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ON SOME NONUNIQUE RANDOM FIXED POINT THEOREMS IN POLISH SPACES

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Abstract. We present some nonumique random fixed point theorems for random mappings in separable complete metric spaces. The present study includes the different categories of orbitally complete metric spaces, ordered metric spaces, metric space with two metrics and metric spaces satisfying the minimal class condition. Our results include the some recent random fixed point theorems of Dhage *et al.* (2013) as special cases.

1. Introduction

Throughout the rest of the paper, let X denote a polish space, i.e., a complete, separable metric space with a metric d. Let $(\Omega, \mathcal{A})$ denote a measurable space with σ-algebra $\mathcal{A}$. A function $x : \Omega \to X$ is said to be a random variable if it is measurable in ω. A mapping $T : \Omega \times X \to X$ is called random mapping if $T(.,x)$ is measurable for each $x \in X$. A random mapping on a metric space X is denoted by $T(\omega, x)$ or simply $T(\omega)x$ for $\omega \in \Omega$ and $x \in X$. A random mapping $T(\omega)$ is said to be continuous on X into itself if the mapping $T(\omega, \cdot)$ is continuous on X for each $\omega \in \Omega$. A measurable function $x : \Omega \to X$ is called a random fixed point of the random mappings $T(\omega)$ if $T(\omega)x(\omega) = x(\omega)$ for all $\omega \in \Omega$. The study of random fixed point theorems is initiated by Spacek [15] and Hans [10], however it is the articles published by Bharucha-Reid [2, 3] which are responsible the multitude development of random fixed point theory.

The following result is useful in the random fixed point theory in Polish spaces.

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Lemma 1.1. *Let X be a Polish space. Then, following statements hold in X.*

(a) *If $\{x_n(\omega)\}$ is a sequence of random variables converging to $x(\omega)$ for all $\omega \in \Omega$, then $x(\omega)$ is also a random variable.*

(b) *If $T(\omega, \cdot)$ is continuous or each $\omega \in \Omega$ and $x : \Omega \to X$ is a random variable, then $T(\omega)x$ is also a random variable.*

The purpose of the present paper is to extend the nonunique fixed point theorems of Ćirić [6] type to random mappings in polish space in different direction. We give our main results in the following section.

2. Nonunique Random Fixed Point Theory

Our first nonunique random fixed point theorem is as follows.

Theorem 2.1. *Let $T(\omega)$ be a continuous random mapping on a complete and separable metric space X into itself satisfying for each $\omega \in \Omega$,*

$$\begin{aligned} 0 \le \min\Big\{ & d(T(\omega)x, T(\omega)y), d(x, T(\omega)x), d(y, T(\omega)y), \\ & \frac{d(x, T(\omega)x)[1 + d(y, T(\omega)y)]}{1 + d(x, y)}, \frac{d(y, T(\omega)y)[1 + d(x, T(\omega)x)]}{1 + d(x, y)} \Big\} \\ & + b(\omega) \min\big\{ d(x, T(\omega)y), d(y, T(\omega)x) \big\} \\ \le\ & q(\omega) \max\big\{ d(x, y), [\min\{ d(x, T(\omega)x), d(y, T(\omega)y) \}] \big\} \end{aligned} \tag{2.1}$$

for all $x, y \in X$, where $b : \Omega \to \mathbb{R}$, $q : \Omega \to \mathbb{R}_+$ are measurable functions such that $0 \le q(\omega) < 1$ for all $\omega \in \Omega$. Then $T(\omega)$ has a random fixed point and which is unique if $b > q$ on Ω.

Proof. Let $x : \Omega \to X$ be an arbitrary measurable function and consider the sequence of successive iterates of $T(\omega)$ at x defined by

$$x_0 = x,\ x_1 = T(\omega)x_0,\ \cdots,\ x_n = T(\omega)x_{n-1} \tag{2.2}$$

for each $n \in \mathbb{N}$. Clearly, $\{x_n\}$ is a sequence of measurable functions on Ω into X. We shall show that $\{x_n\}$ is Cauchy sequence in X. Taking $x = x_0$ and $y = x_1$ in (2.2), we obtain

$$\begin{aligned} 0 \le \min\Big\{ & d(T(\omega)x_0, T(\omega)x_1), d(x_0, T(\omega)x_0), d(x_1, T(\omega)x_1), \\ & \frac{d(x_0, T(\omega)x_0)[1 + d(x_1, T(\omega)x_1)]}{1 + d(x_0, x_1)}, \\ & \frac{d(x_1, T(\omega)x_1)[1 + d(x_0, T(\omega)x_0)]}{1 + d(x_0, x_1)} \Big\} \end{aligned}$$

$$+ b(\omega) \min\left\{d(x_0, T(\omega)x_1), d(x_1, T(\omega)x_0)\right\}$$
$$\le q(\omega) \max\left\{d(x_0, x_1), [\min\{d(x_0, T(\omega)x_0), d(x_1, T(\omega)x_1)\}]\right\}$$

which further gives

$$0 \le \min\left\{d(x_1, x_2), d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_1)[1 + d(x_1, x_2)]}{1 + d(x_0, x_1)}, \frac{d(x_1, x_2)[1 + d(x_0, x_1)]}{1 + d(x_0, x_1)}\right\}$$
$$+ b(\omega) \min\left\{d(x_0, x_2), d(x_1, x_1)\right\}$$
$$\le q(\omega) \max\left\{d(x_0, x_1), [\min\{d(x_0, x_1), d(x_1, x_2)\}]\right\},$$

or,

$$0 \le \min\left\{d(x_1, x_2), d(x_0, x_1), \frac{d(x_0, x_1)[1 + d(x_1, x_2)]}{1 + d(x_0, x_1)}\right\}$$
$$+ b(\omega) \min\left\{d(x_0, x_2), 0\right\}$$
$$\le q(\omega) \max\left\{d(x_0, x_1), [\min\{d(x_0, x_1), d(x_1, x_2)\}]\right\}.$$

This further gives

$$\min\left\{d(x_1, x_2), d(x_0, x_1), \frac{d(x_0, x_1)[1 + d(x_1, x_2)]}{1 + d(x_0, x_1)}\right\}$$
$$\le q(\omega) \max\left\{d(x_0, x_1), [\min\{d(x_0, x_1), d(x_1, x_2)\}]\right\}. \tag{2.3}$$

If

$$\min\{d(x_1, x_2), d(x_0, x_1)\} = d(x_0, x_1),$$

then

$$d(x_0, x_1) \le \frac{d(x_0, x_1)[1 + d(x_1, x_2)]}{1 + d(x_0, x_1)}.$$

Hence, from (2.3) it follows that

$$d(x_0, x_1) \le q d(x_0, x_1)$$

which is a contraction since $q = q(\omega) < 1$ for all $\omega \in \Omega$. So

$$\min\{d(x_1, x_2), d(x_0, x_1)\} = d(x_0, x_1).$$

Now there are two cases. In the first case we have

$$d(x_1, x_2) \le q\, d(x_0, x_1).$$

In the second case we have

$$\frac{d(x_0, x_1)[1 + d(x_1, x_2)]}{1 + d(x_0, x_1)} \le q\, d(x_0, x_1),$$

which further gives

$$d(x_1, x_2) \le q\, d(x_0, x_1).$$

Proceeding in this way, by induction, it follows that

$$d(x_n, x_{n+1}) \leq q d(x_{n-1}, x_n)$$

for each $n \in \mathbb{N}$. From (2.3) it follows that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq q d(x_{n-1}, x_n) \\ &\leq q^2 d(x_{n-2}, x_{n-1}) \\ &\vdots \\ &\leq q^n d(x_0, x_1). \end{aligned} \tag{2.4}$$

Now for any positive integer p, we obtain by triangle inequality,

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + \dots + d(x_{n+p-1}, x_{n+p}) \\ &\leq q^n d(x_0, x_1) + \dots + q^{n+p-1} d(x_0, x_1) \\ &\leq \left[q^n + q^{n+1} + \dots + q^{n+p-1}\right] d(x_0, x_1) \\ &\leq \frac{q^n(1-q^{p-1})}{1-q} \\ &\leq \frac{q^n}{1-q} \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{2.5}$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. The metric space X being $T(\omega)$-orbitally complete, there is a measurable function $x^* : \Omega \to X$ such that $\lim_{n\to\infty} x_n = x^*$. Again as $T(\omega)$ is $T(\omega)$-orbitally continuous, we have

$$T(\omega)x^*(\omega) = \lim_{n\to\infty} T(\omega)x_n(\omega) = \lim_{n\to\infty} x_{n+1}(\omega) = x^*(\omega)$$

for each $\omega \in \Omega$. Thus x^* is a random fixed point of the random mapping $T(\omega)$ on $\Omega \times X$ into X. To prove uniqueness, assume that $b(\omega) > q(\omega)$ for each $\omega \in \Omega$. If $y^*(\neq x^*)$ is another random fixed point of $T(\omega)$, then from condition (2.1) we obtain a contradiction. Hence $T(\omega)$ has a unique random fixed point. This completes the proof. $\square$

Corollary 2.2. *Let T be a continuous mapping on a complete metric space X into itself satisfying*

$$\begin{aligned}
0 \le \min\Big\{ & d(Tx,Ty), d(x,Tx), d(y,Ty), \\
& \frac{d(x,Tx)[1+d(y,Ty)]}{1+d(x,y)}, \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} \Big\} \\
& + b \min\big\{d(x,Ty), d(y,Tx)\big\} \\
& \le q \max\big\{d(x,y), [\min\{d(x,Tx), d(y,Ty)\}]\big\} \qquad (2.6)
\end{aligned}$$

for all $x, y \in X$, where $b \in \mathbb{R}$ and $q \in \mathbb{R}_+$ is such that $0 \le q < 1$. Then T has a fixed point and which is unique if $b > q$.

Corollary 2.2 includes several known fixed point results in the literature including those of Ćirić [6] and Dhage [7] as special cases. Sometimes it possible that a metric space may be complete w.r.t. a metric but may not be complete w.r.t. another metric defined on it. Therefore, it is interesting to obtain the fixed point theorems in such situation. Next we prove a nonunique random fixed point theorem in a metric space with two metrics defined on it.

Theorem 2.3. *Let X be a metric space with two metrics d_1 and d_2. Let $(\Omega, \mathcal{A})$ be a measurable space and let $T : \Omega \times X \to X$ be a random mapping satisfying the condition* (2.1) *w.r.t. d_2 for each $\omega \in \Omega$. Further suppose that*

(i) *$d_1(x,y) \le d_2(x,y)$ for all $x, y \in X$,*
(ii) *$T(\omega)$ is a continuous w.r.t. d_1,*
(iii) *X is complete w.r.t. d_1, and*
(iv) *X is separable metric space.*

Then $T(\omega)$ has a random fixed point and which is unique if $b > q$ on Ω.

Proof. Let $x \in X$ be arbitrary and consider the sequence $\{x_n\}$ of successive iterations of $T(\omega)$ defined by (2.2). Then, $\{x_n\}$ is a sequence of measurable functions from Ω into X. Now proceeding as in the proof of Theorem 2.1, we obtain,

$$d_2(x_n, x_{n+p}) \le \frac{q^n}{(1-q)}$$

for some positive integer p. By hypothesis,

$$\begin{aligned}
d_1(x_n, x_{n+p}) &\le d_2(x_n, x_{n+p}) \\
&\le \frac{q^n}{(1-q)} d_2(x_0, x_1) \\
&\to 0 \quad \text{as} \quad n \to \infty. \qquad (2.7)
\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence w.r.t. the metric d_1. The metric space (X, d_1) being $T(\omega)$-orbitally complete, there is a measurable function $x^* : \Omega \to X$ such that

$$\lim_{n\to\infty} x_{n+1}(\omega) = x^*(\omega)$$

for each $\omega \in \Omega$. From the above limit, it follows that

$$T(\omega)x^*(\omega) = \lim_{n\to\infty} T(\omega)x_n(\omega) = \lim_{n\to\infty} x_{n+1}(\omega) = x^*(\omega)$$

for each $\omega \in \Omega$. Thus $T(\omega)$ has a random fixed point. If $b(\omega) > q(\omega)$ for all $\omega \in \Omega$, then the unicity of random fixed point x^* follows very easily and the proof of Theorem 2.3 is complete. □

3. Random Fixed Points Mappings in Ordered Metric Spaces

We equip the metric space X with an order relation $\leq$ which is a reflexive, antisymmetric and transitive relation in X. The metric space X together with the order relation $\leq$ is called an ordered metric space. A random mapping $T : \Omega \times X \to X$ is called nondecreasing if for any $x, y \in X$ with $x \leq y$ we have that $T(\omega)x \leq T(\omega)y$ for all $\omega \in \Omega$. Similarly random mapping $T : \Omega \times X \to X$ is called non increasing if for any $x, y \in X$, $x \leq y$ implies $T(\omega)x \leq T(\omega)y$ for all $\omega \in \Omega$. A monotone random mapping which is either nondecreasing or nonincreasing on X.

The investigation of the existence of fixed points in partially ordered sets was first considered in Ram and Reuriungs [12]. This study was continued in Nieto and Rodriguer-Lopez [14] by assuming the existence of only lower solution instead of usual approach where both the lower and upper solutions are assumed to exist. These fixed point theorems are then applied to obtain existence and uniqueness results for nonlinear ordinary differential equations in the same paper. A further extension of this idea was considered in Bhaskar and Lakshmikanthan [5]. Below we prove some nonunique random fixed point theorems for monotone random mappings in separable and complete metric spaces.

Theorem 3.1. *Let $(\Omega, \mathcal{A})$ be a measurable space and let X be a partially ordered separable and complete metric space. Let $T : \Omega \times X \to X$ be a monotone nondecreasing random mapping satisfying the contraction condition* (2.1). *Further if $T(\omega)$ is continuous and if there exists a measurable function $x_0 : \Omega \to X$ such that $x_0 \leq T(\omega)x_0$ for all $\omega \in \Omega$, then the random mapping $T(\omega)$ has a random fixed point and which is unique if every pair of elements of X has a lower and an upper bound in X and $b > q$ on Ω.*

Proof. Define a sequence $\{x_n\}$ of successive approximations of $T(\omega)$ by

$$x_{n+1} = T(\omega)x_n, \quad n = 0, 1, 2, ...$$

Clearly $\{x_n\}$ is a sequence of measurable functions from Ω into X such that

$$x_0 \leq x_1 \leq ... \leq x_n \leq ...$$

We show that x_n is a Cauchy sequence in X. Taking $x = x_0$ and $y = x_1$ in (2.1) we obtain

$$d(x_1, x_2) \leq qd(x_0, x_1).$$

Processing in this way, by induction,

$$d(x_n, x_{n+1}) \leq qd(x_{n-1}), x_n)$$

for each $n = 1, 2, ...$ Then by repeated applications of the above inequality, we obtain

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1).$$

Now for any positive integer $m > n$, by triangle inequality, we get

$$\begin{aligned} d(x_m, x_n) &= d(x_n, x_m) \\ &\leq d(x - n, x_{n+1}) + .. + d(x_{m-1}, x_m) \\ &\leq (q^n + q^{n+1} ... + q^{m-n})d(x_0, x_1) \\ &\leq \frac{q^n(1 - q^{m-n})}{1 - q} d(x_0, x_1) \\ &\leq \frac{q^n}{1 - q} d(x_0, x_1) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.1}$$

This shows that $\{x_n\}$ is Cauchy sequence in X. The ordered metric space X being complete, there is a measurable function $x^* : \Omega \to X$ such that $\lim_{n\to\infty} x_n = x^*$. From the continuity of the random mapping $T(\omega)$ it follows that

$$\begin{aligned} x^*(\omega) &= \lim_{n\to\infty} x_{n+1}(\omega) = \lim_{n\to\infty} T(\omega)x_n(\omega) = T(\omega) \lim_{n\to\infty} x_n(\omega) \\ &= T(\omega)x^*(\omega) \end{aligned} \tag{3.2}$$

for all $\omega \in \Omega$. Thus x^* is a random fixed point of the random mapping $T(\omega)$ on X. Further if every pair o elements $x, y \in X$ has a ower and an upper bound, then it can be shown as in the proof of Theorem 2.1 given in Ran and Reurings [12] that $\lim\limits_{n\to\infty} T^n(\omega)x(\omega) = x^*(\omega)$ for all measurable unctions $x : \Omega \to X$. Hence $T(\omega)$ has a unique fixed point. This completes the proof. □

Next, we deal with the case of metric space X with two metrics d_1 and d_2 is defined on it and prove some nonunique random fixed point theorems on separable ordered metric spaces.

Corollary 3.2. *Let X be a partially ordered set and let there exist a metric d such that (X, d) is complete metric space. Let $T : X \to X$ be a monotone nondecreasing mapping satisfying the contraction condition* (2.6)*. Further if T is continuous and if there exists an element $x_0 \in X$ such that $x_0 \leq Tx_0$, then the mapping T has a fixed point and which is unique if every pair of elements of X has a lower and an upper bound in X and $b > q$.*

Corollary 3.2 is new to the literature on fixed point theory on ordered metric spaces and includes a basic fixed point theorem of Nieto and Rodriguez-Lopez [14] as special case under weaker continuity condition.

Theorem 3.3. *Let (Ω, A) be a measurable space and let X be an ordered metric space with two metrics d_1 and d_2. Let $T : \Omega \times X \to X$ be a nondecreasing random mapping satisfying the condition on* (2.1)*. Suppose that the following conditions hold in X.*

(i) *$d_1(x, y) \leq d_2(x, y)$ for all $x, y \in X$.*
(ii) *$T(\omega)$ is continuous w.r.t. d_2.*
(iii) *X is Polish space w.r.t. d_1.*

Further if there exists a measurable function $x_0 : \Omega \to X$ such that $x_0 \leq T(\omega)x_0$ for all $\omega \in \Omega$, then $T(\omega)$ has a random fixed point and which is unique if every pair of elements of X has a lower and an upper bound in X and $b > q$ on Ω.

Proof. Consider the sequence $\{x_n\}$ of successive iterations of $T(\omega)$ at x_0 defined by

$$x_{n+1} = T(\omega)x_n, \ n = 0, 1, 2, ..$$

Clearly, $\{x_n\}$ is a sequence of measurable functions from Ω into X w.r.t. the metric d_1 such that

$$x_0 \leq x_1 \leq \ldots \leq x_n \leq \ldots.$$

Then it can be shown as in the proof of Theorem 3.1 that $\{x_n\}$ is Cauchy sequence in X w.r.t. the metric d, that is, for any positive integer $m > n$,

$$d_2(x_m, x_n) \leq \frac{q^n}{1-q} d_2(x_0, x_1).$$

From hypothesis it follows that

$$d_1(x_m, x_n) \leq \frac{q^n}{1-q} d_2(x_0, x_1) \to 0 \ \text{ as } n \to \infty.$$

This shows that $\{x_n\}$ is a Cauchy sequence w.r.t. the metric d_1. The metric space (X, d_1) being complete and separable, there exists a measurable function $x^* : \Omega \to X$ such that $\lim_{n\to\infty} x_n = x^*$. From the continuity of $T(\omega)$ w.r.t. d_1, it follows that

$$T(\omega)x^*(\omega) = \lim_{n\to\infty} T(\omega)x_{n+1}(\omega) = x^*(\omega)$$

for all $\omega \in \Omega$. This proves that $T(\omega)$ has a random fixed point in X. If every pair of elements of X has a lower and an upper bound in X and $b > q$ on Ω, then the uniqueness follows very easily. This completes the proof. □

4. Nonunique PPF Dependant Random Fixed Point Theory

A fixed point theory of nonlinear operations which are PPF dependent, theory is depending on past, present and future data was developed in Bernfield et.al. [1]. The domain space of the nonlinear operator was taken as $C(I, E)$, $I = [a, b] \subset \mathbb{R}$ and the range space as E, a Banach space. An important example of such a nonlinear operator is a delay differential equation. The PPF fixed point theorems are applied to ordinary nonlinear functional differential equations for proving the existence of solutions. Random fixed point theory for random operator in separable Banach spaces is initiates by Hans [10] and Spacek [15] and further developed by several authors in the literature. A brief survey of such random fixed point theorems appears in Joshi and Bose [11].

In the present section we obtain a successful fusion of above two ideas and prove some nonunique PPF dependent random fixed point theorems for random mappings in separable metric spaces. In the PPF dependent classical fixed point theory, the Razumikkin or minimal class of functions plays a significant role both in proving existence as well as uniqueness of PPF dependent fixed points of the mappings under consideration. Let E be a metric space and let I be a given closed and bounced interval in $\mathbb{R}$, the set of real numbers. Let $E_0 = C(I, \mathbb{R})$ denote the class of continuous mappings from I to E. We equip the class $C(J, E)$ with metric d_0 defined by

$$d_0(x, y) = \sup_{t\in J} d(x(t), y(t)).$$

The following result is obvious.

Lemma 4.1. *If (E, d) is complete then the metric space (E_0, d_0) is also complete.*

When E is a Banach space and let $E_0 = C(J, E)$ be a space of continuous E -valued function defined on J Then minimal class of functions related to a fixed $c \in J$ is defined as

$$\mathcal{M}_c = \{\phi \in E_0 \mid \|\phi\|_{E_0} = \|\phi(c)\|_E\}.$$

Now we are in a position to state our fixed point results concerning the existence of fixed points with PPF dependence. In a metric space X, we define the minimal class $\mathcal{M}_c$ as

$$\mathcal{M}_c = \{\phi, \psi \in E_0 \mid d_0(\phi, \psi) = d(\phi(c), \psi(c))\}.$$

Now we are in a position to state our main result of this section.

Theorem 4.2. *Let $(\Omega, \mathcal{A})$ be a measurable space and E, a separable complete metric space. Let $T : \Omega \times E_0 \to E$ be a continuous random mapping satisfying for each $\omega \in \Omega$,*

$$\begin{aligned}
0 \le \min \Big\{ & d(T(\omega)\phi, T(\omega)\psi), d(\phi(c,\omega), T(\omega)\phi), d(\psi(c,\omega), T(\omega)\psi), \\
& \frac{d(\phi(c,\omega), T(\omega)\phi)[1 + d(\psi(c,\omega), T(\omega)\psi)]}{1 + d_0(\phi,\psi)}, \\
& \frac{d(\psi(c,\omega), T(\omega)\psi)[1 + d(\phi(c,\omega), T(\omega)\phi)]}{1 + d_0(\phi,\psi)} \Big\} \\
& + b(\omega) \min \{ d(\phi(c,\omega), T(\omega)\psi), d(\psi(c,\omega), T(\omega)\phi) \} \\
& \le q(\omega) \max \{ d_0(\phi,\psi), [\min\{ d(\phi(c,\omega), T(\omega)\phi), d(\psi(c,\omega), T(\omega)\psi)\}] \}
\end{aligned} \tag{4.1}$$

for all $\phi, \psi \in E_0$, where $b : \Omega \to \mathbb{R}$ and $q : \Omega \to \mathbb{R}_+$ are measurable functions satisfying $0 \le q(\omega) < 1$ for all $\omega \in \Omega$ and $c \in I$ is a fixed point. Then $T(\omega)$ has a random fixed point with PPF dependence and which is unique if $\mathcal{M}_c$ is closed and $b > q$ on Ω.

Proof. Let $\phi_0 : \Omega \to E_0$ be an arbitrary measurable function and define a sequence $\{x_n\}$ in E_0 as follows. Suppose that $T(\omega)\phi_0 = x_1$ for some $x_1 \in E$ Then choose $\phi_1 \in E_0$ such that $\phi_1(c,\omega) = x_1$ for some fixed $c \in I$ and

$$d_0(\phi_0, \phi_1) = d(\phi_0(c,\omega), \phi_1(c,\omega))$$

for all $\omega \in \Omega$. Again let $T(\omega)\phi_1 = x_2$ for some $x_2 \in E$. Then choose $\phi_2(c,\omega) = x_2$ for each fixed $c \in I$ and

$$d_0(\phi_1, \phi_2) = d(\phi_1(c,\omega), \phi_2(c,\omega))$$

for all $\omega \in \Omega$. Proceeding in this way, we obtain

$$T(\omega)\phi_{n-1} = x_n = \phi_n(c,\omega)$$

with

$$d_0(\phi_{n-1}, \phi_n) = d(\phi_{n-1}(c,\omega), \phi_n(c,\omega), \; n \in \mathbb{N}, \tag{4.2}$$

for all $\omega \in \Omega$. Clearly, $\{\phi_n\}$ and consequently $\{\phi_n(c)\}$ is a sequence of measurable functions from Ω into E_0. Consequently $\{\phi_n(c)\}$ is a sequence of measurable functions from Ω into E. We show that $\phi_n(c,\omega)$ is a Cauchy sequence in E. Taking $\phi = \phi_0$ and $\psi = \phi_1$ in the inequality (4.1) we obtain

$$\begin{aligned} 0 \le \min\Bigg\{ & d(T(\omega)\phi_0, T(\omega)\phi_1), d(\phi_0(c,\omega), T(\omega)\phi_0), d(\phi_1(c,\omega), T(\omega)\phi_1), \\ & \frac{d(\phi_0(c,\omega), T(\omega)\phi_0)[1 + d(\phi_1(c,\omega), T(\omega)\phi_1)]}{1 + d_0(\phi_0, \phi_1)}, \\ & \frac{d(\phi_1(c,\omega), T(\omega)\phi_1)[1 + d(\phi_0(c,\omega), T(\omega)\phi_0)]}{1 + d_0(\phi_0, \phi_1)}\Bigg\} \\ & + b(\omega) \min\left\{d(\phi_0(c,\omega), T(\omega)\phi_1), d(\phi_1(c,\omega), T(\omega)\phi_0)\right\} \\ & \le q(\omega)\ \max\left\{d_0(\phi_0, \psi_1), [\min\{d(\phi_0(c,\omega), T(\omega)\phi_0), d(\phi_1(c,\omega), T(\omega)\phi_1)\}]\right\} \end{aligned} \tag{4.3}$$

which further gives

$$\begin{aligned} 0 \le \min\Bigg\{ & d(\phi_1(c,\omega), \phi_2(c,\omega)), d(\phi_0(c,\omega), \phi_1(c,\omega)), d(\phi_1(c,\omega), \phi_2(c,\omega)), \\ & \frac{d(\phi_0(c,\omega), \phi_1(c,\omega))[1 + d(\phi_1(c,\omega), \phi_2(c,\omega))]}{1 + d_0(\phi_0, \phi_1)}, \\ & \frac{d(\phi_1(c,\omega), \phi_2(c,\omega))[1 + d(\phi_0(c,\omega), \phi_1(c,\omega))]}{1 + d_0(\phi_0, \phi_1)}\Bigg\} \\ & + b(\omega) \min\left\{d(\phi_0(c,\omega), \phi_2(c,\omega)), d(\phi_1(c,\omega), \phi_1(c,\omega))\right\} \\ & \le q(\omega)\ \max\left\{d_0(\phi_0, \phi_1), [\min\{d(\phi_0(c,\omega), \phi_1(c,\omega)), d(\phi_1(c,\omega), \phi_2(c,\omega))\}]\right\}. \end{aligned} \tag{4.4}$$

From expressions (4.2) and (4.4) it follows that

$$\begin{aligned} 0 \le \min\Bigg\{ & d_0(\phi_1, \phi_2), d_0(\phi_0, \phi_1), d_0(\phi_1, \phi_2), \\ & \frac{d_0(\phi_0, \phi_1)[1 + d_0(\phi_1, \phi_2)]}{1 + d_0(\phi_0, \phi_1)}, \frac{d_0(\phi_1, \phi_2)[1 + d_0(\phi_0, \phi_1)]}{1 + d_0(\phi_0, \phi_1)}\Bigg\} \\ & + b(\omega) \min\left\{d_0(\phi_0, \phi_2), d_0(\phi_1, \phi_1)\right\} \\ & \le q(\omega)\ \max\left\{d_0(\phi_0, \phi_1), [\min\{d_0(\phi_0, \phi_1), d_0(\phi_1, \phi_2)\}]\right\}. \end{aligned} \tag{4.5}$$

Now proceeding as in the poof of Theorem 2.1, it can be proved that

$$d_0(\phi_1, \phi_2) \le q d_0(\phi_0, \phi_1).$$

Proceeding in this way, by induction,

$$d_0(\phi_n, \phi_{n+1}) \le q d_0(\phi_{n-1}, \phi_n) \tag{4.6}$$

for each n, $n = 1, 2, \ldots$. By a repeated application of the inequality (4.6), we obtain

$$\begin{aligned} d_0(\phi_n, \phi_{n+1}) &\leq q d_0(\phi_{n-1}, \phi_n) \\ &\ \vdots \\ &\leq q^n d_0(\phi_0, \phi_1). \end{aligned} \tag{4.7}$$

Now for any positive integer p, by triangle inequality,

$$\begin{aligned} d_0(\phi_n, \phi_{n+p}) &\leq d_0(\phi_n, \phi_{n+1}) + \cdots + d_0(\phi_{n+p-1}, \phi_{n+p}) \\ &\leq q^n \big(1 + q + \cdots + q^{p-1}\big) d_0(\phi_0, \phi_1) \\ &\leq \frac{q^n}{(1-q)} d_0(\phi_0, \phi_1) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{4.8}$$

Since

$$d(\phi_n(c, \omega), \phi_{n+p}(c, \omega)) = d_0(\phi_n, \phi_{n+1})$$

for all $\omega \in \Omega$, we have that $\big\{T(\omega)\phi_n\big\}$ is also Cauchy sequence in E. As E is a complete metric space, there exists a measurable function $\phi^* : \Omega \to E_0$ such that $\phi_n \to \phi^*$ and

$$T(\omega)\phi_n = \phi_{n+1}(c, \omega) \to \phi^*(c, \omega)$$

as $n \to \infty$. To prove that ϕ^* is a PPF dependent random fixed point of $T(\omega)$, we first observe that since $T(\omega)$ is continuous on E_0, $T(\omega)$ is a continuous at ϕ^*. Hence for $\epsilon > 0$, there exists a $\delta > 0$ such that

$$d_0(\phi_{n+1}, \phi^*) < \delta \implies d(T\phi_{n+1}, T\phi^*) < \frac{\epsilon}{2}.$$

Also since $T(\omega)\phi_n \to \phi^*(c, \omega)$, for $\gamma = \min\left\{\frac{\epsilon}{2}, \delta\right\}$ there exists $n_0 \in \mathbb{N}$ such that

$$d(T(\omega)\phi_n, \phi^*(c, \omega)) < \gamma$$

for $n \geq n_o$. Thus,

$$\begin{aligned} d(T(\omega)\phi^*, \phi^*(c, \omega)) &\leq d(T(\omega)\phi^*, T(\omega)\phi_n) + d(T(\omega)\phi_n, \phi^*(c, \omega)) \\ &< \frac{\epsilon}{2} + \gamma < \epsilon. \end{aligned} \tag{4.9}$$

Since ϵ is arbitrary, we have

$$T(\omega)\phi^*(\omega) = \phi^*(c, \omega)$$

for all $\omega \in \Omega$. To prove the uniqueness, assume that $\mathcal{M}_c$ is closed in E_0 and $b > q$ on Ω. Then $\phi^* \in \mathcal{M}_c$. Let ψ^* be another PPF dependent fixed point of $T(\omega)$ in $\mathcal{M}_c$. Now by virtue of $\mathcal{M}_c$, we obtain

$$d_0(\phi^*(\omega), \psi^*(\omega)) = d(T(\omega)\phi^*(\omega), T(\omega)\psi^*(\omega)) = d(\phi^*(c, \omega), \psi^*(c, \omega))$$

for all $\omega \in \Omega$. If we substitute $x = \phi^*$ and $y = \psi^*$ in (4.1), then we get a contradiction. Hence, $\phi^*(\omega) = \psi^*(\omega)$ for all $\omega \in \Omega$. This completes the proof. □

Corollary 4.3. *Let E be complete metric space and let $T : E_0 \to E$ be a continuous mapping satisfying for some $b \in \mathbb{R}$,*

$$\begin{aligned}
0 \le \min\Big\{ & d(T\phi, T\psi), d(\phi(c), T\phi), d(\psi(c), T\psi), \\
& \frac{d(\phi(c), T\phi)[1 + d(\psi(c), T\psi)]}{1 + d_0(\phi, \psi)}, \frac{d(\psi(c), T\psi)[1 + d(\phi(c), T\phi)]}{1 + d_0(\phi, \psi)} \Big\} \\
& + b \min\big\{ d(\phi(c), T\psi), d(\psi(c), T\phi) \big\} \\
& \le q \max\big\{ d_0(\phi, \psi), [\min\{ d(\phi(c), T\phi), d(\psi(c), T\psi) \}] \big\}
\end{aligned} \tag{4.10}$$

for all $\phi, \psi \in E_0$, where $c \in I$ is a fixed point and $q \in \mathbb{R}_+$ is a number such that $0 \le q < 1$. Then T has a fixed point with PPF dependence and which is unique if $\mathcal{M}_c$ is closed and $b > q$.

Notice that Corollary 4.3 generalizes PPF dependent fixed point result of Bernfeld *et al.* [1] for the mappings satisfying standard Banach type contraction condition and generalize several other classical PPF dependent fixed point theorems on the lines of Ćirić [6].

5. Conclusion

Finally, while concluding this paper we mention that the random fixed point results of this paper are for only linear contraction which may be generalized to nonlinear contraction on the lines of Boyed and Wong [4]. The contraction condition that has been considered in this paper is the only condition in the literature on metric fixed point theory which generalizes the Banach contraction condition in the both left and right direction. Furthermore, our fixed point theorems may be extended to two, three and four random mappings in Polish spaces to prove the random common fixed point theorems along the similar lines with appropriate modifications. Some of the results along this line will be reported elsewhere.

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