|v_1 - v_2\|_{\mathbb{X}}, \\ &\quad\ \forall\, \gamma_1, \gamma_2 \in \mathscr{P}, u_1, u_2, v_1, v_2 \in \mathscr{K}. \end{aligned} \tag{3.4}$$

(4) $\jmath : \mathscr{P} \times \mathbb{X} \to \mathbb{R}$ is such that

  (a) $\jmath(\gamma, \cdot)$ is locally Lipschitz for all $\gamma \in \mathscr{P}$.

  (b) there exist $\varsigma_0, \varsigma_1, \varsigma_2 \geq 0$ such that

$$\|\partial \jmath(\gamma, u)\|_{\mathbb{X}^\star} \leq \varsigma_0 + \varsigma_1 \|u\|_{\mathbb{X}} + \varsigma_2 \|\gamma\|_{\mathscr{P}}, \quad \forall\, \gamma \in \mathscr{P}, u \in \mathbb{X}. \tag{3.5}$$

  (c) there exist $\alpha_\jmath > 0$ and $\beta_\jmath \geq 0$ such that

$$\begin{aligned} &\jmath^0(\gamma_1, u_1; u_2 - u_1) + \jmath^0(\gamma_2, u_2; u_1 - u_2) \\ &\quad \leq \alpha_\jmath \|u_1 - u_2\|^2_{\mathbb{X}} + \beta_\jmath \|\gamma_1 - \gamma_2\|_{\mathscr{P}} \|u_1 - u_2\|_{\mathbb{X}}, \\ &\quad\ \forall\, \gamma_1, \gamma_2 \in \mathscr{P}, u_1, u_2 \in \mathbb{X}. \end{aligned} \tag{3.6}$$

(5)
$$f(\gamma) \in \mathbb{X}^\star, \ \ \forall \gamma \in \mathscr{P}. \tag{3.7}$$

(6)
$$\alpha_\varphi + \alpha_\jmath < \alpha_\mathscr{A} \rho_\mathscr{A}^2. \tag{3.8}$$

We suppose that $\gamma \in \mathscr{P}$ is given. Let $\eta \in \mathscr{K}$ be fixed, and consider the following auxiliary problem: find $u_\eta = u_\eta(\gamma) \in \mathscr{K}$ and $x_\eta = x_\eta(\gamma) \in \mathscr{A}(\gamma, u_\eta)$ such that

$$\langle x_\eta - f(\gamma), v - u_\eta\rangle_\mathbb{X} + \varphi(\gamma, \eta, v) - \varphi(\gamma, \eta, u_\eta) + \jmath^0(\gamma, u_\eta; v - u_\eta) \geq 0, \ \forall v \in \mathscr{K}. \tag{3.9}$$

Now, we will prove the following results for the main theorems.

Lemma 3.1. *Equation* (3.9) *has an unique solution* $u_\eta \in \mathscr{K}$ *and* $x_\eta \in \mathscr{A}(\gamma, u_\eta)$.

Proof. For the existence part of the proof, we will apply Theorem 2.10. We define $\tilde{\varphi}_{\gamma,\eta} : \mathbb{X} \to \mathbb{R} \bigcup \{+\infty\}$ by

$$\tilde{\varphi}_{\gamma,\eta}(v) = \begin{cases} \varphi(\gamma, \eta, v), & \text{if } v \in \mathscr{K}, \\ +\infty, & \text{otherwise} \end{cases}$$

for $v \in \mathbb{X}$. By using this notation, (3.9) is equivalent to the following one. Find $u_\eta = u_\eta(\gamma) \in \mathscr{K}$, $x_\eta = x_\eta(\gamma) \in \mathscr{A}(\gamma, u_\eta)$ such that

$$\langle x_\eta - f(\gamma), v - u_\eta\rangle_\mathbb{X} + \tilde{\varphi}_{\gamma,\eta}(v) - \tilde{\varphi}_{\gamma,\eta}(u_\eta) + \jmath^0(\gamma, u_\eta; v - u_\eta) \geq 0, \ \forall v \in \mathbb{X}. \tag{3.10}$$

Next, we consider the following operator inclusion for finding $u_\eta = u_\eta(\gamma) \in \mathscr{K}$, $x_\eta = x_\eta(\gamma) \in \mathscr{A}(\gamma, u_\eta)$ such that

$$x_\eta + \partial_c \tilde{\varphi}_{\gamma,\eta}(u_\eta) + \partial \jmath(\gamma, u_\eta) \ni f(\gamma). \tag{3.11}$$

We define two set-valued operators $\mathscr{F}_{\gamma_1}, \mathscr{F}_{\gamma_2} : \mathbb{X} \to 2^{\mathbb{X}^\star}$ by

$$\begin{aligned} &\mathscr{F}_{\gamma_1}(v) = y + \partial \jmath(\gamma, v), \forall \ y \in \mathscr{A}(\gamma, v), \\ &\mathscr{F}_{\gamma_2}(v) = \partial_c \tilde{\varphi}_{\gamma,\eta}(v), \ \text{for } v \in \mathbb{X}, \end{aligned} \tag{3.12}$$

respectively.

Subsequently, we will prove that the equation (3.11) has a solution. To this end, we will verify that the operators $\mathscr{F}_{\gamma_1}$ and $\mathscr{F}_{\gamma_2}$ satisfy the hypotheses of Theorem 2.10. First, we show that the operator $\mathscr{F}_{\gamma_1}$ is bounded and pseudomonotone. Its boundedness follows easily from the boundedness of $\mathscr{A}(\gamma, \cdot)$, *see* (2)(a), and the growth condition (3.5) on $\partial \jmath(\gamma, \cdot)$. Therefore, we demonstrate that the operator $\mathscr{F}(\gamma_1)$ is pseudomonotone. Since for all $v \in \mathbb{X}$ and $y \in \mathscr{A}(\gamma, v)$, the set

$$y + \partial \jmath(\gamma, v) \tag{3.13}$$

is nonempty, closed and convex in $\mathbb{X}^\star$, by Lemma 2.6(ii), it is enough to establish that $\mathscr{F}_{\gamma_1}$ is a generalized pseudomonotone operator. To this goal, let $u_n \in \mathbb{X}$,

$$u_n \rightharpoonup u \in \mathbb{X}$$

and

$$u_n^\star \in \mathscr{F}_{\gamma_1}(u_n),\ u_n^\star \rightharpoonup u^\star \in \mathbb{X}.$$

Then,

$$\limsup \langle u_n^\star, u_n - u\rangle_{\mathbb{X}} \leq 0.$$

We need to prove that $u^\star \in \mathscr{F}_{\gamma_1}(u)$ and

$$\langle u_n^\star, u_n\rangle_{\mathbb{X}} \to \langle u^\star, u\rangle_{\mathbb{X}}.$$

By hypotheses (3.2), (3.3),(3.6) and (3.8), it follows that the operator $\mathscr{F}_{\gamma_1}$ is inverse strongly monotone. Using the inverse strong monotonicity of $\mathscr{F}_{\gamma_1}$, we have

$$(\alpha_{\mathscr{A}}\rho^2_{\mathscr{A}} - \alpha_{\jmath})\|u_n - u\|^2_{\mathbb{X}} \leq \langle u_n^\star, u_n - u\rangle_{\mathbb{X}} - \langle \mathscr{F}_{\gamma_1} u, u_n - u\rangle_{\mathbb{X}}.$$

Hence, we deduce that

$$u_n \to u \in \mathbb{X}.$$

Since $x_n \in \mathscr{A}(\gamma, u_n)$ and $x \in \mathscr{A}(\gamma, u)$ such that

$$\|x_n - x\|_{\mathbb{X}^*} = \check{\mathcal{H}}(\mathscr{A}(\gamma, u_n), \mathscr{A}(\gamma, u)) \leq \rho_{\mathscr{A}}\|u_n - u\|_{\mathbb{X}} \to 0 \ \text{ as } n \to \infty,$$

hence

$$x_n \to x \in \mathbb{X}.$$

Subsequently, by

$$u_n^\star \in \mathscr{F}_{\gamma_1}(u_n),$$

we have

$$u_n^\star = \omega_n + \varpi_n$$

with

$$\omega_n = \mathscr{A}(\gamma, u_n)$$

and

$$\varpi_n \in \partial\jmath(\gamma, u_n).$$

From the boundedness of operators $\mathscr{A}(\gamma, \cdot)$ and $\partial\jmath(\gamma, \cdot)$, by passing to a subsequence if necessary, we may suppose that

$$\omega_n \rightharpoonup \omega \quad \text{and} \quad \varpi_n \rightharpoonup \varpi$$

in $\mathbb{X}^\star$ with some $\omega, \varpi \in \mathbb{X}$, respectively. The condition

$$u_n^\star = \omega_n + \varpi_n$$

immediately implies that

$$u^\star = \omega + \varpi.$$

Applying Lemma 2.9 with $\mathscr{A}(\gamma,\cdot)$, we have

$$\mathscr{A}(\gamma,u_n)\to\mathscr{A}(\gamma,u)\in\mathbb{X}^\star.$$

Hence, it is clear that

$$\omega=\mathscr{A}(\gamma,u).$$

On the other hand, because the set-valued map

$$\mathbb{X}\ni y\rightarrowtail\partial\jmath(\gamma,v)\subset\mathbb{X}^\star$$

has a closed graph with respect to the strong topology in $\mathbb{X}$ and the weak topology in $\mathbb{X}^\star$, we deduce that

$$\varpi\in\partial\jmath(\gamma,u).$$

Hence, we have

$$u^*=\omega+\varpi\in\mathscr{A}(\gamma,u)+\partial\jmath(\gamma,u)=\mathscr{F}_{\gamma_1}(u).$$

Since

$$u_n^*\rightharpoonup u^*\in\mathbb{X}^*\text{ and }u_n\to u\in\mathbb{X},$$

it is clear that

$$\langle u_n^*,u_n\rangle_{\mathbb{X}}\to\langle u^*,u\rangle_{\mathbb{X}}.$$

This proves that $\mathscr{F}_{\gamma_1}$ is generalized pseudomonotone and also that$\mathscr{F}_{\gamma_1}$ is pseudomonotone.

Now, exploiting the hypothesis (3), we know that $\tilde{\varphi}_{\gamma,\eta}$ is a proper, convex and lower semicontinuous function with

$$\mathscr{D}(\tilde{\varphi}_{\gamma,\eta})=\mathscr{K}.$$

Therefore, the operator $\mathscr{F}_{\gamma_2}=\partial_c\tilde{\varphi}_{\gamma,\eta}:\mathbb{X}\to 2^{\mathbb{X}^*}$ is maximal monotone with

$$\mathscr{D}(\mathscr{F}_{\gamma_2})=\mathscr{K}.$$

Next, we will prove that the operator $\mathscr{F}_{\gamma_1}+\mathscr{F}_{\gamma_2}$ satisfies the coercivity condition (2.1) of Theorem 2.10. Since $\varphi(\gamma,\eta,\cdot)$ is convex and l.s.c., it admits an affine minorant, that is, there are $\ell_{\gamma,\eta}\in\mathbb{X}^*$ and $\vartheta_{\gamma,\eta}\in\mathbb{R}$ dependent on γ,η such that

$$\varphi(\gamma,\eta,v)\geq\langle\ell_{\gamma,\eta},v\rangle_{\mathbb{X}}+\vartheta_{\gamma,\eta},\ \forall\,v\in\mathbb{X}. \tag{3.14}$$

Let $u_0,v\in\mathscr{K}$. From (3.14) and Definition 2.2, we have

$$\begin{aligned}\langle\partial_c\tilde{\varphi}_{\gamma,\eta}(v),v-u_0\rangle_{\mathbb{X}}&\geq\varphi(\gamma,\eta,v)-\varphi(\gamma,\eta,u_0)\\&\geq\langle\ell_{\gamma,\eta},v\rangle_{\mathbb{X}}+\vartheta_{\gamma,\eta}-|\varphi(\gamma,\eta,u_0)|.\end{aligned}\tag{3.15}$$

Using (3.15), hypotheses (3.2), (3.3) and (3.6), and the estimate

$$|\langle\partial\jmath(\gamma,u_0),v-u_0\rangle_{\mathbb{X}}|\leq(\varsigma_0+\varsigma_1\|u_0\|_{\mathbb{X}}+\varsigma_2\|\gamma\|_{\mathscr{P}})\|v-u_0\|_{\mathbb{X}},$$

we have

$$\begin{aligned}
&\langle \mathscr{F}_{\gamma_1}(v) + \mathscr{F}_{\gamma_2}(v), v - u_0\rangle_{\mathbb{X}} \\
&\quad = \langle y + \partial \jmath(\gamma, v) + \partial_c \tilde{\varphi}_{\gamma,\eta}(v), v - u_0\rangle_{\mathbb{X}} \\
&\quad = \langle y - x_0, v - u_0\rangle_{\mathbb{X}} + \langle \partial \jmath(\gamma, v) - \partial \jmath(\gamma, u_0), v - u_0\rangle_{\mathbb{X}} \\
&\qquad + \langle \partial_c \tilde{\varphi}_{\gamma,\eta}(v), v - u_0\rangle_{\mathbb{X}} + \langle x_0, v - u_0\rangle_{\mathbb{X}} + \langle \partial \jmath(\gamma, u_0), v - u_0\rangle_{\mathbb{X}} \\
&\quad \geq (\alpha_{\mathscr{A}} \rho_{\mathscr{A}}^2 - \alpha_{\jmath})\|v - u_0\|_{\mathbb{X}}^2 - (\varsigma_0 + \varsigma_1 \|u_0\|_{\mathbb{X}} + \varsigma_2 \|\gamma\|_{\mathscr{P}} + \|x_0\|)\|v - u_0\|_{\mathbb{X}} \\
&\qquad - \|\ell_{\gamma,\eta}\| \|v\|_{\mathbb{X}} + \vartheta_{\gamma,\eta} - |\varphi(\gamma, \eta, u_0)|, \ \forall y \in \mathscr{A}(\gamma, v), x_0 \in \mathscr{A}(\gamma, u_0). \quad (3.16)
\end{aligned}$$

Hence, the coercivity condition (2.1) holds.

We are now in a position to apply Theorem 2.10 and deduce that there exists $u_\eta \in \mathbb{X}$ which solves (3.11). It is clear that every solution to problem (3.11) is a solution to the inequality (3.10). This implies that $u_\eta \in \mathbb{X}$ is a solution of (3.10). It is easy to verify that $u_\eta \in \mathscr{K}$ and hence (3.9) has at least one solution $u_\eta \in \mathscr{K}$.

For the uniqueness part of the lemma, let $u_1, u_2 \in \mathscr{K}$, $x_1 \in \mathscr{A}(\gamma, u_1), x_2 \in \mathscr{A}(\gamma, u_2)$ be the solutions of (3.9) with the same fixed parameters γ and η, that is,

$$\langle x_1 - f(\gamma), v - u_1\rangle_{\mathbb{X}} - \varphi(\gamma, \eta, v) - \varphi(\gamma, \eta, u_1) + \jmath^0(\gamma, u_1; v - u_1) \geq 0, \forall\ v \in \mathscr{K}, \tag{3.17}$$

and

$$\langle x_2 - f(\gamma), v - u_2\rangle_{\mathbb{X}} - \varphi(\gamma, \eta, v) - \varphi(\gamma, \eta, u_2) + \jmath^0(\gamma, u_2; v - u_2) \geq 0, \forall v \in \mathscr{K}. \tag{3.18}$$

Taking $v = u_2$ in (3.17) and $v = u_2$ in (3.18), adding them, we obtain

$$\langle x_1 - x_2, u_2 - u_1\rangle_{\mathbb{X}} + \jmath^0(\gamma, u_1; u_2 - u_1) + \jmath^0(\gamma, u_1; u_1 - u_2) \geq 0. \tag{3.19}$$

From (3.2),(3.3) and (3.6), we have

$$(\alpha_{\mathscr{A}} \rho_{\mathscr{A}}^2 - \alpha_{\jmath})\|u_1 - u_2\|_{\mathbb{X}}^2 \leq 0, \tag{3.20}$$

which due to the smallness condition (3.8), implies

$$u_1 = u_2.$$

Again

$$\|x_1 - x_2\|_{\mathbb{X}^*} = \hat{\mathcal{H}}(\mathscr{A}(\gamma, u_1), \mathscr{A}(\gamma, u_2)) \leq \rho_{\mathscr{A}} \|u_1 - u_2\|_{\mathbb{X}} = 0,$$

hence

$$x_1 = x_2.$$

This completes the proof of the lemma. □

Define now the operator $\Upsilon : \mathscr{K} \to \mathscr{K}$ by

$$\Upsilon\eta = u_\eta \text{ for } \eta \in \mathscr{K}, \tag{3.21}$$

where $u_\eta \in \mathscr{K}$ denotes the unique solution of (3.9).

Lemma 3.2. *The operator Υ has an unique fixed point.*

Proof. Let $\eta_1, \eta_2 \in \mathscr{K}$ and $u_1 = u_{\eta_1}$, $u_2 = u_{\eta_2} \in \mathscr{K}$, $x_1 = x_{\eta_1} \in \mathscr{A}(\gamma, u_{\eta_1})$, $x_2 = x_{\eta_2} \in \mathscr{A}(\gamma, u_{\eta_2})$ be the unique solutions of (3.9) corresponding to η_1, η_2, respectively. From the inequalities

$$\langle x_1 - f(\gamma), v - u_1\rangle_{\mathbb{X}} + \varphi(\gamma, \eta_1, v) - \varphi(\gamma, \eta_1, u_1) + \jmath^0(\gamma, u_1; v - u_1) \geq 0,$$
$$\langle x_2 - f(\gamma), v - u_2\rangle_{\mathbb{X}} + \varphi(\gamma, \eta_2, v) - \varphi(\gamma, \eta_2, u_2) + \jmath^0(\gamma, u_2; v - u_2) \geq 0,$$

we have

$$\begin{aligned}\langle x_1 - x_2, u_1 - u_2\rangle_{\mathbb{X}} \leq \varphi(\gamma, \eta_1, u_2) - \varphi(\gamma, \eta_2, u_1) + \varphi(\gamma, \eta_2, u_1) - \varphi(\gamma, \eta_2, u_2)\\ + \jmath^0(\gamma, u_1; u_2 - u_1) + \jmath^0(\gamma, u_2; u_1 - u_2).\end{aligned} \tag{3.22}$$

We use the hypotheses (3.2), (3.3), (3.4), and (3.6) to obtain

$$\alpha_{\mathscr{A}}\rho_{\mathscr{A}}^2\|u_1 - u_2\|_{\mathbb{X}}^2 \leq \alpha_\varphi\|\eta_1 - \eta_2\|_{\mathbb{X}}\|u_1 - u_2\|_{\mathbb{X}} + \alpha_\jmath\|u_1 - u_2\|_{\mathbb{X}}^2. \tag{3.23}$$

Consequently,

$$\begin{aligned}\|\Upsilon\eta_1 - \Upsilon\eta_2\|_{\mathbb{X}} &= \|u_1 - u_2\|_{\mathbb{X}}\\ &\leq \frac{\alpha_\varphi}{\alpha_{\mathscr{A}}\rho_{\mathscr{A}}^2 - \alpha_\jmath}\|\eta_1 - \eta_2\|_{\mathbb{X}}.\end{aligned} \tag{3.24}$$

Using (3.8) and applying the Banach contraction principle, we deduce that there exists an unique $\eta^* \in \mathscr{K}$ such that

$$\eta^* = \Upsilon\eta^*.$$

This completes the proof of the lemma. □

We now present the main result and provide the proof.

Theorem 3.3. *Assume that conditions* (1)-(5) *hold and the following condition is satisfied:*

$$\alpha_\varphi + \alpha_\jmath < \alpha_{\mathscr{A}}\rho_{\mathscr{A}}^2.$$

Then, for all $\gamma \in \mathscr{P}$, (3.1) *has an unique solution $u = u(\gamma)$ and $x = x(\gamma)$.*

Proof. For the existence, let $\eta^* \in \mathscr{K}$ be the (unique) fixed point of the operator Υ defined by (3.21). Writing inequality (3.9) for $\eta = \eta^*$, we observe that

$$u_{\eta^*} = \Upsilon\eta^* = \eta^*.$$

Hence, we conclude that the function $\eta^* \in \mathscr{K}$ is a solution of (3.1).

The uniqueness of the solution (3.1) is established by a direct computation.

Let $u_1, u_2 \in \mathscr{K}$ be two solutions corresponding to γ, that is, for all $v \in \mathscr{K}, x_1 \in \mathscr{A}(\gamma, u_1), x_2 \in \mathscr{A}(\gamma, u_2)$,

$$\langle x_1 - f(\gamma), v - u_1\rangle_{\mathbb{X}} + \varphi(\gamma, u_1, v) - \varphi(\gamma, u_1, u_1) + \jmath^0(\gamma, u_1; v - u_1) \geq 0, \quad (3.25)$$

and

$$\langle x_2 - f(\gamma), v - u_2\rangle_{\mathbb{X}} + \varphi(\gamma, u_2, v) - \varphi(\gamma, u_2, u_2) + \jmath^0(\gamma, u_2; v - u_2) \geq 0. \quad (3.26)$$

From the above inequalities, similarly as in the proof of Lemma 3.2, we obtain

$$\begin{aligned}\langle x_1 - x_2, u_1 - u_2\rangle_{\mathbb{X}} \leq \varphi(\gamma, u_1, u_2) - \varphi(\gamma, u_1, u_1) + \varphi(\gamma, u_2, u_1) - \varphi(\gamma, u_2, u_2)\\ + \jmath^0(\gamma, u_1; u_2 - u_1) + \jmath^0(\gamma, u_2; u_1 - u_2). \quad (3.27)\end{aligned}$$

Now hypotheses (3.2), (3.3), (3.4) and (4) imply

$$\alpha_{\mathscr{A}}\rho_{\mathscr{A}}^2\|u_1 - u_2\|_{\mathbb{X}}^2 \leq \alpha_\varphi\|u_1 - u_2\|_{\mathbb{X}}^2 + \alpha_\jmath\|u_1 - u_2\|_{\mathbb{X}}^2, \quad (3.28)$$

from which, due to (3.8), it follows that

$$u_1 = u_2.$$

Again $x_1 \in \mathscr{A}(\gamma, u_1)$, $x_2 \in \mathscr{A}(\gamma, u_2)$, we have

$$\|x_1 - x_2\|_{\mathbb{X}^*} = \hat{\mathcal{H}}(\mathscr{A}(\gamma, u_1), \mathscr{A}(\gamma, u_2)) \leq \rho_{\mathscr{A}}\|u_1 - u_2\|_{\mathbb{X}} = 0$$

implies

$$\|x_1 - x_2\|_{\mathbb{X}^*} \leq 0.$$

Therefore,

$$x_1 = x_2.$$

The proof is completed. □

4. Inverse problems

In this section, our objective is to propose a solution for a general identification problem that is framed by the set-valued mixed variational-hemivariational inequality problem explained in (3.1). To achieve this, we will begin by discussing two continuous dependence results that are crucial and groundbreaking in the field of parameter identification problems.

Consider the following set-valued mixed variational-hemivariational inequality problem corresponding to a sequence of parameters $\{\gamma_n\} \subset \mathscr{P}$, $n \in \mathbb{N}$. For given $\gamma_n \in \mathscr{P}$, finding $u_n = u(\gamma_n) \in \mathscr{P}$, $x_n = x(\gamma_n) \in \mathscr{A}(\gamma_n, u_n)$ such that

$$\langle x_n - f(\gamma_n), v - u_n\rangle_{\mathbb{X}} + \varphi(\gamma_n, u_n, v) - \varphi(\gamma_n, u_n, u_n) + \jmath^0(\gamma_n, u_n; v - u_n) \geq 0, \forall v \in \mathscr{P} \quad (4.1)$$

satisfying the following hypotheses:

(7) For any $\{\gamma_n\} \subset \mathscr{P}$, $\{u_n\} \subset \mathbb{X}$, $\{x_n\} \subset \mathbb{X}$ with $\gamma_n \to \gamma \in \mathscr{P}$ and $u_n \rightharpoonup u \in \mathbb{X}$ and $x_n \rightharpoonup x \in \mathscr{A}(\gamma, u)$ and all $v \in \mathbb{X}$, we have

$$\limsup \langle x_n' - x_n, u_n - v\rangle \leq 0, \ \forall x_n' \in \mathscr{A}(\gamma, x_n),\ x_n \in \mathscr{A}(\gamma_n, x_n). \tag{4.2}$$

(8) For any $\{\gamma_n\} \subset \mathscr{P}$, $\{u_n\} \subset \mathbb{X}$ with $\gamma_n \to \gamma \in \mathscr{P}$, $u_n \rightharpoonup u \in \mathbb{X}$ and $x_n \rightharpoonup x \in \mathscr{A}(\gamma, u)$ and all $v \in \mathbb{X}$, we have

$$\begin{aligned} \|x_n - x\|_{\mathbb{X}^*} &\leq \hat{\mathcal{H}}(\mathscr{A}(\gamma_n, u_n), \mathscr{A}(\gamma, u)) \\ &\leq \rho_{\mathscr{A}}\|u_n - u\|_{\mathbb{X}} + r_{\mathscr{A}}\|\gamma_n - \gamma\|_{\mathscr{P}}. \end{aligned} \tag{4.3}$$

(9) For any $\{\gamma_n\} \subset \mathscr{P}, \{u_n\} \subset \mathbb{X}$ with $\gamma_n \to \gamma \in \mathscr{P}$, $u_n \rightharpoonup u \in \mathbb{X}$ and $x_n \rightharpoonup x \in \mathscr{A}(\gamma, u)$ and all $v \in \mathbb{X}$, we have

$$\limsup \jmath^0(\zeta_n, u_n; v - u_n) \leq \jmath^0(\gamma, u; v - u). \tag{4.4}$$

(10) For any $\{\gamma_n\} \subset \mathscr{P}$, $\{u_n\} \subset \mathbb{X}$ with $\gamma_n \to \gamma \in \mathscr{P}$, $u_n \rightharpoonup u \in \mathbb{X}$ and $x_n \rightharpoonup x \in \mathscr{A}(\gamma, u)$ and all $v \in \mathbb{X}$, we have

$$\limsup(\varphi(\gamma_n, u_n, v) - \varphi(\gamma_n, u_n, u_n)) \leq \varphi(\gamma, u, v) - \varphi(\gamma, u, u). \tag{4.5}$$

(11) For any $\{\gamma_n\} \subset \mathscr{P}$ with $\gamma_n \to \gamma \in P$, we have

$$f(\gamma_n) \to f(\gamma) \in \mathbb{X}^*. \tag{4.6}$$

(12)

$$0 \in \mathscr{K}, \tag{4.7}$$

and there exist positive constants ℓ_1, ℓ_2 such that for all $\gamma \in \mathscr{P}, v \in \mathbb{X}$,

$$\begin{aligned} &\|\mathscr{A}(\gamma, 0)\| \leq \ell_1\|\gamma\|_{\mathscr{P}}, \\ &|\varphi(\gamma, 0, v)| \leq \ell_2\|\gamma\|_{\mathscr{P}}\|v\|_{\mathbb{X}}. \end{aligned} \tag{4.8}$$

The first continuous dependence result reads as follows.

Theorem 4.1. *Assume that the hypotheses of Theorem* 3.3 *hold. Suppose that* (7)-(12) *hold and* $\{\gamma_n\} \subset \mathscr{P}$ *with* $\gamma_n \to \gamma \in \mathscr{P}$ *for some* $\gamma \in \mathscr{P}$*. Then the sequences* $\{u_n\} = \{u(\gamma_n)\} \subset \mathscr{K}$, $\{x_n\} = \{x(\gamma_n)\} \in \mathscr{A}(\gamma_n, u_n)$ *of unique solutions to* (4.1) *converge weakly in* $\mathbb{X}$ *to the solution* $u = u(\gamma) \in \mathscr{K}, x = x(\gamma) \in \mathscr{A}(\gamma, u)$ *of* (3.1).

Proof. We begin with the a priori estimate on the solution to (4.1). Let $\gamma_n \in \mathscr{P}$ and $u_n = u(\gamma_n) \in \mathscr{K}$, $x_n = x(\gamma_n) \in \mathscr{A}(\gamma_n, u_n)$ be the unique solution to (4.1). We show that

$$\|u_n\|_{\mathbb{X}} \leq \frac{\varsigma_0 + (\varsigma_2 + \ell_1 + \ell_2)\|\gamma_n\|_{\mathscr{P}} + \|f(\gamma_n)\|_{\mathbb{X}^*}}{\alpha_{\mathscr{A}}\rho_{\mathscr{A}}^2 - \alpha_\varphi - \alpha_\jmath}. \tag{4.9}$$

First, using hypotheses (2) and (4.7), from (4.1), we have

$$\begin{aligned}\alpha_{\mathscr{A}}\rho_{\mathscr{A}}^2\|u_n\|_{\mathbb{X}}^2 &\leq \langle x_n - x_n^0, u_n\rangle \\ &\leq \langle -x_n^0, u_n\rangle + \varphi(\gamma_n, u_n, 0) - \varphi(\gamma_n, u_n, u_n) \\ &\quad + \jmath^0(\gamma_n, u_n; -u_n) + \langle f(\gamma_n), u_n\rangle_{\mathbb{X}} \\ &= \langle -x_n^0, u_n\rangle \\ &\quad + (\varphi(\gamma_n, u_n, 0) - \varphi(\gamma_n, u_n, u_n) + \varphi(\gamma_n, 0, u_n) - \varphi(\gamma_n, 0, 0)) \\ &\quad + (\varphi(\gamma_n, 0, 0) - \varphi(\gamma_n, 0, u_n)) \\ &\quad + (\jmath^0(\gamma_n, u_n; -u_n) + \jmath^0(\gamma_n, 0; u_n)) \\ &\quad - \jmath^0(\gamma_n, 0; u_n) + \langle f(\gamma_n), u_n\rangle_{\mathbb{X}} \end{aligned} \tag{4.10}$$

for all $x_n \in \mathscr{A}(\gamma_n, u_n)$, $x_n^0 \in \mathscr{A}(\gamma_n, 0)$.

Exploiting hypotheses (3)-(4), (4.8) and the estimate

$$\begin{aligned}|\jmath^0(\gamma_n, 0; u_n)| &= |\max\{\langle\zeta_n, u_n\rangle_{\mathbb{X}} | \zeta_n \in \partial\jmath(\gamma_n, 0)\}| \\ &\leq (\varsigma_0 + \varsigma_2\|\gamma_n\|_{\mathscr{P}})\|u_n\|_{\mathbb{X}},\end{aligned} \tag{4.11}$$

we get

$$\begin{aligned}\alpha_{\mathscr{A}}\rho_{\mathscr{A}}^2\|u_n\|_{\mathbb{X}}^2 &\leq \alpha_\varphi\|u_n\|_{\mathbb{X}}^2 + \alpha_\jmath\|u_n\|_{\mathbb{X}}^2 \\ &\quad + (\varsigma_0 + (\varsigma_2 + \ell_1 + \ell_2)\|\gamma_n\|_{\mathscr{P}})\|u_n\|_{\mathbb{X}} \\ &\quad + \|f(\gamma_n)\|_{\mathbb{X}^*}\|u_n\|_{\mathbb{X}}.\end{aligned} \tag{4.12}$$

Therefore, we have

$$(\alpha_{\mathscr{A}}\rho_{\mathscr{A}}^2 - \alpha_\varphi - \alpha_\jmath)\|u_n\|_{\mathbb{X}}^2 \leq (\varsigma_0 + (\varsigma_2 + \ell_1 + \ell_2)\|\gamma_n\|_{\mathscr{P}} + \|f(\gamma_n)\|_{\mathbb{X}^*})\|u_n\|_{\mathbb{X}}, \tag{4.13}$$

and hence the estimate (4.9) follows.

Now, let $\{\gamma_n\} \subset \mathscr{P}$ with $\gamma_n \to \gamma \in \mathscr{P}$ for some $\gamma \in \mathscr{P}$. Let $u_n = u(\gamma_n) \in \mathscr{K}, x_n = x(\gamma_n) \in \mathscr{A}(\gamma_n, u_n)$ be the unique solution of (4.1) guaranteed by Theorem 3.3. It follows from (4.6) and (4.9) that $\{u_n\}$ is a bounded sequence in $\mathbb{X}$ also $\{x_n\}$ is bounded sequence in $\mathbb{X}$. Therefore, by the reflexivity of $\mathbb{X}$, we may assume, passing to a subsequence if necessary, that

$$u_n \rightharpoonup \bar{u} \in \mathbb{X} \quad \text{as } \to \infty$$

with $\bar{u} \in \mathbb{X}$. Therefore,

$$\|x_n - \bar{x}\| \leq \rho_{\mathscr{A}}\|u_n - \bar{u}\| \to 0 \ \text{ as } n \to \infty$$

implies that

$$x_n \rightharpoonup \bar{x} \in \mathscr{A}(\cdot, \bar{u}) \in \mathbb{X} \ \text{ as } \to \infty.$$

Since $u_n \in \mathscr{K}$ and $x'_n \in \mathscr{A}(\gamma, u_n)$, and is weakly closed (being closed and convex), we have

$$\bar{u} \in \mathscr{K} \quad \text{and} \quad \bar{x} \in \mathscr{A}(\gamma, \bar{u}).$$

Choosing $v = \bar{u}$ in (4.1), we have

$$\langle x_n, u_n - \bar{u}\rangle_{\mathbb{X}} \leq \langle f(\gamma_n), u_n - \bar{u}\rangle_{\mathbb{X}} + \varphi(\gamma_n, u_n, \bar{u}) - \varphi(\gamma_n, u_n, u_n) + \jmath^0(\gamma_n, u_n; \bar{u} - u_n). \tag{4.14}$$

Using hypotheses (7)–(11) in (4.14), we have

$$\begin{aligned} \limsup\langle x'_n, u_n - \bar{u}\rangle &\leq \limsup\langle x'_n - x_n, u_n - \bar{u}\rangle + \limsup\langle x_n, u_n - \bar{u}\rangle \\ &\leq \limsup\langle x'_n - x_n, u_n - \bar{u}\rangle + \limsup\langle f(\gamma_n), u_n - \bar{u}\rangle \\ &\quad + \limsup(\varphi(\gamma_n, u_n, \bar{u}) - \varphi(\gamma_n, u_n, u_n)) \\ &\quad + \limsup \jmath^0(\gamma_n, u_n; \bar{u} - u_n) \\ &\leq 0 \end{aligned} \tag{4.15}$$

for all $x'_n \in \mathscr{A}(\gamma, u_n), x_n \in \mathscr{A}(\gamma_n, u_n)$.

Exploiting the facts that the operator $\mathscr{A}(\gamma, \cdot)$ is pseudomonotone, $u_n \rightharpoonup \bar{u} \in \mathbb{X}$ and

$$\limsup\langle x'_n, u_n - \bar{u}\rangle \leq 0, \ \forall\, x'_n \in \mathscr{A}(\gamma, u_n), \tag{4.16}$$

from Lemma 2.9, we infer

$$x'_n \rightharpoonup \bar{x} \in \mathbb{X}^*,$$

or

$$\mathscr{A}(\gamma, u_n) \rightharpoonup \mathscr{A}(\gamma, \bar{u}) \in \mathbb{X}^*, \tag{4.17}$$

$$\langle x'_n, u_n - \bar{u}\rangle_{\mathbb{X}} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.18}$$

Next, conditions (4.17) and (4.18) imply

$$\langle x'_n, u_n\rangle_{\mathbb{X}} \to \langle \bar{x}, \bar{u}\rangle_{\mathbb{X}}, \tag{4.19}$$

$$\begin{aligned} \langle x'_n, u_n - v\rangle_{\mathbb{X}} &= \langle x'_n, u_n\rangle_{\mathbb{X}} - \langle x'_n, v\rangle_{\mathbb{X}} \\ &\to \langle \bar{x}, \bar{u}\rangle_{\mathbb{X}} - \langle \bar{x}, v\rangle_{\mathbb{X}} \\ &= \langle \bar{x}, \bar{u} - v\rangle_{\mathbb{X}} \ \forall v \in \mathscr{K}. \end{aligned} \tag{4.20}$$

Subsequently, we are in a position to pass to the upper limit in (4.1). Let $v \in \mathscr{K}$. Inserting

$$\langle x_n, u_n - v\rangle_{\mathbb{X}} \leq \langle f(\gamma_n), u_n - v\rangle_{\mathbb{X}} + \varphi(\gamma_n, u_n, v) - \varphi(\gamma_n u_n, u_n) + \jmath^0(\gamma_n, u_n, v - u_n) \tag{4.21}$$

into the equality

$$\langle x'_n, u_n - v\rangle_{\mathbb{X}} = \langle x_n - x'_n, v - u_n\rangle_{\mathbb{X}} + \langle x_n, u_n - v\rangle_{\mathbb{X}}, \quad \forall\, x'_n \in \mathscr{A}(\gamma, u_n)$$

and using (4.20), we get

$$\begin{aligned}\langle \bar{x}, \bar{u}-v\rangle_{\mathbb{X}} &= \lim\langle u'_n, u_n - v\rangle_{\mathbb{X}} \\ &\le \limsup\langle x'_n - x_n, u_n - v\rangle_{\mathbb{X}} + \limsup\langle f(\gamma_n), u_n - v\rangle_{\mathbb{X}} \\ &\quad + \limsup(\varphi(\gamma_n, u_n, v) - \varphi(\gamma_n, u_n, u_n)) \\ &\quad + \limsup \jmath^0(\gamma_n, u_n; v - u_n) \\ &\le \varphi(\gamma, \bar{u}, v) - \varphi(\gamma, \bar{u}, \bar{u}) + \jmath^0(\gamma, \bar{u}; v - \bar{u}) + \langle f(\gamma), \bar{u} - v\rangle_{\mathbb{X}}. \end{aligned} \tag{4.22}$$

Again, as $u_n \to \bar{u} \in \mathbb{X}$, $n \to \infty$,

$$\|x'_n - \bar{x}\|_{\mathbb{X}^*} = \hat{\mathcal{H}}(\mathscr{A}(\gamma, u_n), \mathscr{A}(\gamma, \bar{u})) \le \rho_{\mathscr{A}}\|u_n - \bar{u}\|_{\mathbb{X}} = 0,$$

that is,

$$\|x'_n - \bar{x}\|_{\mathbb{X}^*} \le 0$$

implies that

$$x'_n = \bar{x} \in \mathscr{A}(\gamma, \bar{u}).$$

Since $v \in \mathscr{K}$ is an arbitrary, we deduce that $\bar{u} \in \mathscr{K}$, $\bar{x} \in \mathscr{A}(\gamma, \bar{u})$ is a solution of (3.1). Since this problem is uniquely solvable, it follows that

$$\bar{u} = u(\gamma) \text{ and } \bar{x} = x(\gamma),$$

which completes the proof. □

Next, we provide hypotheses on operator $\mathscr{A}$ and element f to obtain the second continuous dependence result.

(13) $\mathscr{A} : \mathscr{P} \times \mathbb{X} \to 2^{\mathbb{X}^*}$ is such that there exists $\mathscr{L}_{\mathscr{A}} > 0$ for all $\gamma_1, \gamma_2 \in \mathscr{P}$, $u \in \mathbb{X}$,

$$\|x_1 - x_2\|_{\mathbb{X}^*} = \hat{\mathcal{H}}(\mathscr{A}(\gamma_1, u), \mathscr{A}(\gamma_2, u)) \le \mathscr{L}_{\mathscr{A}}\|\zeta_1 - \zeta_2\|_{\mathscr{P}}, \tag{4.23}$$

where $\hat{\mathcal{H}}$ is the Hausdorff pseudo-metric.

(14) $f : \mathscr{P} \to \mathbb{X}^*$ is such that there exists $\mathscr{L}_f > 0$ with

$$\|f(\gamma_1) - f(\gamma_2)\|_{\mathbb{X}^*} \le \mathscr{L}_f\|\gamma_1 - \gamma_2\|_{\mathscr{P}}. \tag{4.24}$$

Theorem 4.2. *Assuming the hypotheses of Theorem* 3.3, *and given that* (4.23) *and* (4.24) *hold, then*

$$\|u(\gamma_1) - u(\gamma_2)\|_{\mathbb{X}} \le \frac{\mathscr{L}_{\mathscr{A}} + \beta_\varphi + \beta_\jmath + \mathscr{L}_f}{\alpha_{\mathscr{A}}\rho^2_{\mathscr{A}} - \alpha_\varphi - \alpha_\jmath}\|\gamma_1 - \gamma_2\|_{\mathscr{P}}, \ \forall \gamma_1, \gamma_2 \in \mathscr{P}, \tag{4.25}$$

where $u(\gamma) \in \mathscr{K}$ *denotes the unique solution to* (3.1) *corresponding to* $\gamma \in \mathscr{P}$.

Proof. The existence and uniqueness of solution follow from Theorem 3.3. Let $\gamma_1, \gamma_2 \in \mathscr{P}$ be arbitrary and $u(\gamma_1), u(\gamma_2) \in \mathscr{K}$ and $x(\gamma_1) \in \mathscr{A}(\gamma_1, u(\gamma_1)), x(\gamma_2) \in \mathscr{A}(\gamma_2, u(\gamma_2))$ be the corresponding solutions, that is

$$\langle x(\gamma_1) - f(\gamma_1), v - u(\gamma_1)\rangle_{\mathbb{X}} + \varphi(\gamma_1, u(\gamma_1), v) - \varphi(\gamma_1, u(\gamma_1), u(\gamma_1)) + \jmath^0(\gamma_1, u(\gamma_1); v - u(\gamma_1)) \geq 0, \quad \forall v \in \mathscr{K}, \tag{4.26}$$

$$\langle x(\gamma_2) - f(\gamma_2), v - u(\gamma_2)\rangle_{\mathbb{X}} + \varphi(\gamma_2, u(\gamma_2), v) - \varphi(\gamma_2, u(\gamma_2), u(\gamma_2)) + \jmath^0(\gamma_2, u(\gamma_2); v - u(\gamma_2)) \geq 0, \quad \forall v \in \mathscr{K}. \tag{4.27}$$

Putting $v = u(\gamma_2)$ in (4.26) and $v = u(\gamma_1)$ in (4.27), adding them, we get

$$\begin{aligned}\langle x(\gamma_1) - x(\gamma_2), u(\gamma_1) - u(\gamma_2)\rangle_{\mathbb{X}} \leq{}& \langle f(\gamma_2) - f(\gamma_1), u(\gamma_2) - u(\gamma_1)\rangle_{\mathbb{X}} \\ &+ \varphi(\gamma_1, u(\gamma_1), u(\gamma_2)) - \varphi(\gamma_1, u(\gamma_1), u(\gamma_1)) \\ &+ \varphi(\gamma_2, u(\gamma_2), u(\gamma_1)) - \varphi(\gamma_2, u(\gamma_2), u(\gamma_2)) \\ &+ \jmath^0(\gamma_1, u(\gamma_1); u(\gamma_2) - u(\gamma_1)) \\ &+ \jmath^0(\gamma_2, u(\gamma_2); u(\gamma_1) - u(\gamma_2)).\end{aligned} \tag{4.28}$$

Rearrangement the above inequality, we have

$$\begin{aligned}\langle x'(\gamma_1) - x(\gamma_2), u(\gamma_1) - u(\gamma_2)\rangle_{\mathbb{X}} \leq{}& \langle f(\gamma_2) - f(\gamma_1), u(\gamma_2) - u(\gamma_1)\rangle_{\mathbb{X}} \\ &+ \varphi(\gamma_1, u(\gamma_1), u(\gamma_2)) - \varphi(\gamma_1, u(\gamma_1), u(\gamma_1)) \\ &+ \varphi(\gamma_2, u(\gamma_2), u(\gamma_1)) - \varphi(\gamma_2, u(\gamma_2), u(\gamma_2)) \\ &+ \jmath^0(\gamma_1, u(\gamma_1); u(\gamma_2) - u(\gamma_1)) \\ &+ \jmath^0(\gamma_2, u(\gamma_2); u(\gamma_1) - u(\gamma_2)) \\ &+ \langle x'(\gamma_1) - x(\gamma_1), u(\gamma_1) - u(\gamma_2)\rangle_{\mathbb{X}}\end{aligned} \tag{4.29}$$

for all $x'(\gamma_1) \in \mathscr{A}(\gamma_2, u(\gamma_1))$. Then,using hypotheses (2)-(5), we get

$$\begin{aligned}\alpha_{\mathscr{A}} \rho_{\mathscr{A}}^2 \|u(\gamma_1) - u(\gamma_2)\|_{\mathbb{X}}^2 \leq{}& \alpha_\varphi \|u(\gamma_1) - u(\gamma_2)\|_{\mathbb{X}}^2 + \alpha_\jmath \|u(\gamma_1) - u(\gamma_2)\|_{\mathbb{X}}^2 \\ &+ (\mathscr{L}_{\mathscr{A}} + \beta_\varphi + \beta_\jmath + \mathscr{L}_f)\|\gamma_1 - \gamma_2\|_{\mathscr{P}} \|u(\gamma_1) - u(\gamma_2)\|_{\mathbb{X}},\end{aligned}$$

and hence,

$$(\alpha_{\mathscr{A}} \rho_{\mathscr{A}}^2 - \alpha_\varphi - \alpha_\jmath)\|u(\gamma_1) - u(\gamma_2)\|_{\mathbb{X}}^2 \leq (\mathscr{L}_{\mathscr{A}} + \beta_\varphi + \beta_\jmath + \mathscr{L}_f)\|\gamma_1 - \gamma_2\|_{\mathscr{P}}, \tag{4.30}$$

which proves (4.25). □

Now, we move on to the inverse problem analysis for (3.1). In the context of inverse problems, the set-valued mixed variational-hemivariational inequality problem (3.1) is referred to as the direct problem.

Consider the following inverse problem for given an admissible subset of parameters $\mathscr{P}_{ad} \subset \mathscr{P}$ and a cost functional $\mathscr{F} : \mathscr{P} \times \mathscr{K} \to \mathbb{R}$, find a solution $\gamma^* \in \mathscr{P}_{ad}$ to the following problem

$$\mathscr{F}(\gamma^*, u(\gamma^*)) = \min \mathscr{F}(\gamma, u(\gamma)), \tag{4.31}$$

where $u = u(\gamma) \in \mathscr{K}$ is an unique solution of (3.1) corresponding to a parameter γ.

Now, we are in a position to state the main result regarding the existence of solutions to problem (4.31). We admit the following hypotheses.

$$\mathscr{P}_{ad} \text{ is a compact subset of } \mathscr{P}, \tag{4.32}$$

$$\mathscr{F} : \mathscr{P} \times \mathscr{K} \to \mathbb{R} \text{ is l.s.c. on } \mathscr{P}_{ad} \times \mathbb{X}_w, \tag{4.33}$$

$$\mathscr{F} : \mathscr{P} \times \mathscr{K} \to \mathbb{R} \text{ is l.s.c. on } \mathscr{P}_{ad} \times \mathbb{X}, \tag{4.34}$$

where $\mathbb{X}_w$ denotes the space $\mathbb{X}$ endowed with the weak topology.

Theorem 4.3. *Assume that hypotheses of Theorem* 4.1*, and* (4.32) *and* (4.33) *hold. Then the* (4.31) *has at least one solution.*

Proof. Let $\{(\gamma_n, u_n)\} \subset \mathscr{P}_{ad} \times \mathscr{K}$ be a minimizing sequence of the functional $\mathscr{F}$, that is,

$$\lim \mathscr{F}(\gamma_n, u_n) = \inf\{\mathscr{F}(\gamma, u(\gamma)) | \gamma \in \mathscr{P}\}, \tag{4.35}$$

where $\gamma_n \in \mathscr{P}_{ad}$ and $u_n \in \mathscr{K}$, $x_n \in \mathbb{X}$ is the unique solution of (3.1) that corresponds to γ_n, that is, $u_n = u(\gamma_n)$ and $x_n = x(\gamma_n) \in \mathscr{A}(\gamma_n, u_n)$. From (4.32), there is a subsequence of $\{\gamma_n\}$, denoted in the same way, such that

$$\gamma_n \to \bar{\gamma} \in \mathscr{P} \quad \text{for some } \; \bar{\gamma} \in \mathscr{P}_{ad}.$$

From Theorem 4.1, we infer that the sequence $\{u_n\} \subset \mathscr{K}$ converges weakly in $\mathbb{X}$ to the unique solution $u(\bar{\gamma}) \in \mathscr{K}$ of (3.1). Finally, from (4.33), we have

$$\mathscr{F}(\bar{\gamma}, u(\bar{\gamma})) \leq \liminf \mathscr{F}(\gamma_n, u_n) = \inf\{\mathscr{F}(\gamma, u(\gamma)) | \gamma \in \mathscr{P}_{ad}\}, \tag{4.36}$$

which shows that $\bar{\gamma}$ is a solution of the (4.31). This completes the proof. □

Similarly, we have the following result.

Theorem 4.4. *Assume that hypotheses of Theorem* 4.2 *and* (4.32) *and* (4.34) *hold. Then the* (4.31) *has at least one solution.*

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