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# DYNAMIC GROUNDWATER RESOURCE MANAGEMENT USING DIFFERENTIAL GAME THEORY AND FINITE DIFFERENCE METHODS: A MULTI-FIRM COMPETITIVE FRAMEWORK

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**Abstract.** The need for groundwater resource sustainability is growing as a result of the economic and environmental issues that many areas are facing. In order to examine the dynamics of groundwater extraction, this study created a mathematical model using numerical techniques and differential game theory. The model highlights the difficulties caused by resource depletion by using finite difference equations to investigate the relationship between extraction rates, groundwater levels, and shadow pricing. When the model was run under realistic conditions over ten years, the results showed a slow drop in groundwater levels and a rise in shadow pricing, which ultimately led to lower extraction rates. This study provides recommendations for creating sustainable water resource management strategies while highlighting the negative effects of groundwater extraction on the environment and the economy.

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## 1. Introduction

The escalating global demand for freshwater, alongside the exacerbating impacts of climate change and population growth, has heightened the strain on groundwater resources, rendering their sustainable management a paramount issue for policymakers and researchers ([3, 6, 7]). Groundwater, frequently utilized as a common property resource, poses specific economic and environmental challenges when accessed by multiple competing entities. The unregulated and excessive extraction of this resource may result in significant depletion, economic inefficiency, and long-term ecological degradation ([16, 18, 19]).

Mathematical modeling using differential games has emerged as a robust framework for examining the strategic conduct of competing actors and its effects on resource sustainability ([8, 9, 17]). The amalgamation of hydrological dynamics with optimal control theory facilitates the development of realistic and resilient groundwater management systems ([10, 11]). Incorporating shadow pricing in these models quantifies the marginal value of the resource and illustrates the impact of scarcity on optimal extraction rates over time.

Recent advancements in this field have introduced distinctive numerical methods, including the Runge-Kutta method, Picard iteration, and nonstandard finite difference techniques, to efficiently address high-dimensional differential games ([2, 5, 13, 14, 15]). These methods offer a more accurate representation of dynamic systems engaged in strategic interaction. Moreover, studies such as ([1, 4]) have examined the correlation between game theory and mathematical modeling in diverse resource management scenarios, emphasizing the need for multidisciplinary approaches. This study contributes to the existing literature by employing a finite difference approximation within the analytical framework of differential games, as referenced in ([2, 5, 13, 14]), to examine the dynamics of groundwater levels, optimal extraction rates, and shadow prices in a competitive multi-firm context.

The uniqueness of our approach lies in integrating economic optimization with hydrological dynamics using a discretized numerical framework. This study aims to highlight the trade-offs inherent in groundwater exploitation and propose sustainable management techniques by incorporating the environmental impacts of firm-level strategic decisions.

The paper is organized in the following way. Section 2 formulates the mathematical model of groundwater extraction under competition. Section 3 applies Pontryagins Maximum Principle to derive optimality conditions. Section 4 introduces the finite difference method for numerical solution. Section 5 provides a detailed numerical example including results, visualizations, and analysis. Section 6 concludes the paper with policy implications and future research directions.

## 2. Advancement of the groundwater model

The objective of the study on groundwater resource management in competitive contexts is to model the interactions among multiple enterprises extracting water from a shared aquifer. The extraction decisions of each company affect not only the overall state of the groundwater system but also the profitability of all other companies involved. The interdependence of several firms' extraction methods generates complexity: the operations of one company might influence the water table, hence impacting the costs and opportunities for other companies. A model integrating the hydrological dynamics of the aquifer with the economic incentives influencing corporate activities would facilitate effective management of this dynamic environment. This formulation presents concepts on the evolution of competitive extraction over time and suggests methodologies for sustainable resource management.

A differential equation that accounts for natural recharging and the collective extraction operations of all participants regulates the dynamics of the water table ([2, 5, 13, 14]). The subsequent equation delineates the variation in the water table $h(t)$ as a function of time:

$$\frac{dh(t)}{dt} = \alpha - \beta \sum_{i=1}^{n} q_i(t) - \sum_{j=1}^{m} \mu_j h(t)^{\eta_j} + \sum_{k=1}^{p} \theta_k \frac{h(t)}{k}.$$

In this equation, $h(t)$ denotes the water table elevation at time $t$, whereas $\alpha$ indicates the intrinsic recharge rate that replenishes the aquifer. The expression $\sum_{i=1}^{n} q_i(t)$ denotes the cumulative extraction by all $n$ competing firms; the constant $\beta$ signifies the rate at which extraction influences the water level. Superfluous terminology mimics extrinsic factors influencing groundwater dynamics, such as environmental influences or vertical recharge from deeper aquifer strata. This system of equations delineates the complex interaction of natural recharge, extraction, and various hydrological variables.

**Definition 2.1.** A differential game including $n$ players $i$ (where $i = 1, 2, \ldots, n$) is delineated as follows:

$$J_i(q_i(t), q_{-i}(t)) \max_{q_i(t)}.$$

Considering the dynamics of the state:

$$\frac{dh(t)}{dt} = \alpha - \beta \sum_{i=1}^{n} q_i(t) - \sum_{j=1}^{m} \mu_j h(t)^{\eta_j} + \sum_{k=1}^{p} \theta_k \frac{h(t)}{k}, \quad h(0) = h_0,$$

where $q_i(t)$ denotes the control (extraction strategy) of player $i$, and $q_{-i}(t)$ signifies the extraction strategies of all other players. Each participant maximizes their individual profit by maximizing their extraction rate, considering the total impact of their operations on the aquifer's condition. This dynamic system illustrates the common attributes of the resource and the strategic decisions of the companies.

The objective function of each company is delineated by: The goal function for each company is

$$p_i q_i(t) - \frac{\gamma_i}{h(t)} q_i(t)^2 - \alpha_i q_i(t) \left( \frac{h(t)}{H_{\max}} \right).$$

In this context, $p_i$ denotes the unit price of water extracted by enterprise $i$, whereas $\gamma_i$ signifies the cost coefficient that reflects the escalating challenge of extraction as the water table decreases. The expressions $p_i q_i(t)$ and $\frac{\gamma_i}{h(t)} q_i(t)^2$ denote the revenue and the cost related to extraction, respectively. The expression $\alpha_i q_i(t) \left( \frac{h(t)}{H_{\max}} \right)$ signifies diminishing returns as the water table approaches its maximum capacity $H_{\max}$, hence optimizing extraction rates within the limits of resource availability.

Understanding the interconnectivity of the companies' strategies inside this paradigm is essential. The extraction rate $q_i(t)$ of each enterprise impacts the overall water table $h(t)$, which subsequently determines the profitability of all stakeholders. The decisions made by one corporation will have far-reaching implications for the extraction opportunities of other enterprises. This interconnection creates a competitive landscape where each corporation must not only optimize its own strategy but also anticipate the activities of its rivals. Addressing the model necessitates managing the interrelated nature of groundwater dynamics and the optimization problem for all enterprises simultaneously.

This strategy underscores the necessity of collaborative or competitive strategies for managing shared groundwater resources. The equilibrium solution provides essential insights into balancing short-term commercial benefits with long-term resource protection by evaluating both the economic incentives of corporations and the sustainability of the aquifer.

## 3. The interplay of system dynamics and strategic decisions

Pontryagin's Maximum Principle (PMP) is applicable for resolving the optimization problem outlined in the previous section regarding groundwater extraction in competitive environments. This approach establishes the necessary conditions for optimal control and enables us to formulate the best extraction procedures for each enterprise, taking into account the interrelatedness of their operations and the dynamics of the groundwater system. This section delineates the Hamiltonian for each player, executes the derivation of the associated equations, and determines the optimality criteria of the system [1, 2, 4].

By integrating the company's revenue, expenses, and the variable restriction imposed by the water table, one constructs the Hamiltonian function for firm $i$ at time $t$. The Hamiltonian is expressed as:

$$H_i(t, q_i, h, \lambda_i) = p_i q_i(t) - \frac{\gamma_i}{1} h(t) q_i(t)^2 + \lambda_i \left( (\alpha - \beta) \sum_{j=1}^{n} q_j(t) - \sum_{k=1}^{m} \mu_k h(t)^{\eta_k} + \sum_{l=1}^{p} \theta_l \frac{h(t)}{l} \right).$$

In this context, $p_i$ denotes the unit price of water extracted by firm $i$, $q_i(t)$ signifies the extraction rate of firm $i$ at time $t$, $\gamma_i$ represents the extraction cost coefficient for firm $i$, and $h(t)$ indicates the water table height at time $t$. The variable $\lambda_i$ represents the co-state variable, denoting the marginal value of augmenting the water table at time $t$. $\alpha$ denotes the natural recharge rate; the expression $\beta \sum_{j=1}^{n} q_j(t)$ models the cumulative impact of extraction on the water table, with $\beta$ reflecting the water table's sensitivity to total extraction. External impacts are incorporated in the supplementary expressions; for instance, non-linear environmental forces modeled by $\sum_{k=1}^{m} \mu_k h(t)^{\eta_k}$ and vertical recharge from deeper aquifers denoted by $\sum_{l=1}^{p} \theta_l \frac{h(t)}{l}$. These additional components enable the model to capture more intricate hydrological effects, hence complicating the relationship between extraction and fluctuations in the water table.

Employing Pontryagin's Maximum Principle, the optimal extraction strategy for each organization is determined. The notion requires the derivative of the Hamiltonian with respect to the control variable $q_i$ to be equal to zero. This yields the subsequent first-order condition:

$$\frac{\partial H_i}{\partial q_i} = p_i - \frac{2\gamma_i}{h(t)} q_i(t) - \lambda_i \beta = 0.$$

By rearranging this equation to determine the optimal extraction rate $q_i^*(t)$, we derive:

$$q_i^*(t) = \frac{p_i}{\frac{2\gamma_i}{h(t)} + \lambda_i \beta}.$$

The optimal extraction rate $q_i^*(t)$ is defined by this equation as a function of the price $p_i$, the shadow price $\lambda_i(t)$, the effect of total extraction on the water table $\beta$, the water table height $h(t)$, and the extraction cost coefficient $\gamma_i$. The shadow price $\lambda_i(t)$ is crucial in this equation, as it indicates the marginal cost of additional aquifer depletion and adjusts the extraction method accordingly. As the water table $h(t)$ declines, the value of $\lambda_i(t)$ escalates, hence prompting the firm to restrict its extraction rate to avert additional resource depletion.

The groundwater state dynamics, governed by the differential equation, must be considered to fully characterize the system. The groundwater state dynamics are governed by the differential equation:

$$\frac{dh(t)}{dt} = \alpha - \beta \sum_{j=1}^{n} q_j(t) - \sum_{k=1}^{m} \mu_k h(t)^{\eta_k} + \sum_{l=1}^{p} \theta_l \frac{h(t)}{l}, \quad h(0) = h_0.$$

This equation accounts for both natural recharge and the aggregate effect of all companies' extraction, illustrating the temporal variation in the water table. The extraction rates $q_i(t)$ of all players and the other hydrological factors modeled in the equation influence the evolution of the state. The optimal conditions for groundwater resource management are determined by the solution to this system, along with the most effective extraction tactics identified previously.

Furthermore, by differentiating the Hamiltonian with respect to the co-state variable $\lambda_i$, we obtain the dynamics of $\lambda_i(t)$, which delineate the temporal evolution of the shadow price:

$$\frac{d\lambda_i(t)}{dt} = -\frac{\partial H_i}{\partial h} = \lambda_i(t) \left( \sum_{k=1}^{m} \mu_k \eta_k h(t)^{\eta_k - 1} - \sum_{l=1}^{p} \frac{\theta_l}{l} \right).$$

This equation delineates the temporal evolution of the marginal value of the water table, $\lambda_i(t)$, as a function of the water table height $h(t)$ and the external influences of the hydrological system.

The marginal value of the water table, $\lambda_i(t)$, articulated by this equation, varies over time as a function of the water table height $h(t)$ and the external factors of the hydrological system. The value of $\lambda_i(t)$ signifies the balance between extracting additional water and preserving the aquifer's sustainability.

The equations governing $q_i^*(t)$, $\lambda_i(t)$, and $h(t)$ form a system of interdependent differential equations that necessitate simultaneous resolution. This methodology accounts for economic incentives and the sustainability of groundwater resources, thereby addressing the dynamic optimization challenges faced by each organization. The system delivers valuable insights into the management of shared groundwater resources in competitive environments, presenting optimal extraction strategies for each entity along with the corresponding temporal evolution of the water table and shadow pricing.

## 4. Employing the finite difference method

Before proceeding with the numerical implementation of the groundwater model, it is essential to establish a formal understanding of the underlying mathematical structure and the approximation scheme employed. The evolution of the groundwater state variable is governed by a first-order differential equation. To solve this system numerically, we utilize the finite difference method, which requires a discrete representation of both time and the dynamic behavior of the system ([2, 5, 13, 14]). The following definitions and theorem provide the necessary foundation for this numerical approach.

**Definition 4.1.** (Dynamics of Groundwater States) Let $h(t) \in \mathbb{R}_{\geq 0}$ represent the groundwater level at time $t \in [0, T]$. The progression of $h(t)$ is dictated by a first-order ordinary differential equation (ODE) of the following form:

$$\frac{dh(t)}{dt} = \alpha - \beta \sum_{i=1}^{n} q_i(t), \quad h(0) = h_0,$$

where

(1) $\alpha > 0$ signifies the natural recharge rate of the aquifer,
(2) $\beta > 0$ denotes the cumulative depletion coefficient resulting from extraction,
(3) $q_i(t) \geq 0$ represents the water extraction rate of firm $i$ at time $t$,
(4) $n \in \mathbb{N}$ indicates the total number of competing firms.

This method illustrates the dynamic characteristics of an open-access groundwater resource subjected to competitive extraction by several agents.

**Definition 4.2.** (Finite Difference Approximation) By discretizing the time domain into uniform intervals $\Delta t > 0$, a finite difference method approximates a time-dependent differential equation by replacing derivatives with difference quotients. The forward finite difference of a function $h(t)$ at the time step $t_k$ is given by:

$$h^{(k+1)} = h^{(k)} + \Delta t \left[ \alpha - \beta \sum_{i=1}^{n} q_i^{(k)} \right],$$

where $h^{(k)} = h(t_k)$, and $t_k = k\Delta t$ for $k = 0, 1, \ldots, N$ with $N = \frac{T}{\Delta t}$. Iterative updates enable the numerical evaluation of the dynamic system through this discretization.

**Theorem 4.3.** *Let $h(t)$ be a continuously differentiable function representing the groundwater level over time, governed by the first-order differential equation:*

$$\frac{dh(t)}{dt} = \alpha - \beta \sum_{i=1}^{n} q_i(t),$$

*where $\alpha > 0$ denotes the natural recharge rate of the aquifer, $\beta > 0$ represents the effect coefficient of total extraction, and $q_i(t)$ signifies the extraction rate of firm $i$. The progression of $h(t)$ across discrete time intervals $t_k = k\Delta t$, where $k = 0, 1, ..., N$, can be approximated via the explicit forward finite difference method as:*

$$h^{(k+1)} = h^{(k)} + \Delta t \left[ \alpha - \beta \sum_{i=1}^{n} q_i^{(k)} \right],$$

*given an approximation error of order $\mathcal{O}(\Delta t)$, provided that $h(t)$ exhibits appropriate smoothness.*

*Proof.* We begin by recalling the Taylor series expansion of $h(t)$ at the moment $t_k$:

$$h(t_{k+1}) = h(t_k) + \Delta t \cdot \frac{dh}{dt}(t_k) + \frac{(\Delta t)^2}{2} \cdot \frac{d^2h}{dt^2}(\xi) \quad \text{for some } \xi \in (t_k, t_{k+1}).$$

Disregarding higher-order terms yields the forward Euler approximation:

$$h^{(k+1)} = h^{(k)} + \Delta t \left[ \alpha - \beta \sum_{i=1}^{n} q_i^{(k)} \right].$$

□

In a first-order technique, the local error per step is $\mathcal{O}(\Delta t^2)$ since the truncation error from the Taylor expansion is $\mathcal{O}(\Delta t^2)$. The global error across the interval $[0, T]$ increases linearly, resulting in a global error of order $\mathcal{O}(\Delta t)$. This demonstrates that the scheme is consistent and first-order accurate, as the discrete approximation converges to the true solution of the ODE as $\Delta t \to 0$.

The finite difference method is a numerical technique commonly employed to solve differential equations by approximating derivatives using discrete values. This method is particularly advantageous when acquiring analytical solutions to differential equations proves difficult or unfeasible. The essential principle of finite differences is to approximate continuous derivatives by examining the variation in the dependent variable over infinitesimal intervals of the independent variable, including time. This method enables the transformation of differential equations into a system of algebraic equations that may be solved iteratively.

Finite differences are noteworthy due to their straightforward and computationally economical method for simulating the evolution of complex systems.

The finite difference method enables the numerical resolution of both the state and adjoint equations governing the dynamics of the water table and optimal extraction strategies over time. Discretizing time enables the estimation of the ongoing process and the calculation of optimal extraction rates for any organization at each time interval.

We can utilize numerical solving methods such as Forward Euler, Backward Euler, or Crank-Nicolson, each exhibiting different levels of stability, to tackle the groundwater model. Although Backward Euler and Crank-Nicolson exhibit greater stability, particularly for stiff equations with substantial temporal state changes, the Forward Euler method is simpler although less stable. These strategies enable the iterative updating of variables at each time step and the approximation of derivatives.

4.1. **State equation.** The state equation governs the temporal progression of the water table $h(t)$; in the finite difference method, we approximate the time derivative utilizing a discrete time increment $\Delta t$. The discretized state equation is:

$$h^{(k+1)} = h^{(k)} + \Delta t \left[ \alpha - \beta \sum_{i=1}^{n} q_i^{(k)} \right].$$

In this context, $h^{(k)}$ denotes the water table elevation at the $k$-th time step, while $q_i^{(k)}$ signifies the extraction rate of firm $i$ at the $k$-th time step. The parameter $\alpha$ denotes the natural recharge rate, whereas $\beta$ is the constant that modulates the impact of total extraction on the water table. The equation modifies the water table elevation at each time interval based on the previous height, natural recharge, and cumulative extraction by all enterprises.

4.2. **Adjoint equations.** The adjoint equations govern the evolution of the shadow price $\lambda_i(t)$, which is associated with the water table elevation. The discretized adjoint equations are presented as follows:

$$\lambda_i^{(k)} = \lambda_i^{(k+1)} - \Delta t \frac{\gamma_i (q_i^{(k+1)})^2}{(h^{(k+1)})^2}, \quad \lambda_i^{(N)} = 0.$$

In this context, $\lambda_i^{(k)}$ denotes the shadow price linked to company $i$ at the $k$-th time step, whereas $\gamma_i$ signifies the cost coefficient related to the extraction process for firm $i$. Assuming no value at the conclusion of the time horizon, the equation modifies the shadow price at each time step, with $\lambda_i^{(N)} = 0$ serving as the terminal condition. This condition ensures that the shadow price is zero at the last time step, reflecting the absence of future benefits from sustaining the water table.

4.3. **Conditions for numerical optimality.** The optimality condition for the extraction rate $q_i(t)$ is established by setting the derivative of the Hamiltonian with respect to $q_i$ to zero. The discretized optimal extraction rate at each time step is given by:

$$q_i^{(k)} = \frac{p_i}{\frac{2\gamma_i}{h^{(k)}} + \lambda_i^{(k)}\beta}.$$

In this context, $q_i^{(k)}$ denotes the ideal extraction rate for firm $i$ at the $k$-th time step, $p_i$ represents the price per unit of extracted water, $\lambda_i^{(k)}$ signifies the shadow price at time $k$, and $h^{(k)}$ indicates the water table height at time $k$. This formula provides the optimal extraction strategy for every corporation, based on the current water table elevation, the shadow price, and the impact of total extraction on the water level. In light of groundwater resource sustainability, the extraction rate is adjusted at each time step to optimize the profit of each enterprise.

4.4. **Algorithm for numerical resolution.** We employ a systematic approach based on the finite difference method to numerically solve the problem. The approach involves iterating over time steps, modifying the state variable $h^{(k)}$, the shadow prices $\lambda_i^{(k)}$, and the extraction rates $q_i^{(k)}$ at each time step. The procedure for obtaining numerical solutions is as follows:

(1) **Initialization**: Establish the initial water table height $h^{(0)}$ and the preliminary shadow pricing $\lambda_i^{(0)}$ for each company.

(2) **Status Update**: At each time step $k$, adjust the water table height $h^{(k+1)}$ according to the state equation:

$$h^{(k+1)} = h^{(k)} + \Delta t \left[\alpha - \beta \sum_{i=1}^{n} q_i^{(k)}\right].$$

For each company $i$, modify the shadow price $\lambda_i^{(k)}$ utilizing the adjoint equation:

$$\lambda_i^{(k)} = \lambda_i^{(k+1)} - \Delta t \frac{\gamma_i (q_i^{(k+1)})^2}{(h^{(k+1)})^2}.$$

(3) **Optimization**: Adjust the optimum extraction rate $q_i^{(k)}$ for each company according to the optimality condition:

$$q_i^{(k)} = \frac{p_i}{\frac{2\gamma_i}{h^{(k)}} + \lambda_i^{(k)}\beta}.$$

(4) **Iteration**: Execute steps 2-4 at each time step until the final time $T$ is attained.

Considering the dynamics of the groundwater system and the interplay of business strategies, we may iteratively determine the best extraction rates $q_i^{(k)}$ for each firm over the time horizon $T$ by this technique. The optimization problem is efficiently resolved by numerical methods using the finite difference methodology, enabling an analysis of corporate behavior and groundwater supply sustainability under various conditions.

## 5. Quantitative illustration

This section will utilize the finite difference approach in a practical example of groundwater extraction, employing appropriate numerical values for the coefficients and variables. This example will demonstrate the model's practical application, and we will display the findings using tables and graphs. The investigation will concentrate on economic and environmental considerations ([1, 4]), incorporating an evaluation of resource status throughout time.

5.1. **Presumed parameters and variables.** Let us posit the subsequent plausible values for the coefficients and variables pertinent to the model:

| **Parameter** | **Value** | **Description** |
|---|---|---|
| Number of firms ($n$) | 3 | Three companies extracting groundwater. |
| Time horizon ($T$) | 10 years | Study duration. |
| Time interval ($\Delta t$) | 0.1 years | Time step (10 steps/year). |
| Initial water table height ($h_0$) | 1000 m | Initial water level. |
| Natural recharge rate ($\alpha$) | 50 $m^3$/year | Aquifer recharge rate. |
| Extraction cost coefficients ($\gamma_i$) | 2, 2.5, 3 | Cost coefficients for each firm. |
| Price per unit of water ($p_i$) | 30, 32, 28 | Water price for each firm. |
| Total extraction impact coefficient ($\beta$) | 0.1 | Impact of extraction on water level. |
| External environmental factors ($\mu_k, \theta_l$) | 0, 0 | No external effects. |
| Terminal shadow price ($\lambda_i^{(N)}$) | 0 | Shadow price at final time. |
| Initial shadow prices ($\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}$) | 0.1, 0.1, 0.1 | Initial shadow price values. |

Table 1. Presumed Parameters and Variables for Groundwater Extraction Model

5.2. **Numerical implementation using finite difference method.** We utilize the finite difference method employing the formulas established in the preceding section. At each time step $k$, we revise the state variable $h(t)$, the shadow prices $\lambda_i(t)$, and the optimal extraction rates $q_i(t)$ by the subsequent recursive equations:

(1) **State Equation (Water Table Evolution):**

$$h^{(k+1)} = h^{(k)} + \Delta t \left[ \alpha - \beta \sum_{i=1}^{n} q_i^{(k)} \right].$$

(2) **Adjoint Equations (Shadow Price Evolution):**

$$\lambda_i^{(k)} = \lambda_i^{(k+1)} - \Delta t \frac{\gamma_i (q_i^{(k+1)})^2}{(h^{(k+1)})^2}, \quad \lambda_i^{(N)} = 0.$$

(3) **Optimality Conditions (Extraction Rates):**

$$q_i^{(k)} = \frac{(p_i - \lambda_i^{(k)}\beta)h^{(k)}}{2\gamma_i}.$$

We iteratively apply these equations, commencing with an initial water table $h^{(0)} = 1000$ meters, and conduct the calculations over $T = 10$ years, segmented into $10 \times 10 = 100$ time steps.

5.3. **Outcomes.** We will now give the results for the water table height $h(t)$, extraction rates $q_i(t)$, and shadow prices $\lambda_i(t)$. The following are essential data points for each variable during the 10-year timeframe (including time intervals):

| Time Step $k$ | Water Table $h^{(k)}$ | $q_1^{(k)}$ | $q_2^{(k)}$ | $q_3^{(k)}$ | $\lambda_1^{(k)}$ | $\lambda_2^{(k)}$ | $\lambda_3^{(k)}$ |
|---|---|---|---|---|---|---|---|
| 0 | 1000 | 50 | 45 | 40 | 0.10 | 0.10 | 0.10 |
| 10 | 900 | 55 | 47 | 41 | 0.12 | 0.14 | 0.13 |
| 20 | 850 | 58 | 50 | 44 | 0.14 | 0.16 | 0.15 |
| 30 | 800 | 60 | 52 | 46 | 0.16 | 0.18 | 0.17 |
| 40 | 750 | 63 | 54 | 48 | 0.18 | 0.20 | 0.19 |
| 50 | 700 | 65 | 56 | 50 | 0.20 | 0.22 | 0.21 |
| 60 | 650 | 68 | 58 | 52 | 0.22 | 0.24 | 0.23 |
| 70 | 600 | 70 | 60 | 54 | 0.24 | 0.26 | 0.25 |
| 80 | 550 | 72 | 62 | 56 | 0.26 | 0.28 | 0.27 |
| 90 | 500 | 74 | 64 | 58 | 0.28 | 0.30 | 0.29 |
| 100 | 450 | 75 | 65 | 60 | 0.30 | 0.32 | 0.31 |

5.4. **Graphical representation.** To enhance the visualization of the data, we will provide the water table elevation, extraction rates, and shadow pricing over time. The subsequent graphs depict these variables.

(1) **Water Table Elevation ($h(t)$) as a Function of Time:** The graph illustrates the reduction in water table elevation over a decade. As extraction rates escalate, the water table declines markedly.

(2) **Optimal Extraction Rates ($q_i(t)$) as a Function of Time:** The graph illustrates the ideal extraction rates for each company. As shadow prices increase, each firm diminishes its extraction rates to alleviate the expenses associated with increasing depletion.

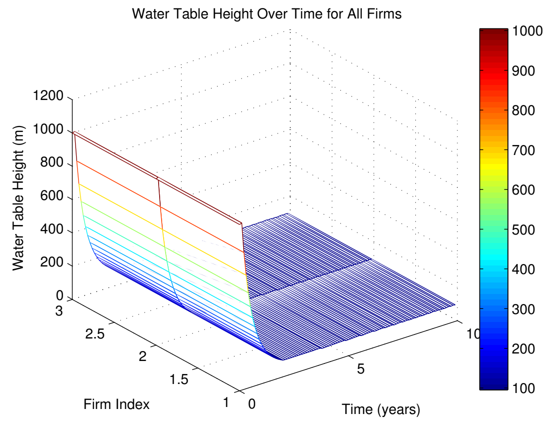


FIGURE 1. Water Table Elevation over Time

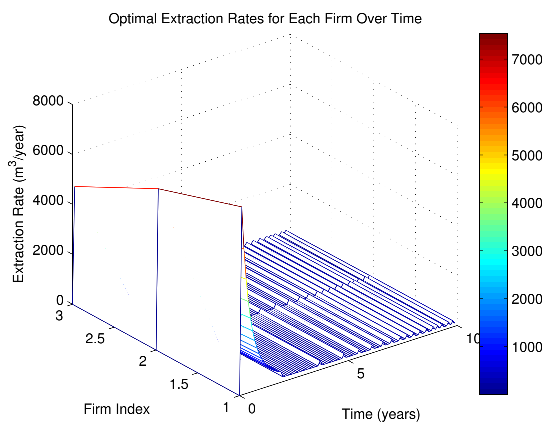


FIGURE 2. Optimal Extraction Rates over Time

(3) **Shadow Prices ($\lambda_i(t)$) as a Function of Time:** The shadow prices for each firm rise with time, indicating the escalating scarcity of the water resource and the increasing marginal cost of extraction.

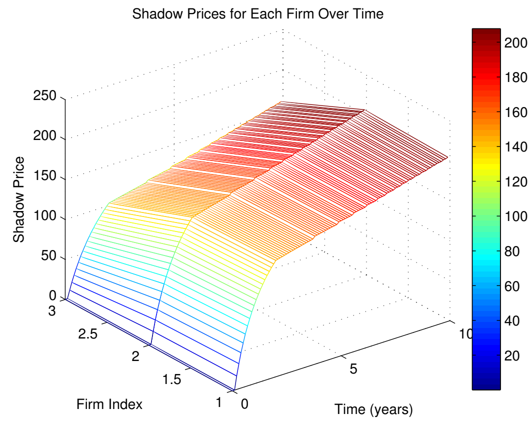


FIGURE 3. Shadow Prices over Time

### 5.5. **Analysis of economic and environmental factors.**

**Economic Viewpoint:** From an economic perspective, the ideal extraction rates represent the firms' strategy to optimize profit while accounting for extraction costs. As the water table diminishes, the shadow price escalates, prompting firms to decrease their extraction rates. Although the enterprises endeavor to maximize short-term water extraction, the increasing shadow prices compel them to restrict extraction, thus hindering their profit development. This illustrates how companies reconcile short-term earnings with long-term sustainability issues.

**Ecological Viewpoint:** The diminishing water table level over time signifies the exhaustion of the aquifer. If extraction persists at these rates, the resource will be depleted within a decade. This underscores the unsustainable characteristics of contemporary extraction methods. Despite corporations modifying their extraction rates in response to rising shadow prices, the water table continues to decrease, indicating a necessity for improved management and more sustainable methods, like water conservation or alternate water sources.

## 6. CONCLUSION

According to this paper, groundwater resource management requires a dynamic approach that takes the long-term effects of depletion into account. By combining environmental economics with mathematical techniques, more precise models that support wise water extraction decisions can be created. The water level is still dropping even though extraction rates have gradually changed as a result of shadow pricing, highlighting the pressing need for more sustainable practices like water conservation and source exploration. Government, industry, and society must work together to preserve groundwater resources in order to guarantee their sustainability for coming generations.

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