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ON SOME NORM AND NUMERICAL RADIUS INEQUALITIES FOR SUMS AND PRODUCT OPERATORS IN SEMI HILBERT SPACE

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Abstract. In this paper, we present novel inequalities concerning the numerical radius and norm of operators in semi-Hilbert spaces. These results not only generalize and refine previously established inequalities but also extend their applicability to a wider class of operators within this framework. By utilizing the structure and properties of semi-Hilbert spaces, we offer new insights into operator behavior, paving the way for further developments in this area of research.

1. Introduction and preliminaries

In functional analysis and operator theory, semi-Hilbert spaces (SHS) arise as a generalization of Hilbert spaces, where a positive operator A induces a semi-inner product on the underlying space. These spaces are particularly relevant in contexts where standard inner product structures prove inadequate, such as in certain mathematical models and physical applications. The study of operator norms, numerical radius inequalities, and related properties within

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semi-Hilbert spaces has garnered significant attention, as it allows researchers to extend classical results and uncover deeper insights into operator behavior under generalized settings. Recent work has emphasized the intricate relationships between operators and the geometry of spaces governed by A, leading to broader generalizations of inequalities and norms that are pivotal in operator theory. This paper contributes to this expanding area of research by deriving new norm and numerical radius inequalities for sums and product operators in semi-Hilbert spaces. By utilizing the concepts of A-bounded operators and their A-adjoints, we build upon earlier studies and propose novel methodologies for analyzing operator interactions. The investigation of inequalities for sums and product operators in Hilbert and semi-Hilbert spaces or other spaces has seen substantial progress in recent years (see, for example, [2, 6, 7, 10, 12, 24, 25, 29, 30, 33, 35, 36, 39, 37, 40, 41]).

Let $B(H)$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|.\|$. Let I be the identity operator. For $T \in B(H)$, we denote by $R(T), N(T)$ and T^* the range, the kernel and the adjoint of T, respectively. For a given linear subspace M of H, its closure in the norm topology of H will be denoted by M. Further, let PS stand for the orthogonal projection onto a closed subspace S of H. An operator $T \in B(H)$ is called positive if

$$\langle Tx, x \rangle \geq 0$$

for all $x \in H$. Furthermore, if $T = 0$, then the square root of T is denoted by $T^{1/2}$. For $T \in B(H)$, the absolute value of T, denoted by $|T|$, is defined as $|T| = (T^*T)^{1/2}$.

Throughout this article, A denotes a non-zero positive operator on H. The positive operator A induces the following semi-inner product

$$\langle ., . \rangle_A : H \times H \to \mathbb{C}$$

$$(x, y) \to \langle x, y \rangle_A = \langle Ax, y \rangle = \langle A^{\frac{1}{2}} x, A^{\frac{1}{2}} y \rangle.$$

We begin by some definitions and concepts in semi Hilbert space.
Let H be a Hilbert space and $B^+(H)$ be the cone of positive (semi definite) operators,

$$B^+(H) = \{A \in B(H), \ \ \langle Ax, x \rangle \geq 0, \ \ x \in H\}.$$

For $A \in B^+(H)$ the mapping,

$$\langle ., . \rangle : H \times H \to \mathbb{C}$$

such that for all $x, y \in H$, $\langle x, y \rangle_A = \langle Ax, y \rangle$ defines a positive semi-definite sesquilinear form. Clearly that $\|x\|_A = \langle x, x \rangle_A^{\frac{1}{2}}$ is a seminorm, so H endowed with this structure said to be a semi helbertian space.

If A is injective, then $\|.\|_A$ becomes a norm. All definitions given in Hilbert space can be extend to semi Hilbert space, for example:

An operator T is said to be bounded if there exits $c > 0$ such that

$$\|Tx\|_A \leq c\|x\|_A$$

for all $x \in H$ and we denote $B_A(H) = \{T \in B(H), \|T\|_A < \infty\}$.

$$\|T\|_A = \sup_{x \in \overline{\mathcal{R}(A)}} \frac{\|Tx\|_A}{\|x\|_A}.$$

If $T \in B_A(H)$ there exists $T^\sharp = A^\dagger T^* A$, where $A^\dagger$ is Moore- Penrose inverse of A and $T^\sharp$ called an A-adjoint of T and we have the following properties: $TT^\sharp, T^\sharp T$ are A-self adjoint and A-positive.

$$\|T\|_A = \|T^\sharp\|_A = \|T^\sharp T\|^{\frac{1}{2}} = \|TT^\sharp\|^{\frac{1}{2}}.$$

The numerical rang is given by $W_A(T) = \sup\{\langle Tx, x\rangle_A, \|x\|_A = 1\}$. If T is an A-self adjoint, then $W_A(T) = \|T\|_A$.

Saadi [42] extended the definition of the concept of normality as follows, an operator $T \in \mathcal{L}_A(H)$ is called an A-normal operator if $TT^\sharp = T^\sharp T$. If $A = I$ or T commute with A we get $T^* = T^\sharp$. It has mentioned in [3] that the classical Schawrz inequality

$$|\langle Tx, y\rangle|^2 \leq \langle Tx, x\rangle\langle Ty, y\rangle,$$

can be easily extended to Semi Hilbert space, for a positive operator and $T \in B_A(H)$ we have

$$|\langle Tx, y\rangle_A|^2 \leq \langle Tx, x\rangle\langle Ty, y\rangle.$$

In the same paper [3] the author define the A-absolute value operator of T as $|T|_A = (AT^\sharp T)^{\frac{1}{2}}$, which is A-self adjoint if T is A self adjoint.

Popescu [34] introduced the Euclidian operator radius as follows

$$w_e(T_1, T_2, ..., T_n) = \sup_{\|x\|=1} \left(\sum_{i=1}^{n} |\langle T_i x, x\rangle|^2 \right)^{\frac{1}{2}}.$$

The same author has proved that for $T = (T_1, ..., T_n)$ we have

$$\frac{1}{\sqrt{n}}\|[T_1, T_2, ..., T_n]\| \leq \|(T_1, T_2, ..., T_n)\|_e \leq \|(T_1, T_2, ..., T_n)\|, \tag{1.1}$$

where $\|[T_1, T_2, ..., T_n]\| = \| \sum_{i=1}^{n} T_i T_i^\sharp\|^{\frac{1}{2}}$.

In [23], the authors define the Euclidian A-radius as follows

$$\omega_{A,e}T_1, T_2, ..., T_n) = \sup_{\|x\|_A=1} \left(\sum_{i=1}^{n} \langle T_i x, x\rangle_A|^2 \right)^{\frac{1}{2}}$$

and the A-semi norm as follows

$$\|(T_1, T_2, ..., T_n)\|_{e,A} = \sup_{\lambda \in B_n} \|\lambda_1 T_1 + ... + \lambda_n T_n\|,$$

where B_n is the unit closed ball in $B_A(H)$.

It has been shown that w_A is a semi norm equivalent to A-operator semi norm and for all $T \in \mathbb{B}_A(H)$ we have

$$\frac{1}{2}\|T\|_A \leq \omega_A(T) \leq \|T\|_A.$$

If T is a A-self adjoint, then $\|T\|_A = w_A(T)$.

Dragomir [19] defined a positive form hermitian on $B^n(H)$, for $S, T \in B^n(H)$ as follows:

$$\langle T, S\rangle_{\varepsilon,x} = \sum_{i=1}^{n} \varepsilon_i \langle T_i, S_i\rangle = \left\langle \sum_{i=1}^{n} \varepsilon_i S_i^* T_i, x \right\rangle,$$

if $S = T$, we get

$$\|T\|_{\varepsilon,x} = \langle T, T\rangle_{\varepsilon,x} = \left\langle \sum_{i=1}^{n} \varepsilon_i T_i^* T_i, x \right\rangle$$

for $\varepsilon = (1, 1, ..., 1)$, we denote by

$$\langle T, S\rangle_x = \langle \sum_{i=1}^{n} S_i^* T_i, x\rangle.$$

Motivated by the previous definition we can easily show for $T \in B_A^n(H)$ the mapping

$$\langle T, S\rangle_{\varepsilon,x,A} = \sum_{i=1}^{n} \varepsilon_i \langle T_i x, S_i x\rangle = \left\langle (\sum_{i=1}^{n} \varepsilon_i S_i^{\sharp} T_i x)x, x \right\rangle.$$

For $\varepsilon = (1, 1, ..., 1)$, we denote by

$$|\langle T, S\rangle_{x,A}| = \left\langle \sum_{i=1}^{n} (S_i^{\sharp} T_i)x, x \right\rangle.$$

2. Main results

Theorem 2.1. *For $T = (T_1, T_2, ..., T_n)$, $S = (S_1, S_2, ..., S_n) \in B_A^n(H)$, we have*

$$w_A^2\left(\sum_{i=1}^n S_i^\sharp T_i\right) \leq \left\|\sum_{i=1}^n T_i^\sharp T_i\right\| \left\|\sum_{i=1}^n S_i^\sharp S_i\right\| \tag{2.1}$$

and

$$\left\|\sum_{i=1}^n (S_i+T_i)^\sharp (S_i+T_i)\right\|_A^{\frac{1}{2}} \leq \left\|\sum_{i=1}^n T_i^\sharp T_i\right\|_A^{\frac{1}{2}} + \left\|\sum_{i=1}^n S_i^\sharp S_i\right\|_A^{\frac{1}{2}}. \tag{2.2}$$

Proof. For $x \in H$ by utilizing Shwartz inequality, we get

$$\begin{aligned}\left|\left\langle (\sum_{i=1}^n (S_i^\sharp T_i)x, x\right\rangle_A\right|^2 &= \left|\left\langle (\sum_{i=1}^n T_i)x, (\sum_{i=1}^n S_i)x\right\rangle_A\right|^2 \\ &\leq \left|\left\langle (\sum_{i=1}^n T_i)x, (\sum_{i=1}^n T_i)x\right\rangle_A\right| \left|\langle (\sum_{i=1}^n S_i)x, (\sum_{i=1}^n S_i)x\right\rangle_A \Bigg| \\ &= \left|\left\langle \sum_{i=1}^n T_i^\sharp T_i)x, x\right\rangle_A\right| \left|\left\langle (\sum_{i=1}^n S_i^\sharp S_i)x, x\right\rangle_A\right|.\end{aligned}$$

Taking the supremum over $x \in H$ with $\|x\|_A = 1$, we get

$$\begin{aligned}w_A^2(\sum_{i=1}^n S^\sharp T_i) &\leq \sup_{\|x\|=1}\left|\left\langle (\sum_{i=1}^n T_i^\sharp T_i)x, x\right\rangle\right|_A^2 . \sup_{\|x\|=1}\left|\left\langle (\sum_{i=1}^n S_i^\sharp S_i)x, x\right\rangle\right|_A^2 \\ &= \left\|\sum_{i=1}^n T_i^\sharp T_i\right\|_A \left\|\sum_{i=1}^n S_i^\sharp S_i\right\|_A \\ &= \left\|\sum_{i=1}^n T_i^\sharp T_i\right\| \left\|\sum_{i=1}^n S_i^\sharp S_i\right\|_A .\end{aligned}$$

Then,

$$w_A^2(\sum_{i=1}^n S^\sharp T_i) \leq \left\|\sum_{i=1}^n T_i^\sharp T_i\right\|_A \left\|\sum_{i=1}^n S_i^\sharp S_i\right\|_A = \left\|\sum_{i=1}^n T_i^\sharp T_i\right\|_A \left\|\sum_{i=1}^n S_i^\sharp S_i\right\|_A .$$

For the second inequality, by using triangular inequality, we get

$$\left\|\sum_{i=1}^{n}(S_i+T_i)^{\sharp}(S_i+T_i)\right\|_A \leq \left\|\sum_{i=1}^{n}T_i^{\sharp}T_i\right\|_A + \left\|\sum_{i=1}^{n}S_i^{\sharp}S_i\right\|_A + \left\|\sum_{i=1}^{n}S_i^{\sharp}T_i\right\|_A + \left\|\sum_{i=1}^{n}T_i^{\sharp}S_i\right\|_A,$$

from the inequality (2.1), we get

$$\begin{aligned}\left\|\sum_{i=1}^{n}(S_i+T_i)^{\sharp}(S_i+T_i)\right\|_A &\leq \left\|\sum_{i=1}^{n}T_i^{\sharp}T_i\right\|_A + \left\|\sum_{i=1}^{n}S_i^{\sharp}S_i\right\|_A + 2\left\|\sum_{i=1}^{n}T_i^{\sharp}T_i\right\|_A \\ &\quad + \left\|\sum_{i=1}^{n}T_i^{\sharp}T_i\right\|_A^{\frac{1}{2}}\left\|\sum_{i=1}^{n}S_i^{\sharp}S_i\right\|_A^{\frac{1}{2}} \\ &= \left(\left\|\sum_{i=1}^{n}T_i^{\sharp}T_i\right\|_A^{\frac{1}{2}} + \left\|\sum_{i=1}^{n}T_i^{\sharp}T_i\right\|_A^{\frac{1}{2}}\right)^2.\end{aligned}$$

Taking the positive root of the two sides, we get

$$\left\|\sum_{i=1}^{n}(S_i+T_i)^{\sharp}(S_i+T_i)\right\|_A^{\frac{1}{2}} \leq \left\|\sum_{i=1}^{n}T_i^{\sharp}T_i\right\|_A^{\frac{1}{2}} + \left\|\sum_{i=1}^{n}S_i^{\sharp}S_i\right\|_A^{\frac{1}{2}}.$$

□

If $S = T^{\sharp}$ in Theorem 2.1, we get

$$w_A^2\left(\sum_{i=1}^{n}T_i^2\right) \leq \left\|\sum_{i=1}^{n}T_i^{\sharp}T_i\right\|_A \left\|\sum_{i=1}^{n}T_i^{\sharp}T_i^{\sharp}\right\|_A \tag{2.3}$$

and

$$\left\|\sum_{i=1}^{n}(T_i+T_i^{\sharp})^2\right\|_A^{\frac{1}{2}} \leq \left\|\sum_{i=1}^{n}T_i^{\sharp}T_i\right\|_A^{\frac{1}{2}} + \left\|\sum_{i=1}^{n}T_iT_i^{\sharp}\right\|_A^{\frac{1}{2}}.$$

If $S = T$, we get

$$w_A^2\left(\sum_{i=1}^{n}T_i^{\sharp}T_i\right) \leq \left\|\sum_{i=1}^{n}T_i^{\sharp}T_i|^2\right\|_A.$$

If $S_i = I$ for all $i \in \{1, 2, ..., n\}$, we get

$$w_A^2(\sum_{i=1}^{n}T_i) \leq n\left\|\sum_{i=1}^{n}T_i^{\sharp}T_i\right\|_A.$$

Corollary 2.2. *For $T = (T_1, T_2, ..., T_n)$ and $S = (S_1, S_2, ..., S_n)$ with*

$$\sum_{i=1}^{n}(S_i + T_i)^\sharp(S_i + T_i) \neq 0,$$

we have

$$\left\|\sum_{i=1}^{n}(S_i + T_i)^\sharp(S_i + T_i)\right\|_A^{\frac{1}{2}} \leq \frac{w_A\left(\sum_{i=1}^{n} T_i^\sharp T_i + \sum_{i=1}^{n} S_i^\sharp T_i\right) + w_A\left(\sum_{i=1}^{n} T\sharp S_i + \sum_{i=1}^{n} S_i^\sharp S_i\right)}{\|\sum_{i=1}^{n}(S_i + T_i)^\sharp(T_i + S_i)\|^{\frac{1}{2}}}$$

Proof. We have,

$$\begin{aligned}
&\left\|\sum_{i=1}^{n}(T_i + S_i)^\sharp(T_i + S_i)\right\|_A \\
&= w_A(\sum_{i=1}^{n}\sum_{i=1}^{n}(T_i + S_i)^\sharp(T_i + S_i)) \\
&= w_A(\sum_{i=1}^{n}(T_i + S_i)^\sharp T_i + \sum_{i=1}^{n}(T_i + S_i)^\sharp S_i \\
&\leq w_A(\sum_{i=1}^{n}(T_i + S_i)^\sharp T_i) + w_A(w_A(\sum_{i=1}^{n}(T_i + S_i)^\sharp S_i) \\
&\leq \left\|\sum_{i=1}^{n}(T_i + S_i)^\sharp(T_i + S_i)\right\|_A^{\frac{1}{2}} \left[\left\|\sum_{i=1}^{n} T_i^\sharp T_i\right\|_A^{\frac{1}{2}} \left\|\sum_{i=1}^{n} S_i^\sharp S_i\right\|_A^{\frac{1}{2}}\right],
\end{aligned}$$

since

$$w_A\left(\sum_{i=1}^{n}(T_i + S_i)^\sharp T_i\right) = w_A\left(\sum_{i=1}^{n} T_i^\sharp T_i + \sum_{i=1}^{n} S_i^\sharp T_i\right)$$

and

$$w_A\left(\sum_{i=1}^{n}(T_i + S_i)^\sharp S_i\right) = w_A\left(\sum_{i=1}^{n} S_i^\sharp S_i + \sum_{i=1}^{n} T_i^\sharp S_i\right).$$

Therefore, we get

$$\left\| \sum_{i=1}^{n} (T_i + S_i)^\sharp (T_i + S_i) \right\|_A \leq w_A \left(\sum_{i=1}^{n} T_i^\sharp T_i + \sum_{i=1}^{n} S_i^\sharp T_i \right) + w_A \left(\sum_{i=1}^{n} T_i^\sharp T_i + \sum_{i=1}^{n} S_i^\sharp T_i \right),$$

which implies the desired inequality. □

If $T = (T_1, T_2)$ and $S = (T_2^\sharp, \pm T_1^\sharp)$, we get

$$w_A(T_2 T_1 \pm T_1 T_2) \leq \left\| T_1^\sharp T_1 + T_2^\sharp T_2 \right\|_A^{\frac{1}{2}} \left\| T_1 T_1^\sharp + T_2 T_2^\sharp \right\|_A^{\frac{1}{2}}.$$

If $T_2 = T_1^\sharp$, we get

$$w_A(T_1^\sharp T_1 \pm T_1 T_1^\sharp) \leq \left\| T_1^\sharp T_1 + T_1 T_1^\sharp \right\|_A.$$

If $T = (T_1, T_1^\sharp)$ and $S = (T_1^\sharp, T_1)$, we get

$$\left\| T_1 + T_1^\sharp \right\|_A \leq \sqrt{2} \left\| T_1^\sharp T_1 + T_1 T_1^\sharp \right\|_A^{\frac{1}{2}}.$$

If $T = (T_1, -T_1^\sharp)$ and $S = (-T_1^\sharp, T_1^\sharp)$, we get

$$\frac{1}{2} \left\| T_1 - T_1^\sharp \right\|_A^2 \leq \left\| T_1^\sharp T_1 + T_1 T_1^\sharp \right\|_A.$$

Theorem 2.3. *For $T = (T_1, T_2, ..., T_n)$, $S = (S_1, S_2, ..., S_n) \in B_A(H)^n$, we have*

$$w_A(\sum_{i=1}^{n} S_i^\sharp T_i) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} T_i^\sharp T_i + S_i^\sharp S_i \right\|_A \tag{2.4}$$

and

$$\left\| \sum_{i=1}^{n} |(S_i + T_i)^\sharp (S_i + T_i) \right\|_A \leq \left\| \sum_{i=1}^{n} T_i^\sharp T_i + S_i^\sharp S_i \right\|_A. \tag{2.5}$$

Proof. For $x \in H$ by mixed Schwartz inequality, we get

$$\left\langle \sum_{i=1}^{n} (S_i^\sharp T_i) x, x \right\rangle_A \leq \left\langle (\sum_{i=1}^{n} T_i) x, x \right\rangle_A^{\frac{1}{2}} \left\langle (\sum_{i=1}^{n} S_i) x, x \right\rangle_A^{\frac{1}{2}}.$$

Since

$$a^{\frac{1}{2}} b^{\frac{1}{2}} \leq \frac{1}{2}(a + b) \quad \text{for} \quad a, b \geq 0,$$

we get

$$\left|\left\langle \sum_{i=1}^{n}(S_i^{\sharp}T_i)x,x\right\rangle_A\right| \leq \frac{1}{2}\left[\left|\left\langle (\sum_{i=1}^{n}T_ix,x\right\rangle_A\right| + \left|\left\langle (\sum_{i=1}^{n}S_ix,x\right\rangle_A\right|\right]$$
$$= \frac{1}{2}\left|\left\langle (\sum_{i=1}^{n}(T_i+S_i))x,x\right\rangle_A\right|.$$

Taking the supermum on x with $\|x\|_A = 1$, we get

$$w_A(\sum_{i=1}^{n}S_i^{\sharp}T_i) \leq \frac{1}{2}\left\|\sum_{i=1}^{n}T_i^{\sharp}T_i + S_i^{\sharp}S_i\right\|_A.$$

For the second inequality, using the result of Theorem 2.1 which gives,

$$\left\langle (\sum_{i=1}^{n}(T_i+S_i)^{\sharp}(T_i+S_i)x,x\right\rangle_A^{\frac{1}{2}} \leq \left(\left\langle (\sum_{i=1}^{n}T_i^{\sharp}T_i)x,x\right\rangle_A^{\frac{1}{2}} + \left\langle (\sum_{i=1}^{n}S_i^{\sharp}S_i)x,x\right\rangle_A^{\frac{1}{2}}\right).$$

Taking the square of the two sides, we get

$$\left\langle (\sum_{i=1}^{n}(T_i+S_i)^{\sharp}(T_i+S_i)x,x\right\rangle_A \leq \left\langle (\sum_{i=1}^{n}T_i^{\sharp}T_i)x,x\right\rangle_A + \left\langle (\sum_{i=1}^{n}S_i^{\sharp}S_i)x,x\right\rangle_A$$
$$+ 2\left\langle (\sum_{i=1}^{n}T_i^{\sharp}T_i)x,x\right\rangle_A^{\frac{1}{2}}\left\langle (\sum_{i=1}^{n}T_i^{\sharp}T_i)x,x\right\rangle_A^{\frac{1}{2}}.$$

Since

$$2\left(\left\langle (\sum_{i=1}^{n}T_i^{\sharp}T_i)x,x\right\rangle_A^{\frac{1}{2}}\left\langle \left(\sum_{i=1}^{n}T_i^{\sharp}T_i\right)x,x\right\rangle_A^{\frac{1}{2}}\right) \leq \left\langle \left(\sum_{i=1}^{n}(T_i^{\sharp}T_i + S_i^{\sharp}S_i)\right)x,x\right\rangle,$$

we get

$$\left\langle (\sum_{i=1}^{n}(T_i+S_i)^{\sharp}(T_i+S_i))x,x\right\rangle \leq 2\left\langle (\sum_{i=1}^{n}(T_i^{\sharp}T_i + S_i^{\sharp}S_i))x,x\right\rangle.$$

Taking the supermum over $x \in H$ with $\|x\|_A = 1$, we get

$$\left\|\sum_{i=1}^{n}(S_i+T_i)^{\sharp}(S_i+T_i)\right\|_A \leq \left\|\sum_{i=1}^{n}T_i^{\sharp}T_i + S_i^{\sharp}S_i\right\|_A.$$

□

Corollary 2.4. *For $T=(T_1,T_2,...,T_n)\in B_A^n(H)$, we have*

$$w_A\left(\sum_{i=1}^{n} T_i^\sharp T_i\right)\leq \frac{1}{2}\left\|\sum_{i=1}^{n}(T_i^\sharp T_i+T_iT_i^\sharp)\right\|_A$$

and

$$\left\|\sum_{i=1}^{n}(T_i^\sharp \pm T_i)^\sharp(T_i\pm T_i^\sharp)\right\|_A\leq \sum_{i=1}^{n}(T_i^\sharp T_iT_iT_i^\sharp).$$

If $S_i=I$ for all $i\in\{1,2,...,n\}$, we get

$$w_A(\sum_{i=1}^{n}T_i)\leq \frac{1}{2}\left\|T_i^\sharp T_i+I\right\|_A.$$

If $T=(T_1,T_2)$ and $S=(T_2,\pm T_1^\sharp)$, we get

$$w_A(T_2T_1\pm T_1T_2)\leq \frac{1}{2}\left\|T_1^\sharp T_1+T_1T_1^\sharp+T_2^\sharp T_2+T_2T_2^\sharp\right\|_A.$$

If $T=(T_2,I)$ and $S=(T_1^\sharp,(T_1T_2)^\sharp)$, we get

$$w_A(T_1T_2)\leq \frac{1}{4}\left\|T_2^\sharp T_2+I+T_1T_1^\sharp+(T_1T_2)(T_1T_2)^\sharp\right\|_A.$$

Lemma 2.5. *For $x,y\in H$, we have*

$$Re\,\langle x,y\rangle_A\leq \frac{1}{2}(\|x\|_A^2+\|y\|_A^2)$$

and

$$Re\,\langle x,y\rangle_A\leq \frac{1}{4}\|x+y\|_A^2.$$

Proof. We have

$$\begin{aligned}Re\,\langle x,y\rangle_A&=Re\left\langle A^{\frac{1}{2}}x,A^{\frac{1}{2}}y\right\rangle\\&\leq \frac{1}{2}(\|A^{\frac{1}{2}}x\|^2+\|A^{\frac{1}{2}}y\|^2)\\&=\frac{1}{2}(\|x\|_A^2+\|y\|_A^2).\end{aligned}$$

For the second inequality, we have

$$\begin{aligned}Re\,\langle x,y\rangle_A&=Re\left\langle A^{\frac{1}{2}}x,A^{\frac{1}{2}}y\right\rangle\\&\leq \frac{1}{4}\|A^{\frac{1}{2}}x+A^{\frac{1}{2}}y\|\\&=\frac{1}{4}\|x+y\|_A.\end{aligned}$$

□

Theorem 2.6. *For $T = (T_1, T_2, ..., T_n)$ in $B_A(H)$, we have*

$$\left\|\sum_{i=1}^{n} T_i\right\|_A^2 \leq \left\|\sum_{i=1}^{n} T_i^\sharp T_i\right\| + \frac{1}{2}\left(\left\|\sum_{1\leq i\neq j\leq n} T_i^\sharp T_j\right\|^2 + 1\right)$$

Proof. We have

$$\begin{aligned}\left\|\sum_{i=1}^{n} T_i x\right\|_A^2 &= Re\left[\sum_{i=1}^{n}\sum_{j=1}^{n} \langle T_i x, T_j x\rangle_A\right] \\ &= \sum_{i=1}^{n} \|T_i\|_A + \sum_{i\neq j=1}^{n} Re\,\langle T_i x, T_j x\rangle_A \\ &= \left\langle \left(\sum_{i=1}^{n} T_i^\sharp\right) x, x\right\rangle_A + Re\left\langle \sum_{i\neq j=1}^{n} \left(T_j^\sharp T_i\right) x, x\right\rangle_A .\end{aligned}$$

Using Lemma 2.5, we get

$$\left\|\sum_{i=1}^{n} T_i x\right\|_A^2 \leq \left\langle \left(\sum_{i=1}^{n} T_i^\sharp\right) x, x\right\rangle_A + \frac{1}{2}\left[\left\|\sum_{i\neq j=1}^{n} \left(T_j^\sharp T_i\right) x\right\|_A^2 + \|x\|_A^2\right].$$

Taking the supermum on $x \in H$ with $\|x\|_A = 1$, we get

$$\left\|\sum_{i=1}^{n} T_i\right\|_A^2 \leq \left\|\sum_{i=1}^{n} T_i^\sharp T_i\right\| + \frac{1}{2}\left(\left\|\sum_{1\leq i\neq j\leq n} T_i^\sharp T_j\right\|^2 + 1\right).$$

□

For $T = (T_1, T_2)$, we get the following result,

$$\|T_1 + T_2\|_A^2 \leq \left\|T_1^\sharp T_1 + T_2^\sharp T_2\right\|_A + \frac{1}{2}\left\|T_1^\sharp T_2 + T_2^\sharp T_1\right\| + \frac{1}{2}.$$

For $T = T_1$ and $T_2 = T^\sharp$, we get

$$\|T + T^\sharp\|^2 \leq \left\|T^\sharp T + TT^\sharp\right\| + \frac{1}{2}\left\|T^2 + (T^\sharp)^2\right) + \frac{1}{2}.$$

Theorem 2.7. *For $T = (T_1, T_2, ..., T_n)$ in $B_A(H)$, we have*

$$\left\|\sum_{i=1}^{n} T_i\right\|_A^2 \leq \left\|(n-1)\sum_{i=1}^{n} T_i^\sharp T_i\right\| + \frac{1}{2}\left\|T_i^\sharp T_i + \sum_{i=1}^{n} T^\sharp \sum_{i=1}^{n} T_i\right\|.$$

Proof. By using Lemma 2.5 (2) and (2.4), we get

$$\left\langle \left(\sum_{i=1}^{n} T_i^{\sharp} T_i \right) x, x \right\rangle_A + Re \left\langle \sum_{i \neq j=1}^{n} \left(T_j^{\sharp} T_i \right) x, x \right\rangle_A$$
$$\leq \left\langle \left(\sum_{i=1}^{n} T_i^{\sharp} T_i \right) x, x \right\rangle_A + \frac{1}{4} \left\langle \sum_{i \neq j=1}^{n} (T_i^{\sharp} + T_j^{\sharp})(T_j^{\sharp} + T_i^{\sharp}) x, x \right\rangle_A .$$

Taking the supermum on $x \in H$ with $\|x\|_A$, we get

$$\left\| \sum_{i=1}^{n} T_i \right\|_A^2 \leq \left\| \sum_{i=1}^{n} T_j^{\sharp} T_i \right\|_A + \frac{1}{4} \left\| \sum_{i \neq j=1}^{n} (T_i^{\sharp} + T_j^{\sharp})(T_j^{\sharp} + T_i^{\sharp}) \right\|_A .$$

On the other hand, we have

$$\sum_{i \neq j=1}^{n} (T_i^{\sharp} + T_j^{\sharp})(T_j^{\sharp} + T_i^{\sharp}) = \sum_{i,j=1}^{n} \left(T_i^{\sharp} T_i + T_j^{\sharp} T_i + T_i^{\sharp} T_j + T_j^{\sharp} T_j \right) - 4 \sum_{i=1}^{n} T_i^{\sharp} T_i$$
$$= 2n \sum_{i=1}^{n} T_i^{\sharp} T_i + 2 \sum_{i=1}^{n} T_i \sharp \sum_{i=1}^{n} T_i - 4 \sum_{i=1}^{n} T_i \sharp T_i$$
$$= 2 \left((n-2) \sum_{i=1}^{n} T_i \sharp T_i + \sum_{i=1}^{n} T_i \sharp \sum_{i=1}^{n} T_i \right) .$$

Substituting in we obtain our desired inequality. □

For $n = 2$, we get

$$\|T_1 + T_2\| \leq \left\| T_1^{\sharp} T_1 + T_2 \sharp T_2 \right\| .$$

Theorem 2.8. *For $T = (T_1, T_2, ..., T_n) \in B_A(H)$, we have*

$$\left\| \sum_{i,j=1}^{n} T_i \right\|_A^2 + \sum_{i,j=1}^{n} \|T_i\|_A^2 \leq \left\| \sum_{i,j=1}^{n} T_i^{\sharp} T_i \right\|_A + \frac{1}{2} \sum_{i,j=1}^{n} \left\| T_i + T_i^{\sharp} \right\|_A .$$

Proof. We have

$$\sum_{i,j=1}^{n} Re \langle T_i x, T_j x \rangle \leq \frac{1}{4} \sum_{1 \leq i \neq j \leq n} \|(T_i + Tj) x\|_A ,$$

then we have

$$\left\| \sum_{i,j=1}^{n} T_i \right\|_A^2 \leq \left\langle \left(\sum_{i=1}^{n} T_i^{\sharp} T_i \right) x, x \right\rangle + \frac{1}{4} \sum_{1 \leq i \neq j \leq n} \|(T_i + Tj) x\|_A .$$

Taking the supermum over $x \in H$ with $\|x\|_A = 1$, we obtain

$$\left\|\sum_{i,j=1}^{n} T_i\right\|_A^2 \leq \left\|\sum_{i=1}^{n} T_i^{\sharp} T_i\right\|_A + \frac{1}{4} \sum_{1\leq i\neq j\leq n} \|T_i + T_j\|_A^2 .$$

□

Theorem 2.9. *For $T = (T_1, T_2, ..., T_n)$, we have*

$$\left\|\sum_{i,j=1}^{n} T_i\right\|_A^2 \leq \left\|\sum_{i=1}^{n} T_i^{\sharp} T_i\right\|_A + \frac{1}{4} \left\|\sum_{1\leq i\neq j\leq n} T_j T_i^{\sharp} + I\right\|_A .$$

Proof. We have

$$Re \langle T_j x, T_i x\rangle \leq \frac{1}{4} \left\|\sum_{1\leq i\neq j\leq n} T_j T_i^{\sharp} + I\right\|_A .$$

Then we get

$$\left\|\sum_{i,j=1}^{n} T_i\right\|_A^2 \leq \left\|\sum_{i=1}^{n} T_i^{\sharp} T_i\right\|_A + \frac{1}{4} \left\|\sum_{1\leq i\neq j\leq n} T_j T_i^{\sharp} + I\right\|_A .$$

□

3. Conclusion

In this work, we have derived and analyzed several new inequalities involving the norm and numerical radius in the context of semi-Hilbertian spaces. The results presented in this paper contribute to the existing body of knowledge by extending classical inequalities in Hilbert space theory to the generalized framework of semi-Hilbertian spaces. Through the study of A-bounded operators and their A-adjoints, we established various norm inequalities, such as the bound on the numerical radius of sums of operators (Theorem 2.1), and the interplay between the operator norm and the semi-inner product induced by a positive operator A. The results provide tighter bounds and offer a deeper understanding of how operator summation and multiplication behave in this extended setting. Specifically, we demonstrated that for sums of operators $T = (T_1, T_2, \ldots, T_n)$, the inequalities relating the A-adjoint operators and their norms offer new insights into the structure of operator sums in semi-Hilbertian spaces. These inequalities provide sharper estimates than those found in earlier studies. Further, we derived results concerning the sum of norms for A-adjoints and their applications in proving new inequalities for operator products and sums. The work presented herein not only generalizes some classical results but also establishes new methods to approach operator inequalities in semi-Hilbertian spaces. These findings provide a stepping stone for further investigations into operator theory in spaces equipped with a semi-inner product, offering avenues for future research into applications in areas such as functional analysis, mathematical physics, and optimization.

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