Nonlinear Functional Analysis and Applications
Vol. 30, No. 4 (2025), pp. 1027-1036
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2025.30.04.03
http://nfaa.kyungnam.ac.kr/journal-nfaa

STABILITY ANALYSIS OF ADDITIVE FUNCTIONAL EQUATIONS IN INTUITIONISTIC FUZZY FRAMEWORKS

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Abstract. In this paper, we investigate the Hyers-Ulam stability of the following additive functional equation

$$f(kx+y)+f(x+ky)=(k+1)f(x)+(k+1)f(y),\ (k=1,2,3,\cdots),$$

in intuitionistic fuzzy normed spaces.

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[0]Received August 6, 2024. Revised December 29, 2024. Accepted December 31, 2024.
[0]2020 Mathematics Subject Classification: 47H10, 39B72, 39A30.
[0]Keywords: Additive functional equation, intuitionistic fuzzy normed space, Hyers-Ulam stability.
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1. Introduction and preliminaries

In 1940, Ulam [19] raised the following question. Under what conditions does there exists an additive mapping near an approximately addition mapping? The case of approximately additive functions was solved by Hyers [6] under certain assumption. In 1978, a generalized version of the theorem of Hyers for approximately linear mapping was given by Rassias [13]. The stability concept that was introduced and investigated by Rassias is called the Hyers-Ulam-Rassias stability. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 14, 15, 16, 18] and references therein.

In the present paper, we determine the stability results concerning the following additive functional equation

$$f(kx+y)+f(x+ky)=(k+1)f(x)+(k+1)f(y),\ (k=1,2,3,\cdots)$$

in intuitionistic fuzzy normed spaces (IFNS).

Here we recall some notations and basic definitions.

Definition 1.1. ([17]) A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is said to be a continuous t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative;
(ii) $*$ is continuous;
(iii) $a*1=a$ for all $a\in[0,1]$;
(iv) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for each $a,b,c,d\in[0,1]$.

Definition 1.2. ([17]) A binary operation $\diamond:[0,1]\times[0,1]\to[0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions:

(i) $\diamond$ is associative and commutative;
(ii) $\diamond$ is continuous;
(iii) $a\diamond 0=a$ for all $a\in[0,1]$;
(iv) $a\diamond b\le c\diamond d$ whenever $a\le c$ and $b\le d$ for each $a,b,c,d\in[0,1]$.

Using the above two definitions, Saadati and Park [17] introduced the concept of intuitionistic fuzzy normed spaces as follows:

Definition 1.3. ([17]) The five-tuple $(X,\mu,\nu,*,\diamond)$ is said to be an intuitionistic fuzzy normed space (IFNS) if X is a vector space, $*$ is continuous t-norm, $\diamond$ is a continuous t-conorm and μ,ν are fuzzy sets on $X\times(0,\infty)$ satisfying the following conditions: For all $x,y\in X$ and $s,t>0$,

(i) $\mu(x,t)+\nu(x,t)\le 1$;
(ii) $\mu(x,t)>0$;
(iii) $\mu(x,t)=1$ iff $x=0$;
(iv) $\mu(\alpha x,t)=\mu\left(x,\frac{t}{|\alpha|}\right)$ for each $\alpha\neq 0$;

(v) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$;
(vi) $\mu(x,.) : (0,\infty) \to [0,1]$ is continuous;
(vii) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$;
(viii) $\nu(x,t) < 1$;
(ix) $\nu(x,t) = 0$ iff $x = 0$;
(x) $\nu(\alpha x, t) = \nu\left(x, \frac{t}{|\alpha|}\right)$ for each $\alpha \neq 0$;
(xi) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s)$;
(xii) $\nu(x,.) : (0,\infty) \to [0,1]$ is continuous;
(xiii) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.

In this case (μ,ν) is called an intuitionistic fuzzy norm.

Definition 1.4. ([12, 20]) Let $(X,\mu,\nu,*,\diamond)$ be an IFNS. Then a sequence (x_n) is said to be intuitionistic fuzzy convergent to $L \in X$ if $\lim \mu(x_n - L, t) = 1$ and $\lim \nu(x_n - L, t) = 0$ for all $t > 0$. In this case, we write $x_n \stackrel{IF}{\to} L$ as $n \to \infty$.

Definition 1.5. ([12, 20]) Let $(X,\mu,\nu,*,\diamond)$ be an IFNS. Then a sequence (x_n) is said to be an intuitionistic fuzzy Cauchy sequence, if $\lim \mu(x_{n+p} - x_n, t) = 1$ and $\lim \nu(x_{n+p} - x_n, t) = 0$ for all $t > 0$ and $p = 1, 2, \cdots$.

Definition 1.6. ([12, 20]) Let $(X,\mu,\nu,*,\diamond)$ be an IFNS. Then $(X,\mu,\nu,*,\diamond)$ is said to be complete, if every intuitionstic fuzzy Cauchy sequence in $(X,\mu,\nu,*,\diamond)$ is intuitionistic fuzzy convergent in $(X,\mu,\nu,*,\diamond)$.

2. Intuitionistic fuzzy stability

Let k be a finite positive integer. The functional equation

$$f(kx+y) + f(x+ky) = (k+1)f(x) + (k+1)f(y) \tag{2.1}$$

is called an additive functional equation, since the function $f(x) = cx$ is its solution. Every solution of the additive functional equation is said to be an additive mapping.

We start with a Hyers-Ulam type theorem in IFNSs for the additive functional equation.

Theorem 2.1. *Let X be a linear space and let $(Z,\mu',\nu',*',\diamond')$ be an IFNS. Let $\varphi : X \times X \to Z$ be a mapping such that for some $\alpha > k$*

$$\mu'\left(\varphi\left(\frac{x}{k}, 0\right), t\right) \geq \mu'\left(\varphi(x,0), \alpha t\right),$$
$$\nu'\left(\varphi\left(\frac{x}{k}, 0\right), t\right) \geq \nu'\left(\varphi(x,0), \alpha t\right),$$

$$\lim_{n\to\infty} \mu'\left(k^n \varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right), t\right) = 1,$$
$$\lim_{n\to\infty} \nu'\left(k^n \varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right), t\right) = 0$$

for all $x, y \in X$ and $t > 0$. Let $(Y, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy Banach space and $f : X \to Y$ be a φ-approximately additive mapping such that

$$\begin{aligned} \mu & kf(kx+y)+f(x+ky)-(k+1)f(x)-(k+1)f(y) \\ &\geq \mu'(\varphi(x,y),t), \\ \nu & kf(kx+y)+f(x+ky)-(k+1)f(x)-(k+1)f(y) \\ &\leq \nu'(\varphi(x,y),t) \end{aligned} \tag{2.2}$$

for all $t > 0$ and all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\begin{aligned} \mu(A(x)-f(x),t) &\geq \mu'\left(\varphi(x,0), \frac{(\alpha-k)t}{2}\right), \\ \nu(A(x)-f(x),t) &\leq \nu'\left(\varphi(x,0), \frac{(\alpha-k)t}{2}\right) \end{aligned} \tag{2.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Put $y = 0$ in (2.2). For all $x \in X$ and $t > 0$,

$$\begin{aligned} \mu(f(kx)-kf(x),t) &\geq \mu'(\varphi(x,0),t), \\ \nu(f(kx)-kf(x),t) &\leq \nu'(\varphi(x,0),t). \end{aligned}$$

Thus

$$\begin{aligned} \mu\left(kf\left(\frac{x}{k}\right)-f(x),t\right) &\geq \mu'\left(\varphi\left(\frac{x}{k},0\right),t\right) \geq \mu'\left(\varphi(x,0),\alpha t\right), \\ \nu\left(kf\left(\frac{x}{k}\right)-f(x),t\right) &\leq \nu'\left(\varphi\left(\frac{x}{k},0\right),t\right) \leq \nu'\left(\varphi(x,0),\alpha t\right). \end{aligned}$$

Replacing x by $\frac{x}{k^n}$ in (2.2), we get

$$\begin{aligned} \mu\left(k^{n+1}f\left(\frac{x}{k^{n+1}}\right)-k^nf\left(\frac{x}{k^n}\right),k^nt\right) &\geq \mu'\left(\varphi\left(\frac{x}{k^n},0\right),\alpha t\right) \\ &\geq \mu'\left(\varphi(x,0),\alpha^{n+1}t\right), \\ \nu\left(k^{n+1}f\left(\frac{x}{k^{n+1}}\right)-k^nf\left(\frac{x}{k^n}\right),k^nt\right) &\leq \nu'\left(\varphi\left(\frac{x}{k^n},0\right),\alpha t\right) \\ &\leq \nu'\left(\varphi(x,0),\alpha^{n+1}t\right). \end{aligned} \tag{2.4}$$

Replacing t by $\frac{t}{\alpha^{n+1}}$ in (2.4), we get

$$\begin{aligned} \mu\left(k^{n+1}f\left(\frac{x}{k^{n+1}}\right)-k^nf\left(\frac{x}{k^n}\right),\frac{k^nt}{\alpha^{n+1}}\right) \geq \mu'\left(\varphi(x,0),t\right), \\ \nu\left(k^{n+1}f\left(\frac{x}{k^{n+1}}\right)-k^nf\left(\frac{x}{k^n}\right),\frac{k^nt}{\alpha^{n+1}}\right) \leq \nu'\left(\varphi(x,0),t\right). \end{aligned} \tag{2.5}$$

It follows from $k^n f\left(\frac{x}{k^n}\right) - f(x) = \sum_{j=0}^{n-1}\left(k^{j+1} f\left(\frac{x}{k^{j+1}}\right) - k^j f\left(\frac{x}{k^j}\right)\right)$ and (2.5) that

$$\begin{aligned} &\mu\left(k^n f\left(\frac{x}{k^n}\right) - f(x), \sum_{j=0}^{n-1}\frac{k^j t}{\alpha^{j+1}}\right) \\ &\quad \geq \prod_{j=0}^{n-1}\mu\left(k^{j+1} f\left(\frac{x}{k^{j+1}}\right) - k^j f\left(\frac{x}{k^j}\right), \frac{k^j t}{\alpha^{j+1}}\right) \geq \mu'\left(\varphi(x,0), t\right), \\ &\nu\left(k^n f\left(\frac{x}{k^n}\right) - f(x), \sum_{j=0}^{n-1}\frac{k^j t}{\alpha^{j+1}}\right) \\ &\quad \leq \coprod_{j=0}^{n-1}\nu\left(k^{j+1} f\left(\frac{x}{k^{j+1}}\right) - k^j f\left(\frac{x}{k^j}\right), \frac{k^j t}{\alpha^{j+1}}\right) \leq \nu'\left(\varphi(x,0), t\right) \end{aligned} \tag{2.6}$$

for all $x \in X, t > 0$ and $n > 0$, where $\prod_{j=0}^{n-1} a_j = a_1 * a_2 * ... * a_n, \coprod_{j=0}^{n-1} b_j = b_1 \diamond b_2 \diamond ... \diamond b_n$.

Replacing x with $\frac{x}{k^m}$ in (2.6), we obtain

$$\begin{aligned} &\mu\left(k^{n+m} f\left(\frac{x}{k^{n+m}}\right) - k^m f\left(\frac{x}{k^m}\right), \sum_{j=0}^{n-1}\frac{k^{j+m} t}{\alpha^{j+m+1}}\right) \\ &\geq \mu'\left(\varphi\left(\frac{x}{k^m}, 0\right), t\right) \geq \mu'\left(\varphi(x,0), t\right), \\ &\nu\left(k^{n+m} f\left(\frac{x}{k^{n+m}}\right) - k^m f\left(\frac{x}{k^m}\right), \sum_{j=0}^{n-1}\frac{k^{j+m} t}{\alpha^{j+m+1}}\right) \\ &\leq \nu'\left(\varphi\left(\frac{x}{k^m}, 0\right), t\right) \leq \nu'\left(\varphi(x,0), t\right). \end{aligned}$$

Thus

$$\begin{aligned} &\mu\left(k^{n+m} f\left(\frac{x}{k^{n+m}}\right) - k^m f\left(\frac{x}{k^m}\right), \sum_{j=m}^{n+m-1}\frac{k^j t}{\alpha^{j+1}}\right) \geq \mu'\left(\varphi(x,0), t\right), \\ &\nu\left(k^{n+m} f\left(\frac{x}{3^{n+m}}\right) - k^m f\left(\frac{x}{k^m}\right), \sum_{j=m}^{n+m-1}\frac{k^j t}{\alpha^{j+1}}\right) \leq \nu'\left(\varphi(x,0), t\right) \end{aligned}$$

for all $x \in X, t > 0, m \geq 0$ and $n \geq 0$. Hence

$$
\begin{aligned}
&\mu\left(k^{n+m} f\left(\frac{x}{k^{n+m}}\right) - k^m f\left(\frac{x}{k^m}\right), t\right) \\
&\quad \geq \mu'\left(\varphi(x,0), \frac{t}{\sum_{j=m}^{n+m-1} \frac{k^j t}{\alpha^{j+1}}}\right), \\
&\nu\left(k^{n+m} f\left(\frac{x}{k^{n+m}}\right) - k^m f\left(\frac{x}{k^m}\right), t\right) \\
&\quad \leq \nu'\left(\varphi(x,0), \frac{t}{\sum_{j=m}^{n+m-1} \frac{k^j t}{\alpha^{j+1}}}\right)
\end{aligned} \tag{2.7}
$$

for all $x \in X$, $t > 0$, $m \geq 0$ and $n \geq 0$. Since $\alpha > k$ and $\sum_{j=0}^{\infty}\left(\frac{k}{\alpha}\right) < \infty$, the Cauchy criterion for convergence in IFNSs shows that $k^n f\left(\frac{x}{k^n}\right)$ is an intuitionistic fuzzy Cauchy sequence in $(Y, \mu, \nu, *, \diamond)$. Since $(Y, \mu, \nu, *, \diamond)$ is complete, this sequence is intuitionistic fuzzy convergent to some point $A(x) \in Y$. Fix $x \in X$ and $m = 0$ in (2.7). Then we obtain

$$
\begin{aligned}
\mu\left(k^n f\left(\frac{x}{k^n}\right) - f(x), t\right) &\geq \mu'\left(\varphi(x,0), \frac{t}{\sum_{j=0}^{n-1} \frac{k^j}{\alpha^{j+1}}}\right), \\
\nu\left(k^n f\left(\frac{x}{k^n}\right) - f(x), t\right) &\leq \nu'\left(\varphi(x,0), \frac{t}{\sum_{j=0}^{n-1} \frac{k^j}{\alpha^{j+1}}}\right)
\end{aligned}
$$

for all $t > 0$ and $n > 0$. Thus we obtain

$$
\begin{aligned}
\mu\left(A(x) - f(x), t\right) &\geq \mu\left(A(x) - k^n f\left(\frac{x}{k^n}\right), \frac{t}{2}\right) * \mu\left(k^n f\left(\frac{x}{k^n} - f(x)\right), \frac{t}{2}\right) \\
&\geq \mu'\left(\varphi(x,0), \frac{t}{2\sum_{j=0}^{n-1} \frac{k^j}{\alpha^{j+1}}}\right), \\
\nu\left(A(x) - f(x), t\right) &\leq \nu\left(A(x) - k^n f\left(\frac{x}{k^n}\right), \frac{t}{2}\right) \diamond \nu\left(k^n f\left(\frac{x}{k^n} - f(x)\right), \frac{t}{2}\right) \\
&\leq \nu'\left(\varphi(x,0), \frac{t}{2\sum_{j=0}^{n-1} \frac{k^j}{\alpha^{j+1}}}\right)
\end{aligned}
$$

for large n. Taking the limit as $n \to \infty$ and using the definition of IFNS, we get

$$
\begin{aligned}
\mu\left(A(x) - f(x), t\right) &\geq \mu'\left(\varphi(x,0), \frac{(\alpha - k)t}{2}\right), \\
\nu\left(A(x) - f(x), t\right) &\leq \nu'\left(\varphi(x,0), \frac{(\alpha - k)t}{2}\right)
\end{aligned}
$$

for all $x \in X$, $t > 0$. Replacing x and y by $\frac{x}{k^n}$ and $\frac{y}{k^n}$ in (2.2), we have

$$\mu\left(k^n f\left(\frac{kx+y}{k^n}\right) + k^n f\left(\frac{x+ky}{k^n}\right) - k^n f\left(\frac{(k+1)x}{k^n}\right) - k^n f\left(\frac{(k+1)y}{k^n}\right)\right) \geq \mu'\left(\varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right), \frac{t}{k^n}\right),$$

$$\nu\left(k^n f\left(\frac{kx+y}{k^n}\right) + k^n f\left(\frac{x+ky}{k^n}\right) - k^n f\left(\frac{(k+1)x}{k^n}\right) - k^n f\left(\frac{(k+1)y}{k^n}\right)\right) \leq \nu'\left(\varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right), \frac{t}{k^n}\right)$$

for all $x, y \in X$, $t > 0$. Since

$$\lim_{n\to\infty} \mu'\left(k^n \varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right), t\right) = 1,$$
$$\lim_{n\to\infty} \nu'\left(k^n \varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right), t\right) = 0$$

for all $x, y \in X$, $t > 0$, A satisfies (2.1). Therefore, A is an additive mapping.

To prove the uniqueness of the additive mapping A, assume that there exists another additve mapping $A' : X \to Y$ which satisfies (2.3). For each $x \in X$, we have $k^n A\left(\frac{x}{k^n}\right) = A(x)$ and $k^n A'\left(\frac{x}{k^n}\right) = A'(x)$ for all $n \in N$. It follows from (2.3) that

$$\begin{aligned}
\mu\left(A(x) - A'(x), t\right) &= \mu\left(k^n A\left(\frac{x}{k^n}\right) - k^n A'\left(\frac{x}{k^n}\right), t\right) \\
&\geq \mu\left(k^n A\left(\frac{x}{k^n}\right) - k^n f\left(\frac{x}{k^n}\right), \frac{t}{2}\right) \\
&\quad * \mu\left(k^n f\left(\frac{x}{k^n}\right) - k^n A'\left(\frac{x}{k^n}\right), \frac{t}{2}\right) \\
&\geq \mu'\left(\varphi\left(\frac{x}{k^n}, 0\right), \frac{(\alpha-k)t}{2\cdot k^n}\right) \\
&\geq \mu'\left(\varphi\left(x, 0\right), \frac{\alpha^n(\alpha-k)t}{2\cdot k^n}\right)
\end{aligned}$$

and similarly

$$\nu\left(A(x) - A'(x), t\right) \leq \nu'\left(\varphi\left(x, 0\right), \frac{\alpha^n(\alpha-k)t}{2\cdot k^n}\right).$$

Since $\lim_{n\to\infty} \frac{\alpha^n(\alpha-k)}{2\cdot k^n} = \infty$ as $\alpha > k$, we get $\lim_{n\to\infty} \mu'\left(\varphi\left(x,0\right), \frac{\alpha^n(\alpha-k)t}{2\cdot k^n}\right) = 1$, and $\lim_{n\to\infty} \nu'\left(\varphi\left(x,0\right), \frac{\alpha^n(\alpha-k)t}{2\cdot k^n}\right) = 0$. Therefore, $\mu\left(A(x) - A'(x), t\right) = 1$ and $\nu\left(A(x) - A'(x), t\right) = 0$ for all $t > 0$. Hence, $A(x) = A'(x)$. This completes the proof. $\square$

In the following theorem, we consider $0 < \alpha < k$.

Theorem 2.2. *Let X be a linear space and $(Z, \mu', \nu', *', \diamond')$ be an IFNS. Let $\varphi : X \times X \to Z$ be a mapping such that for some $0 < \alpha < k$*

$$\mu'\left(\varphi(kx, 0), t\right) \geq \mu'\left(\alpha\varphi(x, 0), t\right),$$
$$\nu'\left(\varphi(kx, 0), t\right) \leq \nu'\left(\alpha\varphi(x, 0), t\right),$$

$\lim_{n\to\infty} \mu'\left(\varphi(k^n x, k^n y), k^n t\right) = 1$ and $\lim_{n\to\infty} \nu'\left(\varphi(k^n x, k^n y), k^n t\right) = 0$ for all $x, y \in X$ and $t > 0$. Let $(Y, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy Banach space and $f : X \to Y$ be a φ-approximately additive mapping in the sense that

$$\begin{aligned} &\mu\left(kf(kx + y) + f(x + ky) - (k + 1)f(x) - (k + 1)f(y), t\right) \\ &\qquad \geq \mu'\left(\varphi(x, y), t\right), \\ &\nu\left(kf(kx + y) + f(x + ky) - (k + 1)f(x) - (k + 1)f(y), t\right) \\ &\qquad \leq \nu'\left(\varphi(x, y), t\right) \end{aligned} \tag{2.8}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\mu\left(A(x) - f(x), t\right) \geq \mu'\left(\varphi\left(x, 0\right), \frac{(k - \alpha)t}{2}\right),$$
$$\nu\left(A(x) - f(x), t\right) \leq \nu'\left(\varphi\left(x, 0\right), \frac{(k - \alpha)t}{2}\right)$$

for all $x \in X$ and $t > 0$.

Proof. The proof of this theorem is similar to the proof of Theorem 2.1. Here we represent the sketch of proof. Putting $y = 0$ in (2.8), we get

$$\mu\left(\frac{f(kx)}{k} - f(x), t\right) \geq \mu'\left(\varphi(x, 0), t\right),$$
$$\nu\left(\frac{f(kx)}{k} - f(x), t\right) \leq \nu'\left(\varphi(x, 0), t\right)$$

for all $x \in X$ and $t > 0$. So

$$\mu\left(\frac{f(k^{n+1}x)}{k} - f(k^n x), t\right) \geq \mu'\left(\varphi(x, 0), \frac{t}{\alpha^n}\right),$$
$$\nu\left(\frac{f(k^{n+1}x)}{k} - f(k^n x), t\right) \leq \nu'\left(\varphi(x, 0), \frac{t}{\alpha^n}\right)$$

for all $x \in X$ and $t > 0$. For each $x \in X, n \geq 0, m \geq 0$ and $t > 0$, we deduce that

$$\begin{aligned} \mu\left(\frac{f(k^{n+m}x)}{k^{n+m}} - \frac{f(k^m x)}{k^m}, t\right) &\geq \mu'\left(\varphi(x,0), \frac{t}{\sum_{j=m}^{n+m-1} \frac{\alpha^j}{k^{j+1}}}\right), \\ \nu\left(\frac{f(k^{n+m}x)}{k^{n+m}} - \frac{f(k^m x)}{k^m}, t\right) &\leq \nu'\left(\varphi(x,0), \frac{t}{\sum_{j=m}^{n+m-1} \frac{\alpha^j}{k^{j+1}}}\right) \end{aligned} \tag{2.9}$$

for all $x \in X$, $t > 0$ and $m, n \geq 0$. Thus $\left\{\frac{f(k^n x)}{kn}\right\}$ is an intuitionistic fuzzy Cauchy sequence in the intuitionistic fuzzy Banach space. So there exists a mapping $A : X \to Y$ defined by $A(x) = \lim_{n\to\infty} \frac{f(k^n x)}{kn}$ and putting $m = 0$ in (2.9), we obtain

$$\begin{aligned} \mu\left(A(x) - f(x), t\right) \geq \mu'\left(\varphi\left(x, 0\right), \frac{(k-\alpha)t}{2}\right), \\ \nu\left(A(x) - f(x), t\right) \leq \nu'\left(\varphi\left(x, 0\right), \frac{(k-\alpha)t}{2}\right) \end{aligned}$$

for all $x \in X$ and $t > 0$. This completes the proof. □

Acknowledgments: This paper was supported by the University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund (UoE) 2026).

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