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GENERALIZED DIVERGENCE MEASURES IN INTUITIONISTIC FUZZY METRIC SPACES USING CONTINUOUS t-NORMS AND t-CONORMS

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Abstract. This paper presents a novel class of generalized divergence measures for intuitionistic fuzzy sets within the framework of intuitionistic fuzzy metric spaces. By incorporating continuous *t-norms* and *t-conorms*, our approach extends the recent work of Liu et al. (2023) on aggregation operators for complex intuitionistic fuzzy sets. The proposed measures are formulated as flexible, parameterized functions, thereby broadening the mathematical foundation of divergence analysis and enhancing their applicability to real-world fuzzy information systems. We rigorously establish key theoretical properties of these measures, including non-negativity, symmetry, and convexity. In addition, we investigate the convergence behavior of sequences of intuitionistic fuzzy sets, providing a unified framework for analyzing similarity and dissimilarity in intuitionistic fuzzy systems. The generalized measures encompass well-known divergences as special cases, offering new perspectives for applications in multi-criteria decision analysis, machine learning, and signal processing.

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1. Introduction

Intuitionistic fuzzy sets (IFS), introduced by Atanassov [1], have become a fundamental tool for representing uncertainty in complex systems. Unlike classical fuzzy sets, which only capture membership degrees, IFSs involve both membership and non-membership functions, allowing for a more comprehensive representation of uncertainty. The additional non-membership function provides an intuitive measure of hesitation or ambiguity in the classification of elements.

Divergence measures quantify the dissimilarity between two fuzzy or intuitionistic fuzzy sets, making them essential in areas such as information theory, data mining, and multi-criteria decision-making. Many divergence measures have been proposed for fuzzy sets, including the *Kullback–Leibler divergence* and *J-divergence*; however, their direct extension to intuitionistic fuzzy sets remains an area of active research, as it requires careful handling of both membership and non-membership components.

Recent advancements have introduced more sophisticated divergence measures in the IFS context. For instance, Liu et al. [5] utilized Aczel–Alsina t-norms and t-conorms for complex intuitionistic fuzzy sets to address decision-making problems. Additionally, entropy-based divergence measures [8, 9] and fuzzy divergence measures using entropy weights have been proposed to improve discrimination between highly similar sets, offering a more sensitive evaluation of dissimilarity. While these approaches represent significant progress, they often lack a unified theoretical framework capable of generalizing across different t-norms and t-conorms.

Our work addresses this gap by defining generalized divergence measures using continuous t-norms and t-conorms within intuitionistic fuzzy metric spaces. This formulation not only unifies previous frameworks [3, 7, 8] but also extends them to a broader class of divergence measures suitable for real-world decision-making problems. Additionally, insights from approximation theory, such as the work of Pradhan and Rao [6], which studied function approximation under Euler–Hausdorff summability in weighted Zygmund classes $W(Z_r^\omega)$, inform the analysis of divergence measures in weighted intuitionistic fuzzy function spaces. Together, these contributions provide a theoretically grounded and adaptable framework for evaluating similarity and dissimilarity between intuitionistic fuzzy sets, overcoming limitations of previous entropy-based and t-norm-based methods.

2. Basic preliminaries and definitions

2.1. **Intuitionistic fuzzy set.** An intuitionistic fuzzy set A in a universe X is defined by a membership function $\mu_A : X \to [0,1]$, a non-membership function $\nu_A : X \to [0,1]$, and an indeterminacy function $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, where $\mu_A(x) + \nu_A(x) \leq 1$.

2.2. **t-Norm and t-Conorm.** A t-norm $T : [0,1]^2 \to [0,1]$ satisfies:

(1) Commutativity: $T(a,b) = T(b,a)$,
(2) Associativity: $T(T(a,b),c) = T(a,T(b,c))$,
(3) Monotonicity: $a \leq c, b \leq d \Rightarrow T(a,b) \leq T(c,d)$,
(4) Identity: $T(a,1) = a$.

A t-conorm $S : [0,1]^2 \to [0,1]$ satisfies:

(1) Commutativity: $S(a,b) = S(b,a)$,
(2) Associativity: $S(S(a,b),c) = S(a,S(b,c))$,
(3) Monotonicity: $a \leq c, b \leq d \Rightarrow S(a,b) \leq S(c,d)$,
(4) Identity: $S(a,0) = a$.

2.3. **Divergence measure.** A divergence measure $D(A,B)$ between intuitionistic fuzzy sets A and B satisfies:

(1) Non-negativity: $D(A,B) \geq 0$,
(2) Identity of indiscernibles: $D(A,B) = 0 \Leftrightarrow A = B$,
(3) Asymmetry in general: $D(A,B) \neq D(B,A)$.

2.4. **Intuitionistic fuzzy metric space.** An intuitionistic fuzzy metric space is a set M with a function $d : M \times M \to [0,1]^3$, assigning membership, non-membership, and indeterminacy degrees for distances between elements.

3. Previous existing results

Atanassov [1] introduced IFSs with key properties enabling advanced divergence analyses. Dubois and Prade [2] studied operations foundational to t-norm/t-conorm-based metrics.

Liu et al. [5] proposed prioritized aggregation using Aczel–Alsina operations, improving intuitionistic fuzzy decision tools. Rani and Garg [7, 3] and Thao [8] further refined these approaches with new aggregation and divergence measures.

Our Contribution: We generalize existing frameworks by defining divergence measures using continuous t-norms and t-conorms, enhancing both theoretical rigor and applicability.

4. Generalized divergence measure

Divergence measures play a crucial role in quantifying the difference between two fuzzy information vectors. Classical measures such as Hamming distance or Kullback–Leibler divergence are often restricted to specific forms. To unify and generalize these approaches, we introduce a generalized divergence measure based on t-norms and t-conorms.

Definition 4.1. (Generalized Divergence Measure) Let $A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ be two finite fuzzy information vectors with $A_i, B_i \in [0, 1]$. For any continuous t-norm $T : [0, 1]^2 \to [0, 1]$ and continuous t-conorm $S : [0, 1]^2 \to [0, 1]$, the *generalized divergence measure* $D(A, B)$ is defined as

$$D(A,B) = \sum_{i=1}^{n} \Big(T(A_i, B_i) + S(A_i, B_i) \Big). \tag{4.1}$$

Example 4.2. Consider two fuzzy information vectors $A = (0.2, 0.7, 0.5)$ and $B = (0.4, 0.6, 0.8)$.

Let the t-norm be

$$T(a, b) = ab, \tag{4.2}$$

and the t-conorm be

$$S(a, b) = \max\{a, b\}. \tag{4.3}$$

Substituting (4.2) and (4.3) into (4.1), we obtain

$$D(A,B) = \sum_{i=1}^{3} \big(T(A_i, B_i) + S(A_i, B_i) \big). \tag{4.4}$$

We compute each term of (4.4):

$$T(0.2, 0.4)+S(0.2, 0.4) = 0.2 \times 0.4+\max\{0.2, 0.4\} = 0.08+0.4 = 0.48, \tag{4.5}$$

$$T(0.7, 0.6)+S(0.7, 0.6) = 0.7 \times 0.6+\max\{0.7, 0.6\} = 0.42+0.7 = 1.12, \tag{4.6}$$

$$T(0.5, 0.8)+S(0.5, 0.8) = 0.5 \times 0.8+\max\{0.5, 0.8\} = 0.40+0.8 = 1.20. \tag{4.7}$$

Finally, by substituting (4.5)–(4.7) into (4.4), we obtain

$$D(A, B) = 0.48 + 1.12 + 1.20 = 2.80. \tag{4.8}$$

Hence, the generalized divergence measure between A and B under the operators (4.2)–(4.3) is $D(A, B) = 2.80$.

5. Main theorems

In this section, we establish fundamental properties of the generalized divergence measure introduced in Definition 4.1, with the formulation given in (4.1).

5.1. **Non-negativity of divergence measures.**

Theorem 5.1. (Non-Negativity) *Let D be a generalized divergence measure for intuitionistic fuzzy sets, defined via a continuous t-norm T and t-conorm S on an intuitionistic fuzzy metric space M. Then, for any two intuitionistic fuzzy sets $A = (A_1, A_2)$ and $B = (B_1, B_2)$ in M, we have*

$$D(A, B) \geq 0, \quad \forall\, A, B \in M. \tag{5.1}$$

Proof. From Definition 4.1, the divergence measure is given by

$$D(A, B) = \sum_{i=1}^{n} \Big(T(A_i, B_i) + S(A_i, B_i) \Big). \tag{5.2}$$

By the axiomatic properties of t-norms and t-conorms (see [4]), we know that

$$T(x, y) \geq 0, \quad S(x, y) \geq 0, \quad \forall\, x, y \in [0, 1].$$

Thus, each summand in (5.2) is non-negative.

Since a finite sum of non-negative terms is itself non-negative, we conclude that

$$D(A, B) \geq 0.$$

This establishes the validity of inequality (5.1). □

5.2. **Symmetry of divergence measures.**

Theorem 5.2. (Symmetry) *Let D be a generalized divergence measure as in Definition 4.1. Then D is symmetric; that is, for any two intuitionistic fuzzy sets A and B,*

$$D(A, B) = D(B, A). \tag{5.3}$$

Proof. Using the definition of D from (4.1), we have

$$D(A, B) = \sum_{i=1}^{n} \Big(T(A_i, B_i) + S(A_i, B_i) \Big).$$

The operations T and S are both commutative, which means

$$T(A_i, B_i) = T(B_i, A_i), \quad S(A_i, B_i) = S(B_i, A_i), \quad \forall\, i.$$

Substituting these relations into the above expression yields

$$D(A, B) = \sum_{i=1}^{n} \Big(T(B_i, A_i) + S(B_i, A_i) \Big) = D(B, A).$$

Therefore, the symmetry property (5.3) holds. □

Intuitively, this property ensures that the divergence measure treats both arguments A and B without bias, making it consistent with the notion of "distance-like" quantities.

5.3. **Convexity of divergence measures.**

Theorem 5.3. (Convexity) *Let D be the generalized divergence measure defined in* (4.1) *using continuous convex t-norms T and continuous convex t-conorms S. Then, for any $A, B, C \in M$ and any $\lambda \in [0, 1]$, we have*

$$D(\lambda A + (1-\lambda)B, C) \leq \lambda D(A, C) + (1-\lambda)D(B, C). \tag{5.4}$$

Proof. From Definition 4.1, the divergence measure is expressed as

$$D(X, C) = \sum_{i=1}^{n} \Big(T(X_i, C_i) + S(X_i, C_i) \Big), \quad X \in M.$$

Since T and S are convex functions on $[0, 1]^2$ (see [4]), we have for each i,

$$T(\lambda A_i + (1-\lambda)B_i, C_i) \leq \lambda T(A_i, C_i) + (1-\lambda)T(B_i, C_i), \tag{5.5}$$

$$S(\lambda A_i + (1-\lambda)B_i, C_i) \leq \lambda S(A_i, C_i) + (1-\lambda)S(B_i, C_i). \tag{5.6}$$

Adding inequalities (5.5) and (5.6), and summing over all $i = 1, 2, \ldots, n$, we obtain

$$D(\lambda A + (1-\lambda)B, C) \leq \lambda D(A, C) + (1-\lambda)D(B, C),$$

which proves inequality (5.4). Thus, D is convex in its first argument with respect to linear combinations of fuzzy information vectors. □

5.4. **Convergence of divergence measures in intuitionistic fuzzy metric spaces.**

Theorem 5.4. (Convergence) *Let D be a generalized divergence measure for intuitionistic fuzzy sets in an intuitionistic fuzzy metric space M. Let $\{A_n\}$ be a sequence of intuitionistic fuzzy sets in M that converges to A in the fuzzy metric sense. Then*

$$\lim_{n \to \infty} D(A_n, A) = 0. \tag{5.7}$$

Proof. By the assumption of convergence in the intuitionistic fuzzy metric, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(A_n, A) < \epsilon, \quad \forall\, n \geq N.$$

Since the divergence measure D in (4.1) is continuous with respect to the fuzzy metric d, small perturbations in the input vectors induce correspondingly small variations in D. Hence,

$$\lim_{n \to \infty} D(A_n, A) = 0,$$

establishing (5.7). □

Intuitively, this property shows that D behaves like a distance: as the sequence (A_n) approaches A, the divergence between them vanishes.

5.5. **Generalized divergence measures as special cases.**

Theorem 5.5. (Special cases) *The generalized divergence measure* $D(A, B)$ *in Definition* 4.1 *includes several well-known divergence measures as special cases. In particular, suitable choices of the* t*-norm* T *and* t*-conorm* S *reduce* D *to classical divergences such as the Hamming and Kullback–Leibler divergence.*

Proof. We consider two illustrative cases.

First, for the *Hamming divergence*, if we choose $T(a, b) = |a - b|$ and $S(a, b) = 0$, then by (4.1) we obtain

$$D(A, B) = \sum_{i=1}^{n} |A_i - B_i|,$$

which coincides with the well-known Hamming divergence.

Next, for the *Kullback–Leibler divergence*, if we take $T(a, b) = a \log \frac{a}{b}$ (for $a, b > 0$) and $S(a, b) = 0$, then

$$D(A, B) = \sum_{i=1}^{n} A_i \log \frac{A_i}{B_i},$$

which is exactly the classical Kullback–Leibler divergence. □

Thus, the generalized framework in (4.1) subsumes a wide class of classical divergence measures, demonstrating both its flexibility and its unifying nature.

6. Applications of the theorems

In this section, we illustrate the practical applications of the theorems presented in Section 5. The properties of generalized divergence measures introduced in Definition 4.1 and formulated in (4.1) are widely applicable in areas such as machine learning, signal processing, and information retrieval.

6.1. **Application of Theorem 5.1.** The non-negativity property ensures that divergence measures provide meaningful quantification of differences between intuitionistic fuzzy sets. In *information retrieval*, this property allows us to compare search results confidently, knowing that the divergence cannot be negative.

Mathematical Illustration: Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be two intuitionistic fuzzy sets. From Theorem 5.1, we have

$$D(A, B) \geq 0.$$

For example, the Kullback–Leibler divergence, a special case of D (Theorem 5.5) satisfies

$$D_{\mathrm{KL}}(A, B) = \sum_{i=1}^{n} A_i \log \frac{A_i}{B_i} \geq 0.$$

6.2. **Application of Theorem 5.2.** Symmetry guarantees that the divergence measure treats both sets equally, which is crucial in *clustering* and *classification.*

Mathematical Illustration: For fuzzy sets A and B, Theorem 5.2 gives

$$D(A, B) = D(B, A),$$

ensuring consistent evaluation of cluster similarity regardless of the order of comparison.

6.3. **Application of Theorem 5.3.** Convexity is valuable in *optimization problems* and *decision-making under uncertainty.* For instance, in *portfolio optimization*, the risk of a weighted combination of assets is controlled by the convexity property.

Mathematical Illustration: For intuitionistic fuzzy sets $A, B, C \in M$ and $\lambda \in [0, 1]$, Theorem 5.3 gives

$$D(\lambda A + (1 - \lambda)B, C) \leq \lambda D(A, C) + (1 - \lambda)D(B, C).$$

This is also applicable in *signal filtering* and *image compression*, where combined data approximations maintain controlled divergence.

6.4. **Application of Theorem 5.4.** Convergence is important in *adaptive algorithms* in *machine learning* and *signal processing.* For example, in adaptive filtering, iterative estimates approach the true signal.

Mathematical Illustration: Let (A_n) be a sequence of intuitionistic fuzzy sets converging to A. By Theorem 5.4, we have

$$\lim_{n\to\infty} D(A_n, A) = 0,$$

ensuring that divergence (error) vanishes as the algorithm iterates.

6.5. **Application of Theorem 5.5.** The generalized divergence measure D encompasses well-known divergences, offering flexibility for diverse applications:

(1) Hamming Distance: Choosing $T(a, b) = |a - b|$ and $S(a, b) = 0$ gives

$$D(A, B) = \sum_{i=1}^{n} |A_i - B_i|,$$

used in error-correcting codes and binary classification.

(2) Kullback–Leibler Divergence: Choosing $T(a, b) = a \log \frac{a}{b}$ and $S(a, b) = 0$ gives

$$D(A, B) = \sum_{i=1}^{n} A_i \log \frac{A_i}{B_i},$$

widely used in statistical modeling and machine learning.

Remark 6.1. Theorems 5.1–5.5 provide the following practical insights: The following summarizes the real-world applications of the theorems discussed:

- Non-Negativity: Ensures that divergence measures are always non-negative, making them reliable in information retrieval and probability comparisons.
- Symmetry: Guarantees that the divergence measure treats both sets equally, which is important for clustering and classification algorithms.
- Convexity: Helps optimize solutions and ensures better approximation and robustness in decision-making problems like portfolio optimization.
- Convergence: Useful in machine learning and adaptive filtering, where a sequence of approximations converges to a target solution.
- Special Cases: The generalized divergence measure can represent various standard measures, offering flexibility in real-world applications like image recognition, error correction, and data compression.

These properties demonstrate that the generalized divergence measure is not only mathematically rigorous but also practically useful across a variety of real-world domains.

7. Conclusion

In this paper, we have introduced a generalized framework for divergence measures in intuitionistic fuzzy metric spaces by incorporating continuous t-norms and t-conorms. Our work builds upon and extends previous studies, particularly the aggregation methods proposed by Liu et al. (2023), by offering a more flexible, parameterized approach to divergence measures. The use of t-norms and t-conorms provides a unified, robust mathematical foundation that enhances the adaptability and theoretical depth of the divergence measures.

We have established important properties such as non-negativity, symmetry, convexity, and convergence for these generalized divergence measures, ensuring their utility in various applications. Furthermore, our approach includes a variety of well-known divergence measures as special cases, which makes it versatile and suitable for real-world problems, including decision-making, information retrieval, machine learning, and signal processing.

The results presented in this paper lay the groundwork for further research in intuitionistic fuzzy set theory, particularly in exploring more complex divergence measures and their applications in multi-criteria decision-making and fuzzy information systems. Future work can focus on refining the framework, studying the computational efficiency of these measures, and applying them to larger, more complex systems for practical applications in diverse fields.

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